2012. 5. 7. @ QM 1 (Grad.)

$$J_- J_+ = J^2 - J_z (J_z + \hbar)$$

$$J_+ J_- = J^2 - J_z (J_z - \hbar)$$

$$| J_\pm |\Lambda, \lambda\rangle |^2 = \langle \Lambda, \lambda | \left(J^2 - J_z (J_z \pm \hbar) \right) | \Lambda, \lambda \rangle$$

$$= [\Lambda - \lambda(\lambda \pm \hbar)] \underbrace{\langle \Lambda, \lambda | \Lambda, \lambda \rangle}_{=1}$$

$$= \Lambda - \lambda(\lambda \pm \hbar)$$

$$\left.\begin{array}{l} \lambda = m\hbar \\ \Lambda = \Lambda' \hbar^2 \end{array}\right) \Rightarrow = [\Lambda' - m(m \pm 1)] \hbar^2 \geq 0$$

there is the upper bound of m

and

there is the lower bound of m

When $m = m_{max}$

$$\Lambda' = m_{max} (m_{max} + 1)$$

$$(J_+ \underbrace{| \Lambda', m_{max} \hbar \rangle}_{\text{upper bound}} = 0)$$

$$j \equiv m_{max}$$

Then $\boxed{\Lambda' = j(j+1)}$

Lower bound
↓

$\Lambda' = j(j+1)$

$\therefore j(j+1) - m_{min}(m_{min} - 1) = 0.$

$\therefore m_{min} = -j$ ($j+1$ is also solution, but.. $m_{min} \leq m_{max}$. $\therefore -j$ is the only solution)

$\therefore -j \leq m \leq j$

$m = -j, -j+1, \cdots, j-1, j.$

$2j$ steps (by 1)

$2j+1$ elements.

↳ natural number. (1, 2, 3, 4, ⋯)

$j = \begin{cases} 0, 1, \cdots & \text{: (orbital angular momentum)} \\ \frac{1}{2}, \frac{3}{2}, \cdots & \text{: (spin..)} \end{cases}$

↳ spherical harmonics..!

↓ electron.. proton...

Total Angular Momentum

$$\vec{J} = \vec{L} + \vec{S}$$

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\ |j, m\pm1\rangle$$

$$= \hbar\sqrt{(j\mp m)(j\pm m+1)}$$

$$\times |j, m\pm1\rangle$$

$\because$ $J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$

$J_z|j,m\rangle = \hbar m|j,m\rangle$

$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \cdots \quad \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

$|j\ j\rangle \quad |j, j-1\rangle \quad |j, -j\rangle$

⊛

$$j^2 + j - m^2 \mp m$$

$$= j^2 - m^2 + j \mp m$$

$$= (j+m)(j-m) + \begin{cases}(j-m)\\(j+m)\end{cases}$$

~~$= \{(j+m)(j-m+1)$~~

$$= \begin{cases}(j-m)(j+m+1)\\(j+m)(j-m+1)\end{cases}$$

$$= (j\mp m)(j\pm m+1)$$

$j=0 \quad (2j+1=1)$

$|0\ 0\rangle = (1)$

$j=\frac{1}{2} \quad (2j+1=2)$

$|\frac{1}{2}\ \frac{1}{2}\rangle \qquad |\frac{1}{2}\ -\frac{1}{2}\rangle$

$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$

$J^2 = \hbar^2 j(j+1) = \frac{3}{4}\hbar^2 \cdot \mathbb{1}$

$\therefore J^2 = \frac{3}{4}\hbar^2 \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\langle\frac{1}{2}\ \frac{1}{2}|J^2|\frac{1}{2}\ \frac{1}{2}\rangle = \frac{3}{4}\hbar^2$

$\langle\frac{1}{2}\ -\frac{1}{2}|J^2|\frac{1}{2}\ \frac{1}{2}\rangle = 0$

$\vdots$

$J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \frac{\hbar}{2}\sigma_3$

~~$J_+ = \hbar\sqrt{j\ldots}$~~

$J_+|j, m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\ |j\ m+1\rangle$

$\therefore \begin{cases} \langle\frac{1}{2}\ \frac{1}{2}|J_+|\frac{1}{2}\ \frac{1}{2}\rangle = \hbar\sqrt{\frac{3}{4}-\frac{3}{4}}\ \langle\frac{1}{2}\ \frac{1}{2}|\frac{1}{2}\ \frac{3}{2}\rangle = 0 \\ \langle\frac{1}{2}\ -\frac{1}{2}|J_+|\frac{1}{2}\ \frac{1}{2}\rangle = 0 \\ \langle\frac{1}{2}\ \frac{1}{2}|J_+|\frac{1}{2}\ -\frac{1}{2}\rangle = \underbrace{\hbar\sqrt{\frac{3}{4}-(-\frac{1}{2})(\frac{1}{2})}}_{\hbar} \cdot \langle\frac{1}{2}\ \frac{1}{2}|\frac{1}{2}\ \frac{1}{2}\rangle \\ \qquad = \hbar\langle\frac{1}{2}\ \frac{1}{2}|\frac{1}{2}\ \frac{1}{2}\rangle = \hbar \end{cases}$

$\therefore \quad J_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Similarly, $\langle \frac{1}{2}\ -\frac{1}{2} | J_- | \frac{1}{2}\ \frac{1}{2} \rangle = \hbar \sqrt{\frac{3}{4} - \frac{1}{2}(-\frac{1}{2})} \langle \frac{1}{2}\ -\frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle$

$= \hbar .$

$\therefore \quad J_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$J_\pm = J_x \pm i J_y .$

$\therefore \quad J_x = \frac{1}{2} (J_+ + J_-)$ $\qquad \therefore \quad J_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_1$

$J_y = \frac{1}{2i} (J_+ - J_-)$ $\qquad J_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_2 .$

$J = 1 \quad (2j+1=3)$

$$J^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$J_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

By brute force calculation.. we can check..

