2012. 5. 16. QM 1. (Grad)

$$\mathcal{L}_{\Theta} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

$$\left[\mathcal{L}_{\Theta} + l(l+1)\right]\Theta = 0.$$

$\downarrow$

$$\Big[\underbrace{\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + l(l+1)\sin^2\theta}_{\Theta \text{ dependent}} + \underbrace{\frac{\partial^2}{\partial\phi^2}}_{\phi \text{ dependent}}\Big]\Theta = 0.$$

$$\therefore \Theta = A(\theta)\,\Phi(\phi)$$

$$\Phi\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + l(l+1)\sin^2\theta\, A\right] + A\cdot\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

Dividing by $A\Phi$,

$$\therefore \frac{\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dA}{d\theta}\right) + l(l+1)\sin^2\theta\, A}{A} = -\frac{\frac{d^2\Phi}{d\phi^2}}{\Phi} = \text{Const.}$$

(ind. of θ & ϕ)

$\boxed{\phi \text{ dependence}}$

$$\langle r, \theta, \phi | L_z | n, l, m \rangle$$

coordinate space representation $L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$

$$= \frac{\hbar}{i}\frac{\partial}{\partial \phi} \langle r, \theta, \phi | n, l, m \rangle$$

$$= m\hbar \langle r, \theta, \phi | n, l, m \rangle.$$

$$= \langle r, \theta, \phi | m\hbar | n, l, m \rangle$$

Since.. $\langle r, \theta, \phi | n, l, m \rangle = R_{nl}(r)\, \Theta(\theta)\, \underbrace{\Phi(\phi)}$

ϕ dependence appears here only!

$$\therefore \frac{\hbar}{i}\frac{\partial}{\partial \phi}\Phi(\phi) = m\hbar\, \Phi(\phi)$$

$$\therefore \frac{d^2\Phi}{d\phi^2} = -m^2\, \Phi.$$

$$\therefore \underbrace{-\frac{1}{\Phi}\cdot\frac{d^2\Phi}{d\phi^2} = m^2}$$

$\boxed{\theta \text{ dependence}}$

$\cancel{\sin\theta \frac{\partial}{\partial\theta}}$

$$\sin\frac{d}{d\theta}\left(\sin\theta \frac{dA}{d\theta}\right) + l(l+1)\sin^2\theta\, A - m^2 A = 0.$$

$$\therefore \quad \frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin^2\theta\,\frac{1}{\sin\theta}\frac{dA}{d\theta}\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)A=0.$$

$$\begin{cases} x = \cos\theta \\ dx = -\sin\theta\, d\theta \\ \sin^2\theta = 1-x^2 \end{cases}$$

$$\downarrow$$

$$\frac{d}{dx}\left((1-x^2)\frac{dA}{dx}\right)+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]A=0.$$

$\Downarrow$ (Associated Legendre equation).

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell^m(x)}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P_\ell^m(x)=0.$$

$$\frac{\hbar}{i}\frac{d}{d\phi}\Phi(\phi)=m\hbar\,\Phi(\phi) \longrightarrow \Phi \propto e^{im\phi}.$$

$$\therefore \quad \Theta = A(\theta)\,\Phi(\phi) \propto P_\ell^m(\cos\theta)\,e^{im\phi}$$

$\downarrow$ normalization

$$Y_\ell^m(\theta,\phi)=\langle \hat{n}(\theta,\phi)\,|\,\ell,m\rangle.$$

(r is decoupled)
$(V(\vec{r}) \Rightarrow V(r))$

$\boxed{Y_{lm}(\theta,\phi)}$ "The complete form"? $\qquad$ $l+m+1$.

$$Y_{ll}(\theta,\phi) = \langle \hat{n}(\theta,\phi) | l\, l \rangle$$

$$\downarrow$$

$$Y_{lm}(\theta,\phi) = \boxed{} \langle \hat{n}(\theta,\phi) | L_-^{\,l-m} | l\, l \rangle$$

$$l-(l-m) = m$$

$$\left\{ \begin{aligned} &| L_- | l\, m \rangle |^2 = \langle l\, m | L_+ L_- | l\, m \rangle \\ &L_- | l\, m \rangle = \hbar \sqrt{(l+m)(l-m+1)}\, | l\; m-1 \rangle \end{aligned} \right]$$

$$L_- | l\, l \rangle = \hbar \sqrt{(l+l)(l-l+1)}\, | l\; l-1 \rangle = \hbar \sqrt{2l \cdot 1}\, | l\; l-1 \rangle$$

