2012. 5. 9. QM 1. (Grad.)

$$\mathcal{D}(R) = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi}$$

$\uparrow$ $R(\hat{n}, \phi)$

if $j = 1/2$ $\quad \vec{J} = \vec{S} = \frac{\hbar}{2}\vec{\sigma}$

$$\mathcal{D}(R) = e^{-\frac{i}{\hbar}\cdot\frac{\hbar}{2}\cdot\vec{\sigma}\cdot\hat{n}\phi}$$

$$= e^{-i\vec{\sigma}\cdot\hat{n}\frac{\phi}{2}}$$

$$= \sum_{k=0}^{\infty}\left(-i\frac{\phi}{2}\right)^{k}\frac{(\vec{\sigma}\cdot\hat{n})^{k}}{k!}$$

$$= \sum_{k=0}^{\infty}\left(-i\frac{\phi}{2}\right)^{2k}\frac{\overbrace{(\vec{\sigma}\cdot\hat{n})^{2k}}^{1}}{(2k)!} + \sum_{k=0}^{\infty}\left(-i\frac{\phi}{2}\right)^{2k+1}\frac{\overbrace{(\vec{\sigma}\cdot\hat{n})^{2k+1}}^{\vec{\sigma}\cdot\hat{n}}}{(2k+1)!}$$

$$(\vec{\sigma}\cdot\hat{n})^2 = \sigma_i\sigma_j\,\hat{n}_i\cdot\hat{n}_j$$

$2\delta_{ij}1$

$$= \frac{1}{2}(\underbrace{\sigma_i\sigma_j + \sigma_j\sigma_i}_{2\delta_{ij}1})\,\hat{n}_i\hat{n}_j$$

$$= \delta_{ij}\,\hat{n}_i\hat{n}_j = \hat{n}\cdot\hat{n} = 1$$

$$\mathcal{D}_{j=1/2}(R) = 1\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\phi}{2}\right)^{2k} - i\vec{\sigma}\cdot\hat{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\phi}{2}\right)^{2k+1}$$

$$= 1\cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore \quad \mathcal{D}_{j=1/2}(R) = \begin{pmatrix} \cos\phi/2 - i\hat{n}_3 \sin\phi/2 & (-i\hat{n}_1 - \hat{n}_2)\sin\phi/2 \\ (-i\hat{n}_1 + \hat{n}_2)\sin\phi/2 & \cos\phi/2 + i\hat{n}_3 \sin\phi/2 \end{pmatrix}$$

ex).

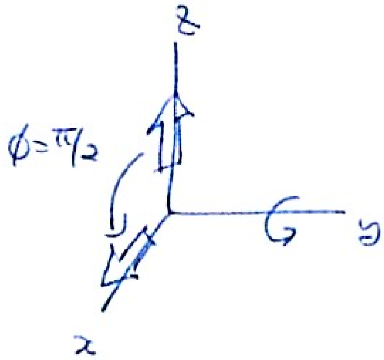
$$\mathcal{D}[R(\hat{n}_3, \phi)] = \begin{pmatrix} e^{-\frac{i}{2}\phi} & 0 \\ 0 & e^{\frac{i}{2}\phi} \end{pmatrix}$$

$$\mathcal{D}[R(\hat{n}_1, \phi)] = \begin{pmatrix} \cos\phi/2 & -i\sin\phi/2 \\ -i\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$\mathcal{D}[R(\hat{n}_2, \phi)] = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

state ket 찾아보자!

$$\left|\tfrac{1}{2}, m_z = \tfrac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$



$$\left|\tfrac{1}{2}, m_z = -\tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$\boxed{\text{rotate?}}$

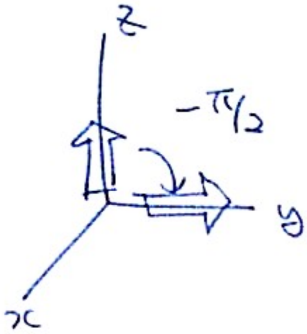
$$\left|\tfrac{1}{2}, m_x = \pm\tfrac{1}{2}\right\rangle \propto \mathcal{D}[R(\hat{y}, \pi/2)] \left|\tfrac{1}{2}, m_z = \pm\tfrac{1}{2}\right\rangle$$

$$= \begin{pmatrix}\cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4\end{pmatrix} \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{(+)\text{ case}} \text{ or } \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{(-)\text{ case}}.$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\sim\end{pmatrix}$$

$$\therefore \left|\tfrac{1}{2}, m_x = \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\left|\tfrac{1}{2}, m_x = -\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$ (phase is not important.)



