Sampling Theorem and PAM

KEEE343 Communication Theory

Lecture #22, May 31, 2011

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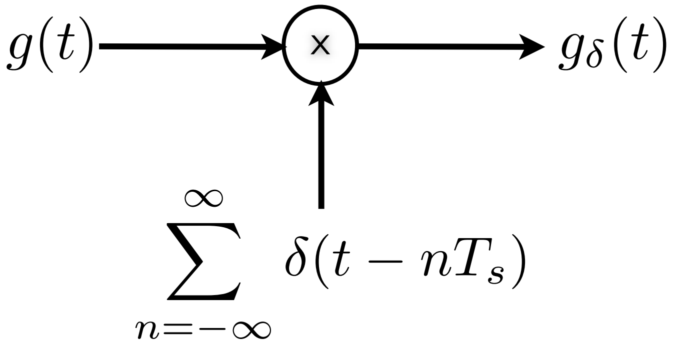
Sampling Process

- Sampling period and sampling rate
  - Sampling period: T_s
  - Sampling rate: $f_s = 1/T_s$



- Instantaneous ideal sampled signal

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

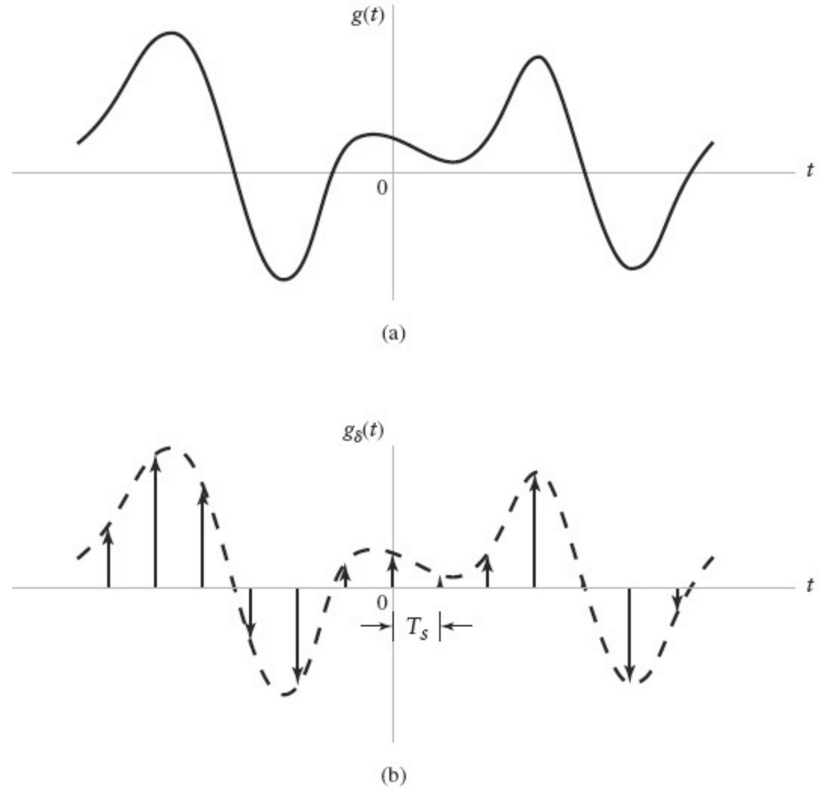
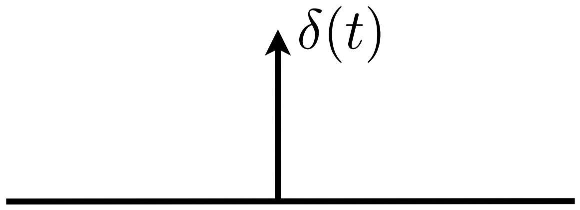


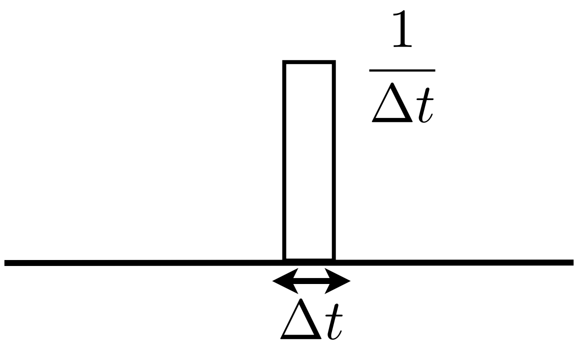
FIGURE 5.1 Illustration of the sampling process. (*a*) Analog waveform $g(t)$. (*b*) Instantaneously sampled representation of $g(t)$.

