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Clebsch-Gordon coefficients.

$$\begin{cases} J^2 |j m\rangle = \hbar^2 j(j+1) |j\ m\rangle \\ J_z |j m\rangle = \hbar m\ |j\ m\rangle. \\ J_\pm |j m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)}\ |j\ m \pm 1\rangle. \\ \langle j' m' | j m\rangle = \delta_{j'j}\delta_{m'm}. \\ \langle j_1^\circ j_2^\circ\ m_1' m_2' | j_1 j_2 m_1 m_2\rangle = \delta_{m_1' m_1}\delta_{m_2' m_2}. \end{cases}$$

$$|j = j_1 + j_2,\ m = j\rangle = |j_1\ m_1 = j_1\rangle \otimes |j_2\ m_2 = j_2\rangle$$

$$= |j_1 j_2 ; j_1 j_2\rangle.$$

$$\Downarrow$$

$$|j\ \ j-1\rangle = \frac{(J_- / \hbar)}{\sqrt{(j+j)(j-j+1)}} |j_1 j_2 ; j_1 j_2\rangle.$$

where.. $J_- = J_{1-} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_{2-}$.

$$\therefore |j\ j-1\rangle = \frac{1}{\sqrt{2j}} \left[\sqrt{2j_1}\ |j_1 j_2 ; j_1 - 1\ j_2\rangle + \sqrt{2j_2}\ |j_1 j_2 ; j_1\ j_2 - 1\rangle \right]$$

$$= \sqrt{\frac{j_1}{j}}\ |j_1 j_2 ; j_1 - 1\ j_2\rangle + \sqrt{\frac{j_2}{j}}\ |j_1 j_2 ; j_1\ j_2 - 1\rangle$$

$$|j\ j-2\rangle = \frac{J_-/\hbar}{\sqrt{(j+j-1)(j-j+1+1)}}\,|j\ j-1\rangle$$

$$= \frac{J_-/\hbar}{\sqrt{(2j-1)\cdot 2}}\,|j\ j-1\rangle$$

$$= \frac{J_-/\hbar}{\sqrt{(2j-1)\cdot 2}}\left[\sqrt{\frac{j_1}{j}}\,|j_1 j_2 ; j_1-1\ j_2\rangle + \sqrt{\frac{j_2}{j}}\,|j_1 j_2 ; j_1\ j_2-1\rangle\right]$$

$$= \frac{1}{\sqrt{(2j-1)\cdot 2}}\sqrt{\frac{j_1}{j}}\left[\sqrt{(j_1+j_1-1)(j_1-j_1+1+1)}\,|j_1 j_2 ; j_1-2\ j_2\rangle + \sqrt{(j_2+j_2)(j_2-j_2+1)}\,|j_1 j_2 ; j_1-1\ j_2-1\rangle\right]$$

$$+ \frac{1}{\sqrt{(2j-1)\cdot 2}}\cdot\sqrt{\frac{j_2}{j}}\left[\sqrt{(j_1+j_1)(j_1-j_1+1)}\,|j_1 j_2 ; j_1-1\ j_2-1\rangle + \sqrt{(j_2+j_2-1)(j_2-j_2+1+1)}\,|j_1 j_2 ; j_1\ j_2-2\rangle\right]$$

$$= \sqrt{\frac{j_1\,(2j_1-1)\cdot 2}{(2j-1)\cdot 2\cdot j}}\,|j_1 j_2 ; j_1-2\ j_2\rangle$$

$$+ \left(\sqrt{\frac{j_1\cdot(2j_2)\cdot 1}{(2j-1)\cdot 2\cdot j}} + \sqrt{\frac{j_2\cdot 2j_1\cdot 1}{(2j-1)\cdot 2\cdot j}}\right)|j_1 j_2 ; j_1-1\ j_2-1\rangle$$

$$+ \sqrt{\frac{j_2\cdot(2j_2-1)\cdot 2}{(2j-1)\cdot 2\cdot j}}\,|j_1 j_2 ; j_1\ j_2-2\rangle$$