$$\left|\tfrac{1}{2}, m_y = \pm\tfrac{1}{2}\right\rangle = \mathcal{D}\left[R(\hat{x}, -\pi/2)\right]\left|\tfrac{1}{2}, m_z = \pm\tfrac{1}{2}\right\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(+) case (−) case

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

up down

$$\mathcal{D}\left[R(\hat{n}, \phi)\right] = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi}$$

$$= \sum_{k=0}^{\infty}\left(-\frac{i}{\hbar}\right)^k \left(\vec{J}\cdot\hat{n}\right)^k \phi^k$$

$$\mathcal{D}\left[R(\hat{n}, -\phi)\right] = e^{+\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi} = \mathcal{D}^{-1}\left[R(\hat{n}, \phi)\right]$$

ϕ is real

$J_i^{\dagger} = J_i$

$$\Rightarrow \underline{\mathcal{D}^{-1}\left[R(\hat{n}, \phi)\right] = \mathcal{D}^{\dagger}\left[R(\hat{n}, \phi)\right]}$$

∴ an operator

$$\Theta \longrightarrow \mathcal{D}\left[R(\hat{n}, \phi)\right]\Theta\,\mathcal{D}^{\dagger}\left[R(\hat{n}, \phi)\right].$$

$|\beta\rangle = \Theta|\alpha\rangle.$

$\mathcal{D}|\beta\rangle = (\mathcal{D}\Theta\mathcal{D}^{+})(\mathcal{D}|\alpha\rangle)$

state가 $|\alpha\rangle \to \mathcal{D}|\alpha\rangle$
$|\beta\rangle \to \mathcal{D}|\beta\rangle$
로 변하면.. operator는
$\Theta \to \mathcal{D}\Theta\mathcal{D}^{+}$ 로 변합니다.

ex)
$j = 1/2$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$ $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

$S_x = \mathcal{D}[R(y,\pi/2)]\, S_z\, \mathcal{D}^{+}[R(y,\pi/2)].$

$\mathcal{D}[R(y,\pi/2)] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\mathcal{D}^{+}[R(y,\pi/2)] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$j=1$

$$J_1 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \qquad |1, m_z=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$J_2 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad |1, m_z=0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad |1, m_z=-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\Downarrow$$

our question

$$|1, m_x=1\rangle = ??$$

$$\left(\frac{J_1}{\hbar}\right)^2 = \frac{1}{2} \begin{pmatrix} 1 & & 1 \\ & 2 & \\ 1 & & 1 \end{pmatrix}$$

$$\left(\frac{J_2}{\hbar}\right)^2 = \frac{1}{2} \begin{pmatrix} 1 & & -1 \\ & 2 & \\ -1 & & 1 \end{pmatrix}$$

$$\left(\frac{J_3}{\hbar}\right)^2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$\longrightarrow \boxed{\left(\frac{J_i}{\hbar}\right)^3 = \frac{J_i}{\hbar} \quad \text{for } i=1,2,3}$$

$$\mathcal{D}[R(\hat{n}_i,\phi)] = e^{-\frac{i}{\hbar}J_i\phi}$$

$$= \sum_{k=0}^{\infty} (-i)^{2k}\left(\frac{J_i}{\hbar}\right)^{2k}\frac{\phi^{2k}}{(2k)!} + \sum_{k=0}^{\infty} (-i)^{2k+1}\left(\frac{J_i}{\hbar}\right)^{2k+1}\frac{\phi^{2k+1}}{(2k+1)!}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$1, J_i^2, J_i^4, \ldots \qquad\qquad J_i, J_i^3, J_i^5, J_i^6 \ldots$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$J_i^2 \qquad\qquad\qquad\qquad J_i \quad J_i \quad J_i$$

$$= 1 + \bigcirc J_i + \bigcirc J_i^2$$

$$= 1 - \left(\frac{J_i}{\hbar}\right)^2 + \left(\frac{J_i}{\hbar}\right)^2 \sum_{k=0}^{\infty}\frac{(-1)^k\phi^{2k}}{(2k)!} - i\left(\frac{J_i}{\hbar}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\phi^{2k+1}}{(2k+1)!}$$

$$= 1 + \left(\frac{J_i}{\hbar}\right)^2(\cos\phi - 1) - i\left(\frac{J_i}{\hbar}\right)\sin\phi.$$

