QM. I. Grad. 2012. 4. 30

Chapter 3. Theory of Angular Momentum.

Rotation of a vector.

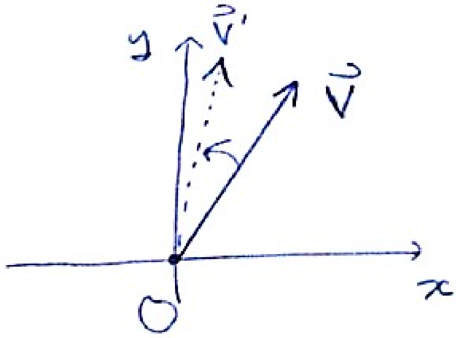
2D → 1 axis of rotation

3D → 3 axes of rotation.

4D → $\frac{4 \cdot 3}{2} = 6$ axes of rotation

⋮

nD → $\frac{n(n-1)}{2}$ axes of rotation.



$V_i' = R_{ij} V_j$ → rotation

⇓

radius is invariant.

∴ The norm is preserved.

$V^2 = V_i^2 = V_i'^2 = V'^2$

$V_i' V_i' = (R_{ij} V_j)(R_{ik} V_k)$

$= R_{ij} R_{ik} V_j V_k = V_j V_k \delta_{jk} = V_j V_j.$

∴ $R_{ij} R_{ik} = \delta_{jk}$

$(R^T)_{ji}(R)_{ik} = \delta_{jk}$

$(R^T R)_{jk} = \delta_{jk}$ $\quad\quad \therefore \underline{R^T R = 1}$

$\uparrow$

$R^T = R^{-1}$ ($\Rightarrow$ R is orthogonal.)

$\therefore$ In matrix form,

$$V' = RV. \longrightarrow R^T V' = V$$

$(R^T V')_i = V_i \rightarrow (R^T)_{ij}(V')_j = V_i$

$\underline{R_{ji}} V_j' = V_i$ (The index is flipped.)

(partial derivative)

$$\begin{cases} x_i' = R_{ij} x_j & \Rightarrow & R_{ij} = \dfrac{\partial x_i'}{\partial x_j} \\ x_i = R_{ji} x_j' & \Rightarrow & R_{ji} = \dfrac{\partial x_i}{\partial x_j'} \end{cases}$$

(chain rule)

$$\frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'} \frac{\partial}{\partial x_j} = R_{ij} \frac{\partial}{\partial x_j}.$$

$$\therefore \vec{\nabla}' = R \vec{\nabla}.$$

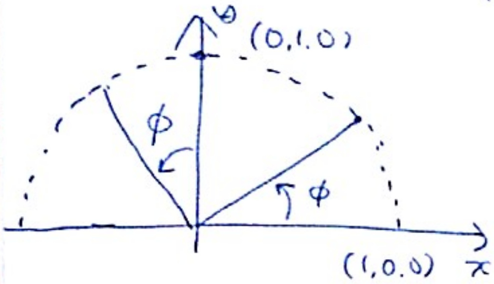
scalar

($\vec{F} = -\vec{\nabla} V(\vec{r})$ $\therefore \vec{F}$ is a vector) angle

$R_3(\phi)$

axis of rot.

$x' = Rx$

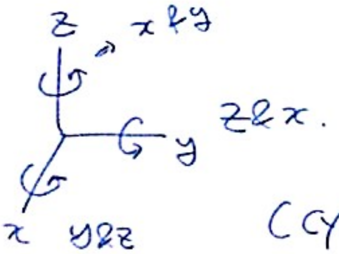


$$\begin{array}{ccc} x & & x' \\ \downarrow & & \downarrow \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \rightarrow & \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \end{array}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

exact??



(cyclic)

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\cancel{R_2(\phi) = \begin{pmatrix} \sin\phi & 0 & \cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & -\sin\phi \end{pmatrix}}$$

$$R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

Infinitesimal rotation

$$e^{-\vec{a}\cdot\vec{\nabla}} = e^{-\frac{i}{\hbar}\vec{a}\cdot\frac{\hbar}{i}\vec{\nabla}} = e^{-\frac{i}{\hbar}\vec{a}\cdot\hat{\vec{p}}}$$

number

operator

(~~infinitesimal~~ translation operator) $= \lim_{N\to\infty}\left(\underbrace{1-\frac{i}{\hbar}\left(\frac{\vec{a}}{N}\right)\cdot\vec{p}}_{\text{infinitesimal translation}}\right)^N$

rotation case?