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k.$$

$J = \frac{3}{2} \quad (2j+1 = 4)$

$$J^2 = \frac{15}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$J_z = \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$J_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$J = 2 \quad (2j+1 = 5)$

$$J^2 = 6\hbar^2 \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$J_z = \hbar \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & -2 \end{pmatrix}$$

$$J_+ = \hbar \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Matrix representation of $\mathcal{D}(R)$

$|\alpha\rangle$

Under rotation..

$$\begin{cases} x \to x' = Rx & \text{(vector)} \\ |\alpha\rangle \to |\alpha\rangle' = \underline{\underline{\mathcal{D}(R)}}\,|\alpha\rangle & \text{(state)} \end{cases}$$

<u>What is $\mathcal{D}(R)$?</u>

$\mathcal{D}(R)$ for $J = 1/2$ system

$\vec{J}_{(1/2)} = \frac{\hbar}{2}\vec{\sigma}$ $\quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

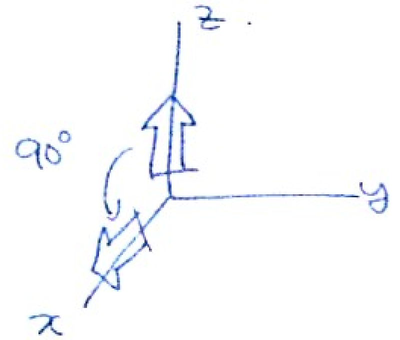
$|\frac{1}{2}\ \frac{1}{2}\rangle_z \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\frac{1}{2}\ -\frac{1}{2}\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\frac{1}{2}\ \frac{1}{2}\rangle_x = ? \qquad |\frac{1}{2}, -\frac{1}{2}\rangle_x = ?$

eigenvalue problem 풀어서
구하게 된다.

$S_x|\frac{1}{2}, \pm\frac{1}{2}\rangle_x = \pm\frac{\hbar}{2}|\frac{1}{2}, \pm\frac{1}{2}\rangle_x$

하지만.. rotation으로



$\mathcal{L}(R) \to 2\times 2$
$R \to 3\times 3$) (하지만 모두 관계가 있을 것..)

$$\begin{cases} (\sigma_i)^\dagger = \sigma_i \\ Tr(\sigma_i) = 0. \end{cases}$$

A : 2×2 hermitian matrix.

$$A^\dagger = A = \begin{pmatrix} c & \overline{a+ib} \\ a+ib & d \end{pmatrix}$$

$(a, b, c, d \in \mathbb{R})$

$$\begin{cases} c = \frac{c+d}{2} + \frac{c-d}{2} \\ d = \frac{c+d}{2} - \frac{c-d}{2} \end{cases}$$

$$\therefore A = \underbrace{\begin{pmatrix} \frac{c-d}{2} & a-ib \\ a+ib & -\frac{c-d}{2} \end{pmatrix}}_{\text{traceless}} + \frac{c+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\therefore Tr(A) = Tr\left(\frac{c+d}{2} \mathbb{1}_{2\times 2}\right) //$$

$\therefore A = a_0 \mathbb{1} + \vec{a}\cdot\vec{\sigma}$ $(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$

$(a_0, a_1, a_2, a_3$: real$)$

any complex 2×2 Hermitian

any complex $n\times n$ Hermitian matrix.

n^2 elements $\times 2$ (complex)

diagonal elements are real

$\therefore 2n^2 - n - \left(\frac{n(n-1)}{2}\right)\times 2 = n^2 \rightarrow$ # of Hermitians

$\frac{n(n-1)}{2}$

↳ (complex 0)

upper off-diagonal elements defines lower off-diagonal elements

$\therefore$ Traceless hermitian (in n-dim)

$= n^2 - 1$

traceless 조건 (1개 조건)

2-dim → $2^2 - 1 = 3$ → $SU(2) \simeq$ traceless hermitian = 3개

(generator 3개)

$[\sigma_i, \sigma_j] = ?$

$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k \Rightarrow \left(\frac{\hbar}{2}\right)^2 [\sigma_i, \sigma_j] = i\hbar \varepsilon_{ijk} \frac{\hbar}{2} \sigma_k$

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

Explicit calculation shows..

$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$

for a given i: $\sigma_i^2 = \mathbb{1}$

$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$

$$\begin{cases} \sigma_i^{2n} = [(\sigma_i)^2]^n = \mathbb{1} \\ \sigma_i^{2n+1} = \sigma_i \end{cases}$$

$$\begin{aligned}(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) &= \sigma_i\sigma_j (a^i b^j) \\ &= (\delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k) a^i b^j \\ &= (\vec{a}\cdot\vec{b})\mathbb{1} + i\,\vec{a}\times\vec{b}\cdot\vec{\sigma}\end{aligned}$$

$(\vec{\sigma}\cdot\vec{a})^2 = \vec{a}^2\mathbb{1}$

$$\mathcal{D}(R) = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi}$$

$$-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\phi \Big|_{J=\frac{\hbar}{2},\ \hat{n}=\hat{z}} = -\frac{i}{\hbar}\frac{\hbar}{2}\sigma_z\phi = -\frac{i}{2}\phi\sigma_z$$

$$\boxed{\mathcal{D}(R)\Big|_{\hat{n}=\hat{z},\ \phi} = e^{-\frac{i}{2}\phi\sigma_z}}$$

$$\begin{cases} \sigma_z^{2n} = \mathbb{1} \\ \sigma_z^{2n+1} = \sigma_z\ , \quad n \geq 0 \end{cases}$$