2012. 5. 30. QM I. (Grad.)

## Tensor operator..

(discrete transformation)

$P$ to vector : $P\vec{x} = -\vec{x}$.

$\uparrow$ parity $\quad (P^2 = \mathbb{1})$

$\big( P(t, \vec{x}) = (t, -\vec{x}) \big)$

ex. $\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\hookrightarrow P$

to axial vector : $P\vec{A} = \vec{A}$. $\rightarrow$ axial vector.

ex. Lorentz force..

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

( $\vec{E}$와 $\vec{B}$가 그냥 vector 였다면,
$\vec{F}$는 axial도 polar vector도 아닌 놈.
하지만 $\vec{F}$는 polar vector!!
$\therefore \vec{B} \rightarrow$ axial vector!! )

$$\left\{ \begin{array}{l} P\vec{E} = -\vec{E}, \quad P\vec{F} = -\vec{F} \\ P\vec{B} = \vec{B}. \end{array} \right.$$

$\downarrow$

From Biot-Savart law.

Scalar $\langle\alpha|\Theta|\alpha\rangle = {}_R\langle\alpha|\Theta|\alpha\rangle_R$

$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle$ $\quad = \langle\alpha|\mathcal{D}^\dagger\Theta\mathcal{D}|\alpha\rangle$

$|\alpha\rangle_P = P|\alpha\rangle$

$\therefore \begin{cases} \Theta = \mathcal{D}^\dagger\Theta\mathcal{D} & \text{if } \Theta \text{ is a scalar.} \\ \Theta = P^{-1}\Theta P & \text{(scalar)} \end{cases}$

(Pseudo-scalar)

$\begin{cases} {}_R\langle\alpha|\Theta_P|\alpha\rangle_R = \langle\alpha|\Theta_P|\alpha\rangle \\ {}_P\langle\alpha|\Theta_P|\alpha\rangle_P = -\langle\alpha|\Theta_P|\alpha\rangle. \end{cases}$

$\begin{cases} \Theta_P = \mathcal{D}^\dagger\Theta_P\mathcal{D} \\ \Theta_P = -P^{-1}\Theta_P P \end{cases}$ if $\Theta_P$ is a pseudoscalar.

vector operator

$\langle\alpha|V^i|\alpha\rangle \to \langle\alpha|\mathcal{D}^\dagger V^i\mathcal{D}|\alpha\rangle = R_{ij}\langle\alpha|V^j|\alpha\rangle$

$= {}_R\langle\alpha|V^i|\alpha\rangle_R$

$\mathcal{D}^{+} V^{i} \mathcal{D}$

$\mathcal{D}[R(\vec{\phi})] = e^{-\frac{i}{\hbar}\vec{\phi}\cdot\vec{J}}$ → 결과 $[V^{i}, J^{j}] = i\hbar\varepsilon_{ijk}V^{k}$

대입하고.. order by order 계산.

special case

$[J^{i}, J^{j}] = i\hbar\varepsilon_{ijk}J^{k}$.

Axial vector rotation & parity

$_{R}\langle\alpha|A^{i}|\alpha\rangle_{R} = R_{ij}\langle\alpha|A^{j}|\alpha\rangle$

$_{P}\langle\alpha|A^{i}|\alpha\rangle_{P} = \langle\alpha|A^{i}|\alpha\rangle$.

$\mathcal{U}^\dagger V^i \mathcal{U} = R_{ij} V^j$

vector $V_i$

- $V_i' = R_{ij} V_j$, $V'^2 = V'^T V' = (RV)^T (RV)$

$$= V^T \underbrace{(R^T R)}_{\mathbb{1}} V = V^T V.$$

a matrix.

- $V^T A V \to V'^T A V' = (RV)^T A (RV)$

$$= V^T \underbrace{(R^T A R)}_{A'} V$$

∴ Under rotation, a matrix $A$ transforms as.

$$A_{ij} \to (R^T A R)_{ij} = (R^T)_{ix} A_{xy} (R)_{yj}$$

$$= R_{xi} R_{yj} A_{xy}$$ (matrix는 같이 취급X)

ex. $V_i \to R_{ij} V_j$ rank 1.

∴ $A_{ij} \to$ Rank 2 ~~ten~~ tensor

If.. $V^T A V$ is a scalar..

$$V^T A V = (V')^T A' \cdot V'$$

$$\downarrow$$

$$= (RV)^T A_R (RV)$$

$$= V^T (\underbrace{R^T A_R R}) V = V^T \underbrace{A} V$$

$$\therefore \quad R^T A_R R = A. \qquad \underline{A_R = R A R^T}$$

(if $V^T A V$ is a scalar)

$$\therefore \ (A_R)_{ij} = \cancel{R_{ix} R} \quad R_{ix} A_{xy} (R^T)_{yj}$$

$$= \underline{R_{ix} R_{jy} A_{xy}}$$

(matrix도 같이 돈다)

⇓

이 경우 tensor라 한다.

