2012. 5. 21. QM I. (Grad).

Addition of Angular Momenta

$A = \{a_1, \cdots, a_m\}$

$B = \{b_1, \cdots, b_n\}$

$\rightarrow A \otimes B = \{(a_i, b_j) \mid a_i \in A, b_j \in B\}$ ← direct product.

↳ ex. $|a\rangle \otimes |b\rangle = |a, b\rangle$

~~$|j_1 m_2\rangle \otimes$~~

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{\oplus} (?) = \sum_{\substack{\oplus \\ j \,\&\, m}} |j\, m\rangle$$

$(2j_1+1)$ cases $(2j_2+1)$ cases.

⊗

$$(|a'\rangle \otimes |b'\rangle)^\dagger (|a\rangle \otimes |b\rangle) = \langle a'|a\rangle \langle b'|b\rangle = \delta_{a'a}\delta_{b'b}$$

$$\begin{cases} \sum_a |a\rangle\langle a| = \mathbb{1}_{m\times m} \\ \sum_b |b\rangle\langle b| = \mathbb{1}_{n\times n} \end{cases}$$

$$\Downarrow$$

$$\sum_{a,b} \left(|a\rangle \otimes |b\rangle \right) \left(\langle a| \otimes \langle b| \right) = \mathbb{1}_{m\times m} \otimes \mathbb{1}_{n\times n}$$

$$= \mathbb{1}_{(m\times n)\times(m\times n)}$$

<u>Direct Sum</u> ?

$$A \oplus B \xrightarrow{\text{(matrix representation)}} \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$$

$$A \rightarrow \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \\ a_{m1} & & a_{mm} \end{pmatrix} \qquad B \rightarrow \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \\ b_{n1} & & b_{nn} \end{pmatrix}$$

※
$$\dim(A \oplus B) = \dim(A) + \dim(B)$$
$$\dim(A \otimes B) = \dim(A) \times \dim(B)$$

dimensional analysis (product가 sum으로 표현된다면...?)

$$(2j_1+1)(2j_2+1) = \sum_{j=a}^{b} (2j+1)$$

dim. of direct product. — dim. of direct sum.

→ (sol.) $\begin{cases} a = |j_1 - j_2| \\ b = j_1 + j_2 \end{cases}$

$\vec{J}_1, \vec{J}_2$ (two - angular momentum operators)

$$\begin{cases} \vec{J}_1^2 |j_1, m_1\rangle = \hbar^2 j_1(j_1+1) |j_1, m_1\rangle \\ \vec{J}_2^2 |j_2, m_2\rangle = \hbar^2 j_2(j_2+1) |j_2, m_2\rangle \\ J_{1z} |j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle \\ J_{2z} |j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle \end{cases}$$

$$\{ |j_1 m_1\rangle \otimes |j_2 m_2\rangle \} = \{ |j_1 j_2 ; m_1 m_2\rangle \}$$

- $\dim \{ \text{"} \} = (2j_1+1)(2j_2+1)$ (dimension)

- $\langle j_1 j_2 ; m_1' m_2' | j_1 j_2 ; m_1 m_2 \rangle = \langle j_1 m_1' | j_1 m_1 \rangle \langle j_2 m_2' | j_2 m_2 \rangle$

(orthonormality)

$$= \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

- (Completeness)

$$\sum_{m_1 m_2} |j_1 j_2 ; m_1 m_2\rangle\langle j_1 j_2 ; m_1 m_2|$$

$$= \sum_{m_1 m_2} |j_1 m_1\rangle\langle j_1 m_1| \otimes |j_2 m_2\rangle\langle j_2 m_2|$$

$$= \mathbb{1}_{(2j_1+1)\times(2j_1+1)} \otimes \mathbb{1}_{(2j_2+1)\times(2j_2+1)}$$

$$= \mathbb{1}_{[(2j_1+1)(2j_1+1)]\times[(2j_2+1)(2j_2+1)]}$$

- $[\vec{J}_1^2, J_{1z}] = [\vec{J}_2^2, J_{2z}] = 0$

$[\vec{J}_1^2, J_{2i}] = [\vec{J}_2^2, J_{1i}] = 0$ (commutation relation)

$i = x, y, z$ $\quad i = x, y, z$.