$$(L_-)^2 | l\, l \rangle = \hbar \sqrt{(l+l)(l-l+1)} \cdot \hbar \sqrt{(l+l-1)(l-l+1+1)}\, | l\; l-2 \rangle$$

$$= \hbar^2 \sqrt{2l \cdot 1} \cdot \sqrt{2l-1 \cdot 2}\, | l\; l-2 \rangle$$

$$\vdots$$

$$(L_-)^{l-m} | l\, l \rangle = (\hbar)^{l-m} \cdot \sqrt{2l \cdot (2l-1) \cdot \cdots \cdot (2l-(l-m+1))}$$

$$\cdot \sqrt{1 \cdot 2 \cdot \cdots \cdot (l-m)}\; | l\, m \rangle$$

$$= (\hbar)^{l-m} \sqrt{\frac{(2l)!\,(l-m)!}{(l+m)!}}\; | l\, m \rangle$$

$$\therefore \; |l\, m\rangle = \left(\frac{L_-}{\hbar}\right)^{l-m} |l\, l\rangle \cdot \sqrt{\frac{1}{(2l)!}\cdot\frac{(l+m)!}{(l-m)!}}$$

$|l\, l\rangle$??

$$L_+ |l\, l\rangle = 0 \qquad (\text{cf. } a|0\rangle = 0).$$

$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Explicitly..

$\langle r, \hat{n}(\theta,\phi)|$

$$\langle \vec{x}|L_+|\alpha\rangle = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\langle \vec{x}|\alpha\rangle$$

$$\therefore \; \cancel{\hbar e^{i\phi}}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) Y_{ll}(\theta,\phi) = 0.$$

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi} Y_{ll} = l\hbar\, Y_{ll}$$

$$\therefore \; \cancel{\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}}$$

$$\therefore \left(\frac{\partial}{\partial\theta} + i\cot\theta \cdot \frac{im\hbar}{\hbar}\, l\right) Y_{ll} = 0$$

$$\therefore \left(\frac{\partial}{\partial\theta} - l\cot\theta\right) Y_{ll} = 0.$$

$$\underline{Y_{\ell\ell}(\theta,\phi) = f(\theta)\, e^{i\ell\phi}}$$

$$\therefore \left(\frac{d}{d\theta} - \ell \cot\theta \right) f(\theta) = 0.$$

$$\therefore \frac{df}{d\theta} = \ell \frac{\cos\theta}{\sin\theta} f(\theta).$$

$$\underline{\cos\theta\, d\theta = d(\overset{x}{\sin\theta})}$$

$$\frac{df(\theta)}{d(\sin\theta)} = \frac{\ell}{\sin\theta} f(\theta).$$

$$\therefore \frac{df}{dx} = \frac{\ell}{x} f. \quad \longrightarrow \quad \frac{df}{f} = \ell \cdot \frac{dx}{x}$$

$$\log \frac{f}{f_0} = \ell \cdot \log \frac{x}{x_0}$$

normalization
-tion

$$\Rightarrow \cancel{f(\theta) =} \quad \cancel{f(x) \propto x^{\ell}} \quad f(x) = N \cdot x^{\ell}$$

$$\therefore \cancel{f(\theta) =}$$

$$\therefore f(\theta) = \underset{C_\ell}{\underset{\parallel}{N}} \cdot x^{\ell} = C_\ell x^{\ell}. = C_\ell \cdot \sin^{\ell}\theta.$$

$\therefore \quad Y_{\ell\ell}(\theta,\phi) = C_\ell \cdot \sin^\ell\theta \; e^{im\phi}$

$\langle\alpha|\alpha\rangle = 1.$

$\langle\alpha| \int d^3x \, |\vec{x}\rangle\langle\vec{x}|\alpha\rangle = 1.$

$V(\vec{r}) = V(r).$

$|r, \hat{n}(\theta,\phi)\rangle = |r\rangle \otimes |1 \cdot \hat{n}(\theta,\phi)\rangle$

$$\left[\int_0^\infty dr \cdot r^2 \, |r\rangle\langle r| \; \int d\Omega \, |\hat{n}\rangle\langle\hat{n}| \right.$$

$$\left. \left(\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi = \int d\Omega = 4\pi \right) \right]$$

$\langle \ell m | \ell m \rangle = 1$

$\downarrow$

$\langle \ell m | \int d\Omega \, |\hat{n}\rangle\langle\hat{n}|\ell m\rangle = \int d\Omega \, \langle \ell m|\hat{n}\rangle\langle\hat{n}|\ell m\rangle$
$= 1$