$$= \sqrt{\frac{(2j_1-1)\cdot j_1}{(2j-1)\cdot j}}\ |j_1\ j_2 ;\ j_1-2\ \ j_2\rangle$$

$$+\ 2\sqrt{\frac{j_1 j_2}{(2j-1)\cdot j}}\ |j_1\ j_2 ;\ j_1-1\ \ j_2-1\rangle$$

$$+\ \sqrt{\frac{(2j_2-1)\cdot j_2}{(2j-1)\cdot j}}\ |j_1\ j_2 ;\ j_1\ \ j_2-2\rangle .$$

$\Downarrow$ $j_1 = 1/2$, $j_2 = 1/2$ 인 경우에 대입 //

$$|j=1,\ m=1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2} ; \tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$|j=1,\ m=0\rangle = \sqrt{\frac{1/2}{1}}\ |\tfrac{1}{2}\ \tfrac{1}{2} ; -\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$+ \sqrt{\frac{1/2}{1}}\ |\tfrac{1}{2}\ \tfrac{1}{2} ; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2} ; -\tfrac{1}{2}\ \tfrac{1}{2}\rangle + |\tfrac{1}{2}\ \tfrac{1}{2} ; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle\right)$$

$$|j=1,\ m=-1\rangle = 2\sqrt{\frac{1/2\cdot 1/2}{(2\cdot 1-1)\cdot 1}}\ |\tfrac{1}{2}\ \tfrac{1}{2} ; -\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$= |\tfrac{1}{2}\ \tfrac{1}{2} ; -\tfrac{1}{2}\ -\tfrac{1}{2}\rangle .$$

triplet은 다구함. $(2 \otimes 2 = 1 \oplus 3)$

singlet은?

$\sqrt{\frac{1}{2}}\left[\,|\tfrac{1}{2}\,\tfrac{1}{2}; -\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{1}{2}; \tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right]$ 와 orthogonal.

$\Downarrow$

Convention.

$$|j=0,\ m=0\rangle = \sqrt{\frac{1}{2}}\left[\,|\tfrac{1}{2}\,\tfrac{1}{2}; +\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{1}{2}\,\tfrac{1}{2}; -\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]$$

In general..

$J_{1-}\otimes 1_2$ $\qquad$ $1_1 \otimes J_{2-}$

$$|j=j_1+j_2,\ m\rangle = \sqrt{\frac{(j_1+j_2+m)!}{[2(j_1+j_2)]!\,(j_1+j_2-m)!}}\left(\frac{J_{1-}+J_{2-}}{\hbar}\right)^{j_1+j_2-m}|j_1 j_2; j_1 j_2\rangle$$

$j_1 = 1$, $j_2 = \frac{1}{2}$

$j_{max} = \frac{3}{2}$. $\qquad j_{min} = \frac{1}{2}$.

$$\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = \left|1\ \tfrac{1}{2} ; 1\ \tfrac{1}{2}\right\rangle.$$

$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(\left|1\ \tfrac{1}{2} ; 1\ -\tfrac{1}{2}\right\rangle + \sqrt{2}\left|1\ \tfrac{1}{2} ; 0\ \tfrac{1}{2}\right\rangle \right)$$

$$\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left[\left|1\ \tfrac{1}{2} ; -1\ +\tfrac{1}{2}\right\rangle + \sqrt{2}\left|1\ \tfrac{1}{2} ; 0\ -\tfrac{1}{2}\right\rangle \right]$$

$$\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = \left|1\ \tfrac{1}{2} ; -1\ -\tfrac{1}{2}\right\rangle.$$

? ⇒ orthogonality?

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\left|1\ \tfrac{1}{2} ; 1\ -\tfrac{1}{2}\right\rangle - \left|1\ \tfrac{1}{2} ; 0\ \tfrac{1}{2}\right\rangle \right]$$

$$\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left[\sqrt{2}\left|1\ \tfrac{1}{2} ; -1\ \tfrac{1}{2}\right\rangle - \left|1\ \tfrac{1}{2} ; 0\ -\tfrac{1}{2}\right\rangle \right]$$

"apply ladder operator"

(for conserving the sign convention).