⇓

Rank 2 tensor.

## Tensor

$$\underbrace{T'^{ijk\cdots} = \overbrace{R_{ix}R_{jy}R_{kz}\cdots}^{n\text{개} \to \text{Rank-}n\text{ tensor}}\, T^{xyz\cdots}}_{\text{definition}}$$

(길이 rotate 오오오오오오오오오)

## Examples of tensors

- $\mathbb{1}$

$$R\mathbb{1}R^T = RR^T = \mathbb{1}.$$

$\therefore$ Identity matrix is invariant under rotation.

(이 조건이가.. 다른 vector의 길이 가 invariant 하도록 보장오오)

길이 → metric

$$V'^T V' = V^T V.$$

$$V'^T \mathbb{1} V' \qquad (RV)^T(RV) = V^T \underbrace{(R^T R)}_{\mathbb{1}} V$$

$\Updownarrow$

metric tensor of the Euclidean space ($\mathbb{1}$) is invariant under rotation.

- $\delta_{ij} \xrightarrow{\;\;} \overset{\to\, \delta'_{ab}}{R_{ai} R_{bj} \delta_{ij}}$

$$= R_{ai} R_{bi} = R_{ai} (R^T)_{ib}$$

$$= (RR^T)_{ab} = \delta_{ab}.$$

(identity matrix !!)

$\delta_{ij}$ is symmetric and invariant under rotation

<u>Tensor operator</u> $V^{i_1, \cdots, i_n}$.

$${}_R\langle \alpha | V^{i_1 \cdots i_n} | \alpha \rangle_R = R_{i_1 j_1} R_{i_2 j_2} \cdots \langle \alpha | V^{j_1 j_2 \cdots} | \alpha \rangle.$$

ex. $\underline{A^i B^j}$ ?

(Rank 2)

$$(A^i B^j)_R \Rightarrow A'^i B'^j = R^{ia} R^{jb} A^a B^b$$

<u>symmetric 과 antisymmetric 으로 분리 가능.</u>

$\Downarrow$

$$\frac{1}{2} (A^i B^j + A^j B^i) - \frac{\delta^{ij}}{3} \vec{A} \cdot \vec{B} \quad \text{(traceless)}$$

$$\frac{1}{2} (A^i B^j - A^j B^i) \quad \text{(traceless)}$$

$$\frac{\delta^{ij}}{3} \vec{A} \cdot \vec{B} \quad \text{(trace !!)}$$

$\therefore$ $A^i B^j = \frac{1}{3}\delta^{ij}\vec{A}\cdot\vec{B}$ → 1 degree of freedom ⇒ traceful symmetric tensor

9 degrees of freedom

remaining 5 degrees of freedom

$+ \frac{1}{2}(A^i B^j + A^j B^i) - \frac{\delta^{ij}}{3}\vec{A}\cdot\vec{B}$ → traceless symmetric tensor

$+ \frac{1}{2}(A^i B^j - A^j B^i)$ → antisymmetric tensor

cross product

3 degrees of freedom

⊗ special notation

- $A^{(i}B^{j)}$ : symmetric traceful.

  $\left(\frac{\delta^{ij}}{3}\vec{A}\cdot\vec{B}\right)$

- $A^{[i}B^{j]}$ : antisymmetric traceless

  $\left(\frac{1}{2}(A^i B^j - A^j B^i)\right)$

⊗ $A^i B^j - A^j B^i = \varepsilon^{ijk}(\vec{A}\times\vec{B})^k$

rank 1. axial vector?

$\boxed{A^i B^j + A^j B^i : \text{ really rank 2.}}$

## Under rotation

- $\frac{1}{3}\delta_{ij}\,\vec{A}\cdot\vec{B} \to \frac{1}{3}\delta_{ij}\,\vec{A}\cdot\vec{B}$ (invariant)

- $\frac{1}{2}\varepsilon^{ijk}(\vec{A}\times\vec{B})^k \to \frac{1}{2}\varepsilon^{ijk}R^{kl}(\vec{A}\times\vec{B})^l$

- $A^{(i}B^{j)} \to R_{ia}R_{jb}\,A^{(a}B^{b)}$

$\Downarrow$

$3 \otimes 3 = 1 \oplus 3 \oplus 5$

# Spherical Harmonics and Tensor

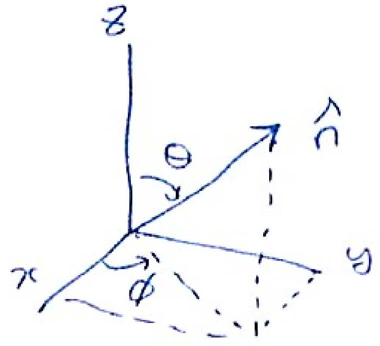
$Y_{lm}(\theta, \phi) \equiv \langle \hat{n}(\theta, \phi) | l\, m \rangle$

$\downarrow$

Cartesian coordinate로..