$\int d\Omega \, \langle \ell \ell | \hat{n}\rangle\langle\hat{n}|\ell \ell\rangle = 1$

$$\boxed{\int d\Omega \; Y_{\ell\ell}^*(\theta,\phi) \, Y_{\ell\ell}(\theta,\phi) = 1}$$

$$\int_{-1}^{1} d\cos\theta \int_{0}^{2\pi} d\phi \; |C_\ell|^2 \cdot (\underbrace{\sin^2\theta}_{1-x^2})^{\ell} = 1 \qquad x = \cos\theta .$$

$$\therefore \; |C_\ell|^2 \cdot 2\pi \cdot \int_{-1}^{1} dx \, (1-x^2)^{\ell} = 1$$

$$\downarrow$$

$$4\pi \, |C_\ell|^2 \int_{0}^{1} dx \, (1-x^2)^{\ell} = 1 .$$

$$\left(\begin{array}{l} x^2 = t \qquad x = t^{1/2} \\ 2x \, dx = dt \\ \quad dx = \dfrac{dt}{2x} = \dfrac{1}{2} t^{-1/2} \, dt \end{array} \right)$$

$$\cancel{\int_0^1 d\phi}$$

$$\int_0^1 dx \, (1-x^2)^{\ell} = \int_0^1 dt \cdot \frac{t^{-1/2}}{2} \cdot (1-t)^{\ell}$$

$$= \frac{1}{2} \int_0^1 dt \;\; t^{\frac{1}{2}-1} \;\; (1-t)^{(\ell+1)-1}$$

$$= \frac{1}{2} \, B\left(\frac{1}{2}, \ell+1\right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\ell+\frac{1}{2}\right)}{2 \cdot \Gamma\left(\ell+\frac{3}{2}\right)}$$

있다
이렇게도 풀 수 있다
하지만 더 좋은 방법!

$$2\pi\,|C_\ell|^2 \int_{-1}^{1} dx\,(1-x^2)^\ell = 1.$$

$$\left(\begin{array}{l} dx = 2dt \\ 1+x = 2t \\ x = 2t-1. \\ 1-x = 2-2t \\ 1-x^2 = 4t(1-t) \end{array}\right)$$

$$2\pi\,|C_\ell|^2 \cdot \int_0^1 (2dt)\cdot\left[4t(1-t)\right]^\ell$$

$$\cancel{= 2^{2\ell+1}}$$

$$= 2\pi\,|C_\ell|^2 \cdot\ 2^{2\ell+1} \cdot\ B(\ell+1, \ell+1)$$

$$= 2\pi\,|C_\ell|^2 \cdot\ 2^{2\ell+1} \cdot\ \frac{\Gamma(\ell+1)\Gamma(\ell+1)}{\Gamma(2\ell+2)}$$

$$= 1.$$

$$\therefore\ |C_\ell|^2 = \frac{(2\ell+1)!}{4\pi\cdot 2^{2\ell}\cdot (\ell!)^2}$$

$$\boxed{Y_{\ell\ell}(\theta,\phi) = \sqrt{\frac{(2\ell+1)!}{4\pi}}\ \frac{e^{i\ell\phi}}{2^\ell\cdot \ell!}\sin^\ell\theta}$$

$$\therefore \quad Y_{lm}(\theta,\phi) = \frac{1}{2^l\, l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sqrt{\frac{1}{(2l)!}\cdot\frac{(l+m)!}{(l-m)!}}\left(\frac{L_-}{\hbar}\right)^{l-m}\left[e^{il\phi}\sin^l\theta\right]$$

$$\boxed{Y_{lm}(\theta,\phi) = \frac{1}{2^l\, l!}\sqrt{\frac{(2l+1)\cdot(l+m)!}{4\pi\cdot(l-m)!}}\left(\frac{L_-}{\hbar}\right)^{l-m}\left[e^{il\phi}\sin^l\theta\right]}$$

$$\boxed{\left(\frac{L_-}{\hbar}\right) = -e^{-i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)}$$

$$\langle \vec{x}\,|\,n,l,m\rangle = R_{nl}(r)\,Y_{lm}(\theta,\phi) \qquad (V(\vec{r}) = V(r))$$

$$= \langle r\,|\,n,l\rangle\,\langle \hat{n}(\theta,\phi)\,|\,l,m\rangle$$

Norm.
$$\begin{cases} \displaystyle\int_0^\infty dr\cdot r^2\,|R_{nl}(r)|^2 = 1 \\ \displaystyle\underbrace{\int d\Omega}_{\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi}\,|Y_{lm}(\theta,\phi)|^2 = 1 \end{cases}$$

l = half integer 이면..
매우 이상한 결과 나옴 (존재 X)
∴ spherical harmonics 는
l = 정수 일때만 존재!!