$$= \mathcal{D}$$

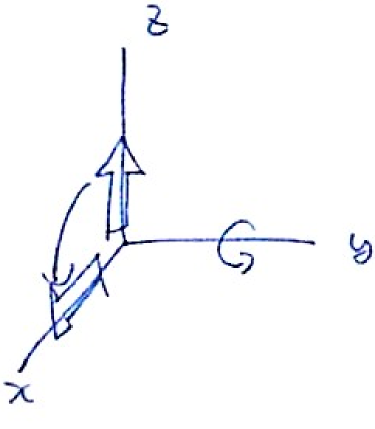
$$\mathcal{D}[R(\hat{x},\phi)] = \begin{pmatrix} \frac{1}{2}(1+c) & -\frac{i}{\sqrt{2}}s & -\frac{1}{2}(1-c) \\ -\frac{i}{\sqrt{2}}s & c & -\frac{i}{\sqrt{2}}s \\ -\frac{1}{2}(1-c) & -\frac{i}{\sqrt{2}}s & \frac{1}{2}(1+c) \end{pmatrix}$$

$$\mathcal{D}[R(\hat{y},\phi)] = \begin{pmatrix} \frac{1}{2}(1+c) & -\frac{1}{\sqrt{2}}s & \frac{1}{2}(1-c) \\ \frac{1}{\sqrt{2}}s & c & -\frac{1}{\sqrt{2}}s \\ \frac{1}{2}(1-c) & \frac{1}{\sqrt{2}}s & \frac{1}{2}(1+c) \end{pmatrix}$$

$$\mathcal{D}[R(\hat{z},\phi)] = \begin{pmatrix} e^{-i\phi} & & \\ & 1 & \\ & & e^{i\phi} \end{pmatrix}$$

$c : \cos\phi$

$s : \sin\phi$



$$\begin{cases} \hat{n} = \hat{y}. \\ \phi = \pi/2. \end{cases}$$

$\cos \pi/2 = 0$

$\sin \pi/2 = 0$

$$\mathcal{D}[R(\hat{y},\tfrac{\pi}{2})] = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

$$\mathcal{D}[R(\hat{y},\pi/2)]\,|1, m_z=1\rangle = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow m_x = 1.\ \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad m_x = 0\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad m_x = -1.\ \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Check

$$J_1 = \mathcal{D}[R(\hat{y},\pi/2)]\, J_3\, \mathcal{D}^{+}[R(\hat{y},\pi/2)]$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & 0 & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Correct

$$\mathcal{D}[R(\hat{z},\phi)] = e^{-\frac{i}{\hbar}J_3\phi} \qquad J_3|j\,m\rangle = m\hbar\,|j\,m\rangle$$

$$\mathcal{D}[R(\hat{z},\phi)]\,|j\,m\rangle = e^{-\frac{i}{\hbar}J_3\phi}\,|j\,m\rangle$$

$$= e^{-\frac{i}{\hbar}m\hbar\phi}\,|j\,m\rangle$$

$$= e^{-im\phi}\,|j\,m\rangle.$$

$$J_z = \hbar\begin{pmatrix} j & & & 0 \\ & j-1 & & \\ & & \ddots & \\ 0 & & & -j \end{pmatrix}$$

$$\langle j\,m'|\,\mathcal{D}[R(\hat{z},\phi)]\,|j\,m\rangle = \delta_{mm'}\,e^{-im\phi}$$

$$\therefore\ \mathcal{D}[R(\hat{z},\phi)] = \begin{pmatrix} e^{-ij\phi} & & & & 0 \\ & e^{-i(j-1)\phi} & & & \\ & & \ddots & & \\ & & & e^{-im\phi} & \\ & & & & \ddots \\ 0 & & & & & e^{ij\phi} \end{pmatrix}$$

ex.) $$\mathcal{D}_{j=1/2}[R(\hat{z},\phi)] = \begin{pmatrix} e^{-\frac{i}{2}\phi} & 0 \\ 0 & e^{\frac{i}{2}\phi} \end{pmatrix}$$

$$\mathcal{D}_{j=1}[R(\hat{z},\phi)] = \begin{pmatrix} e^{-i\phi} & & 0 \\ & 1 & \\ 0 & & e^{i\phi} \end{pmatrix}$$

$$\mathcal{D}_{j=3/2}[R(\hat{z},\phi)] = \begin{pmatrix} e^{-i\frac{3}{2}\phi} & & & 0 \\ & e^{-i\phi/2} & & \\ & & e^{i\phi/2} & \\ 0 & & & e^{i\frac{3}{2}\phi} \end{pmatrix}$$

H.W.

$$\mathcal{D}[R_z(\alpha)R_y(\beta)R_z(\gamma)] \stackrel{?}{=} \mathcal{D}[R_z(\alpha)]\mathcal{D}[R_y(\beta)]\mathcal{D}[R_z(\gamma)]$$