2012 6. 4. QM1. (Grad.)

$$\langle \hat{e}_i | 1\, m \rangle = \sqrt{\frac{3}{4\pi}}\, \hat{e}_m^{\,i}$$

$$\begin{cases} \hat{e}_\pm = \mp \frac{1}{\sqrt{2}} (\hat{e}_x \pm i \hat{e}_y) & \text{: transverse polarization vector} \\ \hat{e}_0 = \hat{e}_z . & \text{: longitudinal P. V.} \\ \hat{e}_\lambda^{*} \cdot \hat{e}_{\lambda'} = \delta_{\lambda\lambda'} & \\ \hat{e}_m^{\,i} = \hat{e}^i \cdot \hat{e}_m & \\ \hat{e}^1 = \hat{e}_x ,\ \hat{e}^2 = \hat{e}_y ,\ \hat{e}^3 = \hat{e}_z . & \end{cases}$$

We learned that..

$$Y_{ll}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(2l)!}{2^{2l} \cdot (l!)^2}} \cdot \sin^l\theta \; e^{il\phi} .$$

$$\begin{cases} \cos\theta = \hat{n} \cdot \hat{e}_0 \\ \mp \frac{1}{\sqrt{2}} \sin\theta \, e^{\pm i\phi} = \hat{n} \cdot \hat{e}_\pm \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -\frac{1}{\sqrt{2}} \sin\theta \, e^{+i\phi} = \hat{n} \cdot \hat{e}_+ \\ \sin\theta \, e^{i\phi} = -\sqrt{2}\, \hat{n} \cdot \hat{e}_+ . \\ \sin^l\theta \cdot e^{il\phi} = (-1)^l \cdot \sqrt{2^l} \cdot (\hat{n} \cdot \hat{e}_+)^l \end{cases}$$

$$\therefore\ Y_{\ell\ell}(\theta,\phi) = (-1)^{\ell}\cdot\sqrt{\frac{2\ell+1}{4\pi}\cdot\frac{(2\ell)!}{2^{\ell}\cdot(\ell!)^2}}\cdot(\hat{n}\cdot\hat{e}_+)^{\ell}.$$

$$\left.\begin{aligned}
Y_{00}(\theta,\phi) &= \frac{1}{\sqrt{4\pi}}\\
Y_{11}(\theta,\phi) &= -\sqrt{\frac{3}{4\pi}}\cdot(\hat{n}\cdot\hat{e}_+)\\
Y_{22}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}}\,(\hat{n}\cdot\hat{e}_+)^2\\
Y_{33}(\theta,\phi) &= -\sqrt{\frac{35}{8\pi}}\,(\hat{n}\cdot\hat{e}_+)^3
\end{aligned}\right\} \Rightarrow Y_{\ell\ell}(\theta,\phi) = \alpha_\ell\,(\hat{n}\cdot\hat{e}_+)^{\ell}.$$

$$\langle \ell\ell|\ell\ell\rangle = 1.$$

$$\Downarrow$$

$$\langle \ell\ell|\left[\int d\Omega\,|\hat{n}(\theta,\phi)\rangle\langle\hat{n}(\theta,\phi)|\right]|\ell\ell\rangle.$$

$$= \int d\Omega\ Y_{\ell\ell}^{*}(\theta,\phi)\,Y_{\ell\ell}(\theta,\phi)$$

Ω dependence comes from here.

$$= |\alpha_\ell|^2\cdot\int d\Omega\ (\hat{n}\cdot\hat{e}_+^{*})^{\ell}\,(\hat{n}\cdot\hat{e}_+)^{\ell}$$

fixed!

$$= |\alpha_\ell|^2 \cdot \hat{e}_+^{*i_1} \cdot \hat{e}_+^{*i_2} \cdots \hat{e}_+^{*i_\ell}$$

$$\times \hat{e}_+^{j_1} \cdot \hat{e}_+^{j_2} \cdots \hat{e}_+^{j_\ell}$$

$$\times \int d\Omega\, \hat{n}_{i_1} \hat{n}_{i_2} \cdots \hat{n}_{i_\ell} \times \hat{n}_{j_1} \cdots \hat{n}_{j_\ell} . = 1 .$$

$\boxed{\ell = 1}$.

$$1 = |\alpha_1|^2 \cdot \hat{e}_+^{*i} \cdot \hat{e}_+^{j} \underbrace{\int d\Omega\, \hat{n}^i \hat{n}^j}_{\text{angle integration}}$$

↳ angle dependence vanishes!!

no directional information!!

invariant under rotation

$\Rightarrow$ Scalar Tensor!!