- $[J_{1i}, J_{2j}] = 0$ (x, y, z)

$[J_{1i}, J_{1j}] = i\hbar \varepsilon_{ijk} J_{1k}$

$[J_{2i}, J_{2j}] = i\hbar \varepsilon_{ijk} J_{2k}$

(Commutation relation)

$\mathcal{D}[R(\hat{n}, \phi)] \left(|j_1, m_1\rangle \otimes |j_2 m_2\rangle \right)$

$= \left(\mathcal{D}^{(j_1)}[R(\hat{n}, \phi)] \otimes \mathcal{D}^{(j_2)}[R(\hat{n}, \phi)] \right)$

$\left(|j_1 m_1\rangle \otimes |j_2 m_2\rangle \right)$

$$\mathcal{D}[R(\hat{n}, \phi)] = \lim_{n \to \infty} \left(1 - \frac{i}{n\hbar} \hat{n} \cdot \vec{J} \phi \right)^n = e^{-\frac{i}{\hbar} \hat{n} \cdot \vec{J} \phi}$$

Using this idea..

$$\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)} = \lim_{n \to \infty} \left[\left(\mathbb{1}_{(2j_1+1) \times (2j_1+1)} - \frac{i}{n\hbar} \hat{n} \cdot \vec{J}_1 \phi \right) \otimes \left(\mathbb{1}_{(2j_2+1) \times (2j_2+1)} - \frac{i}{n\hbar} \hat{n} \cdot \vec{J}_2 \phi \right) \right]^n$$

$$\overset{?}{=} \mathcal{D}^{(\otimes)(j)}[R(\hat{n},\phi)]$$

$$= \lim_{n\to\infty}\left[\mathbb{1}_{[(2j_1+1)(2j_2+1)]^2} - \frac{i}{n\hbar}\hat{n}\cdot\vec{J}\phi\right]^n$$

↕ (Compare!)

(check)

$$\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)} = \lim_{n\to\infty}\Big[\mathbb{1}_{[(2j_1+1)(2j_2+1)]\times[(2j_1+1)(2j_2+1)]} - \frac{i}{n\hbar}\Big(\mathbb{1}_{(2j_1+1)\times(2j_1+1)} \otimes \vec{J}_2 + \vec{J}_1 \otimes \mathbb{1}_{(2j_2+1)\times(2j_2+1)}\Big)\phi\Big]^n$$

∴ The total angular momentum

$$\vec{J} = \mathbb{1}_{(2j_1+1)\times(2j_1+1)} \otimes \vec{J}_2 + \vec{J}_1 \otimes \mathbb{1}_{(2j_2+1)\times(2j_2+1)}.$$

(in short, $\vec{J} = \vec{J}_1 + \vec{J}_2$)

$$= \mathbb{1}_1 \otimes \vec{J}_2 + \vec{J}_1 \otimes \mathbb{1}_2.$$

The ladder operator

$$J_\pm = J_x \pm i J_y.$$

$\Downarrow$

(and)
$$\begin{cases} J_\pm = J_{1\pm} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_{2\pm}. \\ J_z = J_{1z} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_{2z}. \\ J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) \end{cases}$$

$$J_{1+}J_{2-} = J_{2-}J_{1+} = J_{1x}J_{2x} + J_{1y}J_{2y} + i(J_{1x}J_{2y} - J_{1y}J_{2x})$$

(commute)

$$J_{1-}J_{2+} \quad = \quad " \quad -i(\quad " \quad)$$

$$\therefore \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+}) = [\vec{J}_1 \cdot \vec{J}_2 - J_{1z}J_{2z}]$$

$$\therefore \vec{J}_1 \cdot \vec{J}_2 = J_{1z}J_{2z} + \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+})$$

$$\left[\begin{aligned} \therefore \vec{J}^2 &= (\vec{J}_1 + \vec{J}_2)^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 \\ &= \vec{J}_1^2 + \vec{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}. \end{aligned}\right]$$

Check if $|j_1 j_2 ; m_1 m_2\rangle$ is an eigenket of $\vec{J}^2$ (?)

