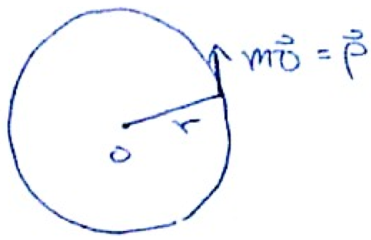
2012. 5.2. QM. 1 (Grad.)

Orbital Angular momentum



$\vec{L} = \vec{r} \times \vec{p}$. $\quad (\vec{r})^i = x_i$

$\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$

$\left((\vec{A} \times \vec{B})^i = \varepsilon^{ijk} A^j B^k \right)$

$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ $\qquad ([\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk}\hbar \hat{J}_k$ 확인 $)$

$\Downarrow$

$$[\hat{L}_i, \hat{L}_j] = \varepsilon_{iab}\varepsilon_{jcd} [\hat{x}_a \hat{p}_b, \hat{x}_c \hat{p}_d]$$

$$\begin{aligned}[A, BC] \\ = [A,B]C \\ + B[A,C]\end{aligned}$$

$$= \varepsilon_{iab}\varepsilon_{jcd} \left\{ [\hat{x}_a \hat{p}_b, \hat{x}_c] \hat{p}_d + \hat{x}_c [\hat{x}_a \hat{p}_b, \hat{p}_d] \right\}$$

$$= \varepsilon_{iab}\varepsilon_{jcd} \left\{ \cancel{[\hat{x}_a, \hat{x}_c]} \hat{p}_b \hat{p}_d + \hat{x}_a [\hat{p}_b, \hat{x}_c] \hat{p}_d + \hat{x}_c [\hat{x}_a, \hat{p}_d] \hat{p}_b + \hat{x}_c \cancel{\hat{x}_a [\hat{p}_b, \hat{p}_d]} \right\}$$

$$= -i\hbar \, \varepsilon_{iab}\varepsilon_{jcd} \left\{ \delta_{bc} \hat{x}_a \hat{p}_d - \delta_{ad} \hat{x}_c \hat{p}_b \right\}$$

$$= i\hbar \left\{ -\varepsilon_{iab}\varepsilon_{jcd} \delta_{bc} \hat{x}_a \hat{p}_d + \varepsilon_{iab}\varepsilon_{jcd} \delta_{ad} \hat{x}_c \hat{p}_b \right\}$$

$$= i\hbar \left\{ -\varepsilon_{iab}\varepsilon_{jbd} \hat{x}_a \hat{p}_d + \varepsilon_{iab}\varepsilon_{jca} \hat{x}_c \hat{p}_b \right\}$$

$$= i\hbar \left\{ \varepsilon_{iab}\varepsilon_{jdb}\,\hat{x}_a \hat{p}_d \mp \varepsilon_{iba}\varepsilon_{jca}\,\hat{x}_c \hat{p}_b \right\}$$

$$\varepsilon_{iab}\varepsilon_{jcb} = \delta_{ij}\delta_{ac} - \delta_{ic}\delta_{aj}$$

$$= i\hbar \left\{ (\delta_{ij}\delta_{ad} - \delta_{id}\delta_{aj})\,\hat{x}_a \hat{p}_d - (\delta_{ij}\delta_{bc} - \delta_{ic}\delta_{bj})\,\hat{x}_c \hat{p}_b \right\}$$

$$= i\hbar \left\{ \cancel{\delta_{ij}\,\hat{x}_a \hat{p}_a} - \hat{x}_j \hat{p}_i - \cancel{\delta_{ij}\,\hat{x}_c \hat{p}_c} + \hat{x}_i \hat{p}_j \right\}$$

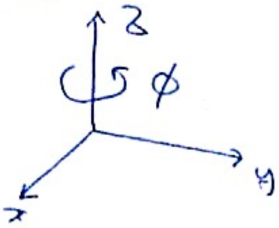
dummy indices.

$$= i\hbar \left\{ \hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i \right\} = i\hbar\, \varepsilon_{ijk} \hat{L}_k .$$

$$\boxed{\therefore \quad [\hat{L}_i, \hat{L}_j] = i\hbar\, \varepsilon_{ijk} \hat{L}_k}$$

Explicit?