(ex. 1 !)

$\Downarrow$

$$\therefore \int d\Omega\, \hat{n}^i \hat{n}^j = C \cdot \delta^{ij} .$$

↓ trace!!

$$\int d\Omega\, \underbrace{\hat{n}^i \hat{n}^i}_{= \hat{n}\cdot\hat{n} = 1.} = C \cdot \mathrm{Tr}(\delta^{ii}) = C \cdot 3$$

$$= 4\pi = 3 \cdot C$$

$$\therefore C = \frac{4\pi}{3}$$

$\therefore \cancel{1 = |\alpha_1|^2}$

자세히

$$\int d\Omega\, \hat{n}^i \hat{n}^j = C \cdot \delta^{ij}$$

양변에 δ^{ij} 곱

$$\delta^{ij} \int d\Omega\, \hat{n}^i \hat{n}^j = \int d\Omega\, \hat{n}^i \hat{n}^j \delta^{ij} = C \cdot \delta^{ij} \delta^{ij}$$

$$\Rightarrow \int d\Omega\, (\hat{n} \cdot \hat{n}) = C \cdot \delta^{ii}$$

$$\Downarrow$$

$$4\pi = 3C$$

$$\therefore \quad 1 = |\alpha_1|^2 \cdot \hat{e}_+^{*i} \hat{e}_+^{j} \frac{4\pi}{3} \delta^{ij}$$

$$= |\alpha_1|^2 \cdot \underbrace{(\hat{e}_+^{*} \cdot \hat{e}_+)}_{=1} \cdot \frac{4\pi}{3}$$

$$\therefore \quad \alpha_1 = \sqrt{\frac{3}{4\pi}}$$

$$\therefore \quad \langle \hat{n} | \ell \ell \rangle = Y_{\ell\ell}(\theta, \phi) = \alpha_1 (\hat{n} \cdot \hat{e}_+)$$

$$= \sqrt{\frac{3}{4\pi}} (\hat{n} \cdot \hat{e}_+).$$

$\boxed{l=2}$

$$I = |\alpha_2|^2 \cdot \hat{e}_+^{*i}\, \hat{e}_+^{*j}\, \hat{e}_+^{k}\, \hat{e}_+^{l} \underbrace{\int d\Omega\, \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_l}.$$

Same way,

$$\int d\Omega\, \hat{n}^i \hat{n}^j \hat{n}^k \hat{n}^l = C\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)$$

rotational invariant는 identity??

i, j, k, l 은 symmetric

(서로 바꿔도 변화 없어야?)

∴ δ 앞 계수 모두 동일?

이런 이유 때문에 Kronecker delta 도 모두 symmetric

양변 δ^{ij} 곱

$$\delta^{ij} \int d\Omega\, \hat{n}^i \hat{n}^j \hat{n}^k \hat{n}^l = \int d\Omega\, \hat{n}^i \hat{n}^i \hat{n}^k \hat{n}^l$$

$$= C\left[\delta^{ii}\delta^{kl} + \delta^{ii} + \delta^{ii}\right]$$

양변 δ^{kl} 곱

$$\therefore \int d\Omega\, (\hat{n}\cdot\hat{n})^2 = C\left[\underset{3\times 3}{\delta^{ii}\delta^{kk}} + \underset{3}{\delta^{ii}} + \underset{3}{\delta^{ii}}\right]$$

$$= C \cdot 15$$

$\therefore \quad C = \dfrac{4\pi}{15}.$

$\therefore \quad \langle \hat{n} | 22 \rangle = Y_{22}(\theta, \phi) = \sqrt{\dfrac{15}{8\pi}} \, (\hat{n} \cdot \hat{e}_+)^2.$

$$1 = |\alpha_2|^2 \, \hat{e}_+^{*i} \, \hat{e}_+^{*j} \, \hat{e}_+^{k} \, \hat{e}_+^{l}$$

$$\times \underbrace{\frac{4\pi}{15} \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} \right)}_{= \int d\Omega \, \hat{n}^i \hat{n}^j \hat{n}^k \hat{n}^l}$$