<u>If it is true..</u>

$$[\vec{J}^2, \vec{J}_1^2] = [\vec{J}^2, \vec{J}_2^2] = [\vec{J}^2, J_{1z}] = [\vec{J}^2, J_{2z}] = 0$$

However it is not true!

$$\dim \{ \underbrace{|j_1 m_1\rangle \otimes |j_2 m_2\rangle}_{\downarrow} \} = (2j_1+1) \times (2j_2+1)$$

// must be presered

$$\dim \{ \sum_{j \oplus} |j \underbrace{m}_{(2j+1)}\rangle \}$$

$$\sum_{j=a}^{b} (2j+1) = (2j_1+1)(2j_2+1)$$

⊛ $\sum_{a}^{b} 1 = b-a+1$

$$\sum_{j=a}^{b} j = \left(\sum_{j=1}^{b} j\right) - \left(\sum_{j=1}^{a-1} j\right)$$

$$= \frac{b(b+1)}{2} - \frac{1}{2}(a-1)a$$

$$= \frac{1}{2}(b+a)(b-a+1)$$

⊛ $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\therefore \sum_{j=a}^{b} (2j+1) = (b-a+1) + (b+a)(b-a+1)$$

$$= (b-a+1)\left[1 + (b+a)\right]$$

$$= (b-a+1)(b+a+1)$$

$$= (b-a+1)(b+a+1)$$

$$\therefore \quad (b+a+1)(b-a+1) = (2j_1+1)(2j_2+1)$$

$$\begin{cases} b = \max(j) = j_1+j_2 \\ a = \min(j) = |j_1-j_2| \end{cases}$$

$$b+a = 2j_1 \quad (j_1 \geq j_2)$$

$$b-a = 2j_2$$

$$\therefore \begin{cases} b = j_1+j_2 \\ a = j_1-j_2 \end{cases}$$

$$\therefore \; b = j_1+j_2$$

$$a = |j_1-j_2|$$

//

$$\therefore \quad \dim\left\{ \sum_{j=|j_1-j_2|}^{j_1+j_2} {}^{\oplus} \, |j\,m\rangle \right\}$$

$$\therefore \quad \{ |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle \} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \{ |j\, m\rangle \} \Big|_{m=m_1+m_2}.$$

ex.

$$\{ \underbrace{|\tfrac{1}{2}\, m_1\rangle}_{2개} \otimes \underbrace{|\tfrac{1}{2}\, m_2\rangle}_{2개} \} = \underbrace{\{ |0\,0\rangle \}}_{1} \oplus \underbrace{\{ |1, m\rangle \}}_{3}$$

1: spin singlet

3: spin triplet.

$\therefore\ |0\ 0\rangle = ?$

total 4 cases

Convention : real

$|1\ 1\ 1\rangle = ?$

$\boxed{?}$: Clebsch-Gordon Coefficient

$j_1\ j_2\ m_1\ m_2$

$$|0\ 0\rangle = \cancel{|\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle} + \boxed{?}\, |\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

($m=1$.)

$$+ \boxed{?}\, |\tfrac{1}{2}\ \tfrac{1}{2};\ -\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \cancel{|\tfrac{1}{2}\ \tfrac{1}{2};\ -\tfrac{1}{2}\ -\tfrac{1}{2}\rangle}.$$

($m=-1$.)

$|1\ 1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$. determined

$|1\ 0\rangle = \boxed{?}\, |\tfrac{1}{2}\ \tfrac{1}{2};\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle + \boxed{?}\, |\tfrac{1}{2}\ \tfrac{1}{2};\ -\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$|1\ -1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2};\ -\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$. determined.

$$|j\,m\rangle = \sum_{m_1} C_{m_1} \cdot |j_1 j_2 ; m_1, m-m_1\rangle$$

$$= \sum_{m_1} |j_1 j_2 ; m_1, m-m_1\rangle\langle j_1 j_2 ; m_1, m-m_1 | j\, m\rangle$$

Clebsch-Gordon Coefficient 구하는 방법

결정된것 (the higtest or lowest) 에서 부터
ladder operator $J_\pm = 1_1 \otimes J_{2\pm} + J_{1\pm} \otimes 1_2$
걸어가며 하나씩 구해 간다.

⇓ 결과

$$\begin{cases} |1\ 0\rangle = \dfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2} ; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle + |\tfrac{1}{2}\ \tfrac{1}{2} ; -\tfrac{1}{2}\ \tfrac{1}{2}\rangle\right) \\ |0\ 0\rangle = \dfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2} ; \tfrac{1}{2}\ -\tfrac{1}{2}\rangle - |\tfrac{1}{2}\ \tfrac{1}{2} ; -\tfrac{1}{2}\ \tfrac{1}{2}\rangle\right). \end{cases}$$