Polar coordinates representation



$$L_3 = \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

(z축을 중심으로..
φ 만큼 회전의 generator)

$$\vec{\nabla} = \frac{\hat{e}_i}{h_i} \cdot \frac{\partial}{\partial \xi_i}$$

$$\begin{cases} \hat{r} \times \hat{\theta} = \hat{\phi} \\ \hat{\theta} \times \hat{\phi} = \hat{r} \\ \hat{\phi} \times \hat{r} = \hat{\theta} \end{cases}$$

$$\hat{r} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta$$

$$\hat{\theta} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z}\, \cancel{\cos} \sin\theta$$

$$\hat{\phi} = -\hat{x} \sin\phi + \hat{y} \cos\phi$$

$$\hookrightarrow \frac{\partial \hat{r}}{\partial \phi} \Big/ \left|\frac{\partial \hat{r}}{\partial \phi}\right|$$

H.W. Prove that..

$$\begin{cases} \hat{L}_1 = \frac{\hbar}{i}\left[-\sin\phi \frac{\partial}{\partial \theta} - \cos\phi \cot\theta \frac{\partial}{\partial \phi}\right] \\ \hat{L}_2 = \frac{\hbar}{i}\left[\cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi}\right] \\ \hat{L}_3 = \frac{\hbar}{i}\frac{\partial}{\partial \phi} \end{cases}$$

Ladder operator

$$\hat{L}_{\pm} = \hat{L}_1 \pm i \hat{L}_2 .$$

$$\therefore \hat{L}_{\pm} = \pm \hbar e^{\pm i\phi} \left[\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi} \right]$$

$$\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k .$$

↳ Hermitian conjugate

$$\hat{L}_i^{+} = \varepsilon_{ijk} \hat{p}_k^{+} \hat{x}_j^{+}$$, $\hat{x}$ & $\hat{p}$ are hermitian.

$$= \varepsilon_{ijk} \hat{p}_k \hat{x}_j$$

$$= \varepsilon_{ijk} \left\{ \cancel{[\hat{p}_k , \hat{x}_j]}^{\,0} + x_j \hat{p}_k \right\} \quad (k \neq j)$$

$$= \hat{L}_i \longrightarrow \text{Hermitian !!}$$

$$\therefore (\hat{L}_{\pm})^{+} = (\hat{L}_1 \pm i \hat{L}_2)^{+} = \hat{L}_1 \mp i \hat{L}_2 = \hat{L}_{\mp} .$$

⇓

Not hermitian.

$\vec{L} = \vec{r} \times \vec{p}$ → orbital angular momentum

$\downarrow$

$\vec{J}$ → general angular momentum.

defined as.. $[\hat{J}_i, \hat{J}_j] = i\hbar \varepsilon_{ijk} \hat{J}_k$.

$\hat{J}_\pm = \hat{J}_1 \pm i \hat{J}_2$.

↳ $J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2)$

$$= J_1^2 + J_2^2 - i(J_1 J_2 - J_2 J_1)$$

$$= J_1^2 + J_2^2 - i \cdot i\hbar J_3$$

$$= J_1^2 + J_2^2 + \hbar J_3$$

$\therefore \vec{J}^2 = J_1^2 + J_2^2 + J_3^2$

$$= J_+ J_- + J_3^2 - \hbar J_3$$

$$[A, B^2] = ABB - BBA = ABB - BAB + BAB - BBA = [A,B]B + B[A,B]$$

$$[J_3, \vec{J}^2] = [J_3, J_1^2 + J_2^2 + J_3^2]$$

$$= [J_3, J_1^2] + [J_3, J_2^2]$$

$$= [J_3, J_1]J_1 + J_1[J_3, J_1] + [J_3, J_2]J_2 + J_2[J_3, J_2]$$

$$= i\hbar J_2 J_1 + J_1 \, i\hbar J_2 - i\hbar J_1 J_2 - J_2 \, i\hbar J_1 = 0.$$

$$[\vec{J}^2, J_\pm] = [J^2, J_1] \pm i[J^2, J_2] = 0$$

commute?