$\hat{e}^+ \cdot \hat{e}^+ = 0$

$\therefore \delta^{ij}\delta^{kl}$ 의 contribution 없음

$$= |\alpha_2|^2 \, \frac{4\pi}{15} \cdot 2$$

$$\therefore \alpha_2 = \sqrt{\frac{15}{8\pi}}$$

~~[illegible]~~

reducible.

$3 \otimes 3 = 1 \oplus 3 \oplus 5$

scalar, axial vector, spin-2 tensor

~~Cartesian~~

Cartesian Tensor $\longrightarrow$ Spherial Tensor.

↳ reducible $\uparrow$ ⤴ irreducible

$\downarrow$

$$\boxed{\begin{array}{l} \text{ex). vector } \cancel{\text{facto}} \text{ indices.} \\ \hat{n}^i \hat{n}^j \hat{n}^k \hat{n}^l \\ \text{: rank- 4} \\ \text{Cartesian} \\ \text{Tensor.} \end{array}}$$

↳ angle integral

(⇒ lost directional information

⇒ becom scalar.)

ex). $A^i B^j$ (Cartesian rank-2)

rotation 에 의해

$$= \frac{\delta^{ij}}{3}(A \cdot B)$$
$$+ \frac{1}{2}(A^i B^j - A^j B^i)$$
$$+ \frac{1}{2}(A^i B^j + A^j B^i) - \frac{\delta^{ij}}{3}(A \cdot B)$$

(Spherial Tensor로

간단?

(Spherial Tensor는

anglular momentum의

eigenstate임

⇓

rotation 표현하기

적합)

Rank -n Cartesian Tensor

$\updownarrow$ 서로 다름

Rank - n Spherical Tensor 하지만..

변환가능.

왜냐하면.. Rank-1 Cartesian Tensor 가

Rank-1 Spherical Tenser 와 동등 하기 때문이다.

(~~spin~~ ∴ spin-1 particle

⇒ vector particle)

(massive spin-1)

의 angular momentum

state.

$$J_- |\ell\, m\rangle = \hbar \sqrt{J(J+1) - m(m-1)}\; |J\;\, m-1\rangle$$

$$\Downarrow$$

$$\langle \hat{n} | J\, m\rangle = \langle \hat{n} | \sqrt{\frac{(J+m)!}{(2J)!\,(J-m)!}} \left(\frac{J_-}{\hbar}\right)^{J-m} |J\, J\rangle$$

$J=1$:

$$\left.\begin{aligned} |1\, 0\rangle &= \frac{J_-}{\sqrt{2}\,\hbar} |1\, +\rangle \\ |1\, -1 &= \frac{J_-^2}{2\hbar^2} |1\, +\rangle \end{aligned}\right\}$$

$$\left\{\begin{aligned} \langle \hat{e}_\lambda | 1\, \pm\rangle &= \hat{e}_\pm^{\lambda} \\ \langle \hat{e}_\lambda | 1\, 0\rangle &= \hat{e}_0^{\lambda} \end{aligned}\right\}$$

($|J\, m\rangle$를 $\langle \hat{n}|$과 붙이면 spherical harmonics)

$\langle \hat{e}_\lambda |$과 붙이면??

$$\left(\langle \hat{n} | 1\, m\rangle = Y_{1m}(\theta, \phi)\right)$$

$$\left\{ \begin{aligned} \langle \hat{e}_i | 1 + \rangle &= \hat{e}_+{}^i \\ \langle \hat{e}_i | J_- | 1 + \rangle &= \sqrt{2}\hbar \, \hat{e}_0{}^i \\ \langle \hat{e}_i | J_-^2 | 1 + \rangle &= 2\hbar^2 \, \hat{e}_-{}^i . \end{aligned} \right.$$