표현해보자!

그렇다면. ~~Cate~~ Cartesian coordinate로 표현된 tensor들을 어떻게 다룰지 Idea를 얻을 수 있을 것!




$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\begin{pmatrix} \hat{e}_1 \cdot \hat{n} = \sin\theta\cos\phi \\ \hat{e}_2 \cdot \hat{n} = \sin\theta\sin\phi \\ \hat{e}_3 \cdot \hat{n} = \cos\theta \end{pmatrix}$$

$(\hat{e}_i \cdot \hat{e}_j = \delta_{ij})$

$$\left( \begin{array}{l} \hat{e}_1 \pm i\hat{e}_2 = \sin\theta\, e^{\pm i\phi}, \\ \hat{e}_\pm \equiv \mp \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2), \quad \hat{e}_0 \equiv \hat{e}_3. \end{array} \right)$$ 새로운 정의

?

- $\hat{e}_\pm \cdot \hat{e}_0 = 0.$
- $\hat{e}_\pm \cdot \hat{e}_\pm = \frac{1}{2}(\hat{e}_1 \pm i\hat{e}_2)\cdot(\hat{e}_1 \pm i\hat{e}_2)$

$= 0.$

- $\hat{e}_\pm \cdot \hat{e}_\mp = -\frac{1}{2}(\hat{e}_1 \pm i\hat{e}_2)\cdot(\hat{e}_1 \mp i\hat{e}_2)$

$= -\frac{1}{2}(1+1) = -1.$

$\therefore$

$$\begin{cases} (\hat{e}_\lambda)^* \cdot (\hat{e}_\pm) = -\delta_{\lambda\pm} & (\lambda = + \text{ or } - \text{ or } 0) \\ (\hat{e}_\lambda)^* \cdot (\hat{e}_0) = +\delta_{\lambda 0} \end{cases}$$

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$$\langle \hat{n} | 00 \rangle = Y_{00}(\theta, \phi)$$

$$= \frac{1}{\sqrt{4\pi}} \text{ (no direction)}$$

(spin 0 : nothing to rotate. scalar.)
(no direction).

**Normalization**

$$\boxed{\begin{aligned} \langle lm | lm \rangle &= \int d\Omega \, \langle lm | \hat{n} \rangle \langle \hat{n} | lm \rangle \\ &= \int |Y_{lm}|^2 \, d\Omega = 1. \end{aligned}}$$

$l = 1.$

$$Y_{11}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta = \sqrt{\frac{3}{4\pi}} \, \hat{n} \cdot \hat{e}_+$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \, \hat{n} \cdot \hat{e}_0$$

$$Y_{1-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta = \sqrt{\frac{3}{4\pi}} \, \hat{n} \cdot \hat{e}_- .$$

$\therefore\ Y_{11}(\theta.\phi) = \sqrt{\frac{3}{4\pi}}\ \hat{n}\cdot\boxed{\hat{e}_+} \Rightarrow -\frac{1}{2}(\hat{e}_1 + i\hat{e}_2)$

(positive angular momentum state!)

$Y_{10}(\theta.\phi) = \sqrt{\frac{3}{4\pi}}\ \hat{n}\cdot\hat{e}_0$

$Y_{1-1}(\theta.\phi) = \sqrt{\frac{3}{4\pi}}\ \hat{n}\cdot\boxed{\hat{e}_-} \Rightarrow \frac{1}{\sqrt{2}}(\hat{e}_1 - i\hat{e}_2)$

Spin = 1 particle ⇒ vector particle!

(vector $\hat{e}_+$, $\hat{e}_0$, $\hat{e}_-$ 의 projection 으로 표현된다!!)

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$\therefore\ T^i_{1m} = \langle \hat{e}_i | 1m \rangle = \sqrt{\frac{3}{4\pi}}\ \hat{e}_m{}^i.$ $\quad \left(m = \underset{(1)}{+},\ 0,\ \underset{(-1)}{-}\right).$

ex. $Y_{ll} = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(2l)!}{2^l \cdot l!}}\ (\hat{n}\cdot\hat{e}_+)^l.$

Main goal : Prove the Wigner-Eckart Theorem