$\{ J_3, J_\pm, J^2 \}$: commute..?

commute

commute

check.

$$[J_3, J_\pm] = [J_3, J_1] \pm i[J_3, J_2]$$
$$= i\hbar J_2 \pm i[-i\hbar J_1]$$
$$= \pm\hbar(J_1 \pm iJ_2)$$
$$= \pm\hbar J_\pm.$$

참고

$$\begin{cases} [a, a^+] = 1. \\ [\hat{N}, a] = -\hat{a} \\ [\hat{N}, \hat{a}^+] = \hat{a}^+ \end{cases}$$

⇓

$$\begin{Bmatrix} a^+ \leftrightarrow J_+ \\ a \leftrightarrow J_- \\ H \leftrightarrow J_3. \end{Bmatrix}$$

(or N).

$\therefore \{J^2, J_3\}$ commute.

$$\begin{cases} [J^2, J_\pm] = 0. \\ [J_3, J_\pm] = \pm\hbar J_\pm. \end{cases}$$

∴ Harmonic oscillator와 비슷한 논리 전개!

$[J_3, J_\pm] = \pm\hbar J_\pm$.

$[J_+, J_-] = 2\hbar J_3$

$[J^2, J_\pm] = 0$.

J^2과 J_3 simultaneous eigen state 있다!

$J_3 J_+ - J_+ J_3 = \hbar J_+$

$J_+ J_3 = (J_3 - \hbar) J_+$.

$J_3|\lambda\rangle = \lambda|\lambda\rangle$ 라 하자.

$$J_3(J_+|\lambda\rangle) = J_3 J_+|\lambda\rangle = (J_+ J_3 + \hbar J_+)|\lambda\rangle$$

$$= (\lambda + \hbar) J_+|\lambda\rangle$$

Using similar method...

$$J_3(J_-|\lambda\rangle) = \cdots = (\lambda-\hbar)(J_-|\lambda\rangle)$$

$$\therefore \Rightarrow \begin{cases} J_3|\lambda\rangle = \lambda|\lambda\rangle \\ J_+|\lambda\rangle \propto |\lambda+\hbar\rangle \\ J_-|\lambda\rangle \propto |\lambda-\hbar\rangle \end{cases}$$

$J_\pm|\lambda\rangle = C_\pm\,\underbrace{|\lambda\pm\hbar\rangle}_{\text{normalized}}$ 라 하자.

$\langle\lambda|J_\mp J_\pm|\lambda\rangle = |C_\pm|^2$.

여기서.. $|\lambda\rangle$를 $|\Lambda,\lambda\rangle$라 쓰자.

(J_3과 J^2의 simultaneous eigenstate)

$$\begin{cases} J^2|\Lambda,\lambda\rangle = \Lambda|\Lambda,\lambda\rangle \\ J_3|\Lambda,\lambda\rangle = \lambda|\Lambda,\lambda\rangle \end{cases}$$

$$\begin{cases} J_-J_+ = J^2 - J_3(J_3+\hbar) \\ J_+J_- = J^2 - J_3(J_3-\hbar) \end{cases}$$ 이용..

J_3 eigenvalue의 upper & lower bound 존재

$\Downarrow$

$$[\Lambda - \lambda(\lambda+\hbar)]\langle\Lambda,\lambda|\Lambda,\lambda\rangle = |C_+|^2 > 0.$$

$$[\Lambda - \lambda(\lambda-\hbar)] = |C_-|^2 > 0.$$

$\therefore J_3 \Rightarrow m\hbar$ ... $[\Lambda - m(m+1)\hbar^2]$

$\therefore l(l+1)\hbar^2 \quad (l \geq 0)$.