Spin-2 ?

$$|2, 1\rangle = \frac{J_-}{2\hbar} \, |2 \; 2\rangle$$

$$|2 \; 0\rangle = \frac{J_-^2}{2\sqrt{6}\,\hbar^2} \, |2 \; 2\rangle$$

$$|2 \; {-1}\rangle = \frac{J_-^3}{12\hbar^3} \, |2 \; 2\rangle$$

$$|2 \; {-2}\rangle = \frac{J_-^4}{24\hbar^4} \, |2, 2\rangle$$

$$Y_{22}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\, e^{2i\phi} \sin^2\theta = \sqrt{\frac{15}{8\pi}}\, (\hat{n}\cdot\hat{e}_+)^2$$

$$Y_{21}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\, e^{i\phi} \sin\theta\cos\theta = \sqrt{\frac{15}{8\pi}} \cdot \sqrt{2}\, (\hat{n}\cdot\hat{e}_+)(\hat{n}\cdot\hat{e}_0)$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\, (2\cos^2\theta - \sin^2\theta) = \sqrt{\frac{15}{8\pi}}\sqrt{\frac{2}{3}}\left[(\hat{n}\cdot\hat{e}_0)^2 + (\hat{n}\cdot\hat{e}_+)(\hat{n}\cdot\hat{e}_-)\right]$$

~~Y_{2-1}~~

$$Y_{2-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\, \sqrt{2}\, (\hat{n}\cdot\hat{e}_-)(\hat{n}\cdot\hat{e}_0)$$

$$Y_{2-2}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\, (\hat{n}\cdot\hat{e}_-)^2.$$

∴ $\hat{n}$을 제거하면.. vector indices 2개.

하지만.. vector indices 가 2개 있더라도

이미 irreducible 상태임!!

즉.. $l=2$ 인 state 와 관계!!

∴ irreducible rank-2 Cartesian tensor를 spherical harmonics로부터 다 찾아 낼 수 있어!!

$$\hat{n}^i \hat{n}^j \times \sqrt{\frac{15}{8\pi}}\, \hat{e}_+^i \hat{e}_+^j = Y_{22}$$

$$\hat{n}^i \hat{n}^j \times \sqrt{\frac{15}{8\pi}} \sqrt{2}\, \hat{e}_+^i \hat{e}_0^j = Y_{21}$$

$$\hat{n}^i \hat{n}^j \times \sqrt{\frac{15}{8\pi}} \sqrt{\frac{2}{3}} \left(\hat{e}_0^i \hat{e}_0^j + \hat{e}_+^i \hat{e}_-^j \right) = Y_{20}$$

$$\hat{n}^i \hat{n}^j \times \sqrt{\frac{15}{8\pi}} \sqrt{2} \left(\hat{e}_-^i \hat{e}_0^j \right) = Y_{2-1}$$

$$\hat{n}^i \hat{n}^j \times \sqrt{\frac{15}{8\pi}}\, \hat{e}_-^i \hat{e}_-^j = Y_{2-2}.$$

$$\Downarrow$$

2)

$$T_{2m}^{ij} = \left(\langle e^i | \otimes \langle e^j | \right) | l m \rangle$$

이때.. $|l\, m\rangle$ 에 대한 정보는..
$\hat{e}_+$, $\hat{e}_0$, $\hat{e}_-$ 가 가지고 있다!

$$T_{1m}^{i} = \langle e^i | 1 m \rangle$$

$$T_{2m}^{ij} = \left(\langle e^i | \otimes \langle e^j | \right) | 2 m \rangle$$

$\Downarrow$

총 9개 하지만 $|2m\rangle$이 5개만 남겨둠

$$T_{3m}^{ijk} = \left(\langle e^i | \otimes \langle e^j | \otimes \langle e^k | \right) | 3 m \rangle$$

↳ Spherical Tensors (irreducible)