as $\phi\to 0$

$$\cos\phi = 1-\frac{\phi^2}{2}+O(\phi^3)$$

$$\sin\phi = \phi + O(\phi^3)$$

$$\therefore\ R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\phi^2}{2} & -\phi \\ 0 & \phi & 1-\frac{\phi^2}{2} \end{pmatrix} + O(\phi^3)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\phi^2}{2} & 0 \\ 0 & 0 & 1-\frac{\phi^2}{2} \end{pmatrix} + \phi\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + O(\phi^3)$$

$$R_2(\phi) = \begin{pmatrix} 1-\phi^2/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\phi^2/2 \end{pmatrix} + \phi\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + O(\phi^3)$$

$$R_3(\phi) = \begin{pmatrix} 1-\phi^2/2 & 0 & 0 \\ 0 & 1-\phi^2/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\phi^3)$$

$$A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_3 \end{pmatrix} \quad \text{(diagonal)}$$

$$B = \begin{pmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & b_3 \end{pmatrix} \quad \text{(diagonal)}$$

$$AB = \begin{pmatrix} a_1 b_1 & & 0 \\ & a_2 b_2 & \\ 0 & & a_3 b_3 \end{pmatrix} \quad \therefore [A, B] = 0.$$

$$\therefore [R_1(\phi_1), R_2(\phi_2)] = \phi_1 \phi_2 \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right]$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix} \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \left[R_1(\phi_1),\ R_2(\phi_2) \right] = \phi_1 \phi_2 \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{?}$$

$$\left(R_3(\phi_3) = \mathbb{1} + \phi_3 \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{generator of rotation.}} + \cancel{\phi}\ O(\phi^2) \right)$$

$$\therefore \left[R_1(\phi_1),\ R_2(\phi_2) \right] = \phi_1 \phi_2 \left(R_3(\phi_3) - \mathbb{1} \right) / \phi_3 + O(\phi^3)$$

↳ (the reason that x & y rot. & y & x rot. don't commute.)

$$\boxed{R_3(\phi_3) = \lim_{n\to\infty} \left[\mathbb{1} + \frac{\phi}{n} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^n .}$$

(exact result!)

$$\underbrace{\phi \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{not hermitian}} = \underbrace{-i}_{\substack{\text{Convention} \\ (\text{agrees with } e^{-i\vec{a}\cdot\hat{\vec{p}}/\hbar})}} \phi \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\frac{i}{\hbar} \phi \, \underbrace{\hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{G_3 \ (\text{angular momentum generator})}$$

$$\therefore R_3(\phi_3) = \lim_{n\to\infty} \left[\mathbb{1} - \frac{i}{\hbar}\left(\frac{\phi_3}{n}\right) \underbrace{G_3}_{\hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \right]^n$$

Likewise,

$$R_1(\phi_1) = \lim_{n\to\infty} \left[\mathbb{1} - \frac{i}{\hbar}\left(\frac{\phi_1}{n}\right) \underbrace{G_1}_{\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}} \right]^n$$

$$R_2(\phi_2) = \lim_{n\to\infty} \left[\mathbb{1} - \frac{i}{\hbar}\left(\frac{\phi_2}{n}\right) \underbrace{G_2}_{\hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}} \right]^n$$

$$\therefore \begin{cases} R_1(\phi_1) = e^{-\frac{i}{\hbar}\phi_1 G_1} \\ R_2(\phi_2) = e^{-\frac{i}{\hbar}\phi_2 G_2} \\ R_3(\phi_3) = e^{-\frac{i}{\hbar}\phi_3 G_3} \end{cases}$$

Properties $\rightarrow \begin{cases} Tr\,(G_i) = 0. \\ (G_i)^\dagger = G_i \quad (\text{hermitian}). \end{cases}$

$$G_1 = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$G_2 = \hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$G_3 = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

G_i : Hermitian.

$Tr\,(G_i) = 0$ (traceless)

$\therefore$ traceless hermitian!

$$G_1 G_2 = \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad G_2 G_1 = \hbar^2 \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\therefore [G_1, G_2] = \hbar^2 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\hbar G_3.$$

In general,

$$[G_a, G_b] = i\hbar\, \underbrace{\varepsilon_{abc}}\, G_c$$

(the fundamental property of structure constant. rot.)

(Fundamental commutation relation of angular momentum generator)

$\boxed{O(3)}$ (orthogonal group)

3 dim. Euclidean space.

R is orthogonal ($R^{-1} = R^T$).

$\Downarrow$ Generalization

$$[\hat{J}_i, \hat{J}_j] = i\hbar\, \varepsilon_{ijk} J_k.$$

$$U(t, t_0) = e^{-\frac{i}{\hbar} H(t-t_0)} \quad (t \geq t_0) \quad (H \text{ is ind. of time})$$

$$U(\vec{a}) = e^{-\frac{i}{\hbar} \vec{a} \cdot \hat{\vec{p}}}$$

$$\mathcal{D}(\hat{n}, \phi) = e^{-\frac{i}{\hbar} \phi\, \hat{n} \cdot \hat{\vec{J}}}.$$