$j_1 = 1. \quad j_2 = 1 \quad \rightarrow j_{max} = 2. \quad j_{min} = 0.$

$3 \otimes 3 = 5 \oplus 3 \oplus 1.$

$|2\,2\rangle = |1\,1;1\,1\rangle.$

$|2\,1\rangle = \frac{1}{\sqrt{2}}\left[|1\,1;1\,0\rangle + |1\,1;0\,1\rangle\right]$

$|2\,0\rangle = \frac{1}{\sqrt{6}}\left[|\;;1\,-1\rangle + |\;;-1\,1\rangle + 2|\;;0\,0\rangle\right]$

$|2\,-1\rangle = \frac{1}{\sqrt{2}}\left[|\;;-1\,0\rangle + |\;;0\,-1\rangle\right]$

$|2\,-2\rangle = |1\,1;-1\,-1\rangle.$

orthogonality

$\rightarrow |1\,1\rangle = \frac{1}{\sqrt{2}}\left[|\;;1\,0\rangle - |\;;0\,1\rangle\right] \quad \mathcal{J}$ ladder.

$|1\,0\rangle = \frac{1}{\sqrt{2}}\left[|\;;1\,-1\rangle - |\;;-1\,1\rangle\right] \quad \mathcal{J}$ "

$|1\,-1\rangle = \frac{1}{\sqrt{2}}\left[|\;;-1\,0\rangle - |\;;0\,-1\rangle\right]$

orthogonality (with $|2\,0\rangle$ & $|1\,0\rangle$)

$\rightarrow |0\,0\rangle = a|\;;1\,-1\rangle + b|\;;-1\,1\rangle + c|\;;0\,0\rangle$

$= \frac{1}{\sqrt{3}}\left[|\;;1\,-1\rangle + |\;;-1\,1\rangle - |\;;0\,0\rangle\right] \quad a^2+b^2+c^2=1.$

$(a, b, c \in \mathbb{R})$

~~$|2\,0$~~

$\langle 2\,0|0\,0\rangle = 0 \Rightarrow a+b+2c=0$

$\langle 1\,0|0\,0\rangle = 0 \Rightarrow a-b=0.$

$\therefore a = b = -c$

[Convention of C-G coefficient]

$$\underline{j_1 = l \quad j_2 = \tfrac{1}{2}}$$

$$j_{max} = l + \tfrac{1}{2} \qquad j_{min} = l - \tfrac{1}{2}$$

$$\Rightarrow \quad |\, j_{max} = l + \tfrac{1}{2},\ m_{max} = l + \tfrac{1}{2} \rangle = |\, l\ \tfrac{1}{2} ; l\ \tfrac{1}{2} \rangle$$

$$/\!/$$

$$|\, l + \tfrac{1}{2},\ l + \tfrac{1}{2} \rangle = |\, l\ \tfrac{1}{2} ; l\ \tfrac{1}{2} \rangle$$

$$|\, l + \tfrac{1}{2},\ l - \tfrac{1}{2} \rangle = \frac{(J_- / \hbar)}{\sqrt{(l + \frac{1}{2} + l + \frac{1}{2})(l + \frac{1}{2} - l - \frac{1}{2} + 1)}}\ |\, l + \tfrac{1}{2},\ l + \tfrac{1}{2} \rangle$$

$$= \frac{1}{\sqrt{(2l+1)}} \left(\frac{J_-}{\hbar} \right) |\, l\ \tfrac{1}{2} : l\ \tfrac{1}{2} \rangle$$

$$= \frac{1}{\sqrt{2l+1}} \Big[\sqrt{(l+l)(l-l+1)}\ |\, l\ \tfrac{1}{2} : l-1\ \tfrac{1}{2} \rangle$$

$$+ \sqrt{(\tfrac{1}{2} + \tfrac{1}{2})(\tfrac{1}{2} - \tfrac{1}{2} + 1)}\ |\, l\ \tfrac{1}{2} : l\ -\tfrac{1}{2} \rangle \Big]$$

$$= \frac{1}{\sqrt{2l+1}} \Big[\sqrt{2l}\ |\, l\ \tfrac{1}{2} : l-1\ \tfrac{1}{2} \rangle + |\, l\ \tfrac{1}{2} : l\ -\tfrac{1}{2} \rangle \Big]$$

$$= \sqrt{\frac{2l}{2l+1}}\ |\, l\ \tfrac{1}{2} : l-1\ \tfrac{1}{2} \rangle + \sqrt{\frac{1}{2l+1}}\ |\, l\ \tfrac{1}{2} : l\ -\tfrac{1}{2} \rangle$$

$$|\, l+\tfrac{1}{2},\ l+\tfrac{1}{2} \rangle = |\, l\ \tfrac{1}{2} ;\ l\ \tfrac{1}{2} \rangle$$

$$|\, l+\tfrac{1}{2},\ l-\tfrac{1}{2} \rangle = \sqrt{\frac{2l}{2l+1}}\ |\, l\ \tfrac{1}{2} ;\ l-1\ \tfrac{1}{2} \rangle + \sqrt{\frac{1}{2l+1}}\ |\, l\ \tfrac{1}{2} ;\ l\ -\tfrac{1}{2} \rangle$$

orthogonal

$$|\, l-\tfrac{1}{2},\ l-\tfrac{1}{2} \rangle = -\sqrt{\frac{1}{2l+1}}\ |\, l\ \tfrac{1}{2} ;\ l-1\ \tfrac{1}{2} \rangle + \sqrt{\frac{2l}{2l+1}}\ |\, l\ \tfrac{1}{2} ;\ l\ -\tfrac{1}{2} \rangle$$

$$\begin{cases} aX + bY \\ \Downarrow \text{orthogonal} \\ -bX + aY \end{cases}$$

↓ 내리기만 하면 된다!

$$|\, l+\tfrac{1}{2},\ l-\tfrac{3}{2} \rangle = \frac{(J_-/\hbar)}{\sqrt{(l+\frac{1}{2}+l-\frac{1}{2})(l+\frac{1}{2}-l+\frac{1}{2}+1)}}\ |\, l+\tfrac{1}{2},\ l-\tfrac{1}{2} \rangle$$

⋮

Show that..

$$|l+\tfrac{1}{2}\ m\rangle = \sqrt{\frac{l+\frac{1}{2}+m}{2l+1}}\ |l\ \tfrac{1}{2};\ m-\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\frac{l+\frac{1}{2}-m}{2l+1}}\ |l\ \tfrac{1}{2};\ m+\tfrac{1}{2},\ -\tfrac{1}{2}\rangle$$

$$|l-\tfrac{1}{2},\ m\rangle = -\sqrt{\frac{l+\frac{1}{2}-m}{2l+1}}\ |l\ \tfrac{1}{2};\ m-\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\frac{l+\frac{1}{2}+m}{2l+1}}\ |l\ \tfrac{1}{2};\ m+\tfrac{1}{2},\ -\tfrac{1}{2}\rangle.$$

$$\left[\begin{aligned}
&\mathcal{D}^{(j_1)}[R(\hat{n},\phi)]\,\mathcal{D}^{(j_2)}[R(\hat{n},\phi)]\circ \\
&= \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{D}^{(j)}[R(\hat{n},\phi)] \\
&= \left(\begin{array}{c|c|c}
\mathcal{D}^{(j=|j_1-j_2|)}(R) & 0 & 0 \\ \hline
0 & \mathcal{D}^{|j_1-j_2|+1}(R) & 0 \\ \hline
0 & 0 & \ddots \\ \hline
 & & \mathcal{D}^{(j_1+j_2)}(R)
\end{array}\right)
\end{aligned}\right]$$