[Ref: Haykin & Moher, Textbook]

Implementation of the Delta Function

- The delta function can be implemented by a rectangular pulse with small width





- Smaller Δt , better approximation

Fourier Transform of a Sampled Signal

- Fourier transform

$$
\begin{aligned}
\mathcal{F}[g_{\delta(t)}] &= \mathcal{F}\left[\sum_{n=-\infty}^{n=\infty} g(nT_s)\delta(t-nT_s)\right] \\
&= \sum_{n=-\infty}^{\infty} g(nT_s)\mathcal{F}\left[\delta(t-nT_s)\right] \\
&= \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi nT_s f}
\end{aligned}
$$

Review of the Fourier Transform of a Periodic Signal

- Consider a periodic signal $g_{T_0}(t)$ with a fundamental period T_0 and let $g(t)$

$$g(t) = \begin{cases} g_{T_0}(t), & -\frac{T_0}{2} \leq t \leq \frac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$$

- Then we can show the Fourier transform of $g_{T_0}(t)$ as

$$\begin{aligned} g_{T_0}(t) &= \sum_{n=-\infty}^{\infty} g(t - nT_0) \\ &= f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0) \Longleftrightarrow f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0) \end{aligned}$$

Duality Property

- Compare the Fourier transformed signals of the sampled signal and the periodic signal

$$\sum_{n=-\infty}^{\infty} g(t-nT_0) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)e^{j2\pi n f_0 t} \iff f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f-nf_0)$$

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s) \iff \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi n T_s f}$$

- Or we can rewrite

$$T_0 \sum_{n=-\infty}^{\infty} g(t-nT_0) = \sum_{n=-\infty}^{\infty} G(nf_0)e^{j2\pi n f_0 t} \iff \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f-nf_0)$$

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s) \iff \sum_{n=-\infty}^{\infty} g(nT_s)e^{-j2\pi n T_s f}$$

- Recall the duality property

$$\begin{aligned} g(t) &\iff G(f) \\ G(t) &\iff g(-f) \end{aligned}$$

- Using the duality property, we can express the Fourier transform of the sampled signal as

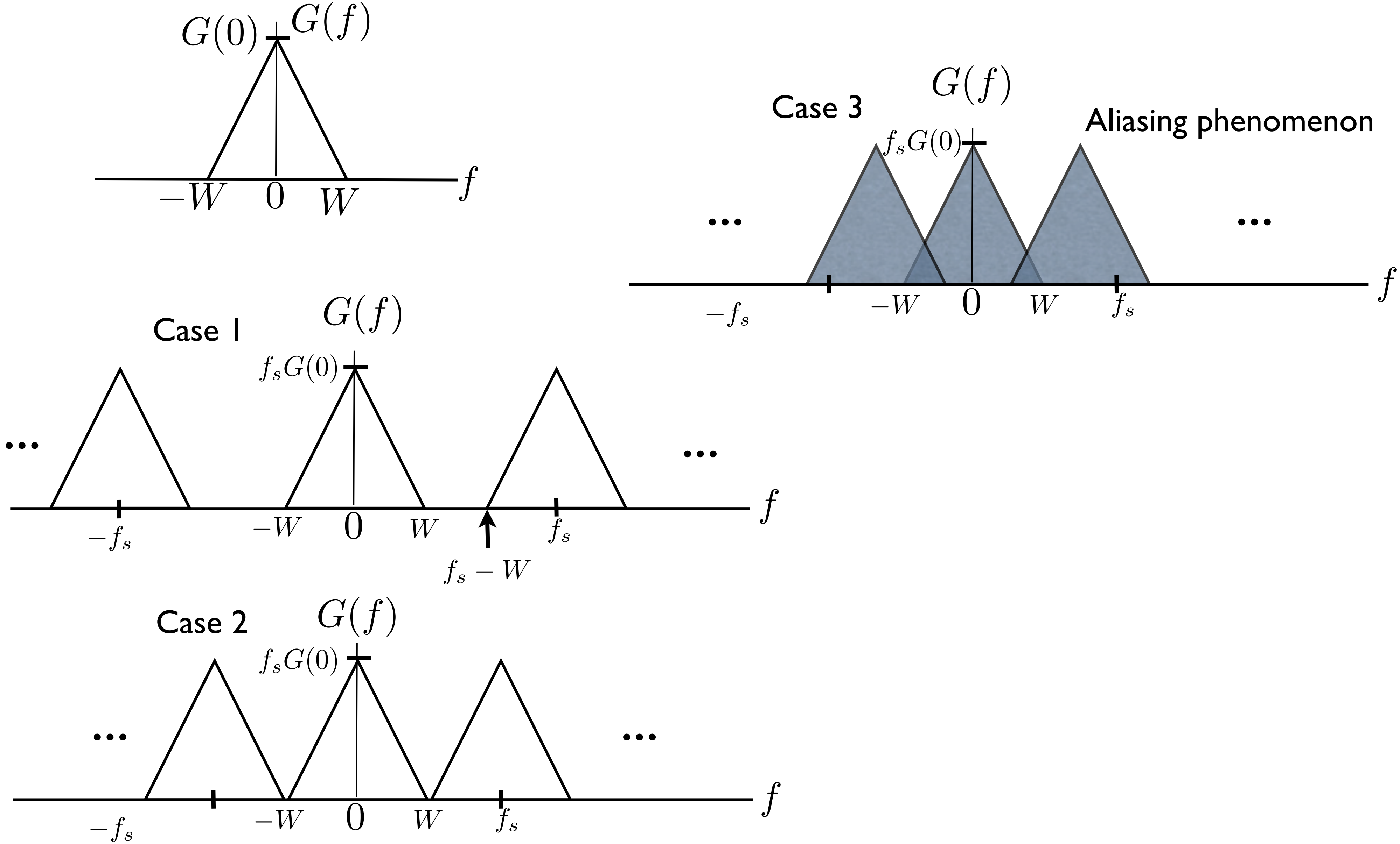
$$\begin{aligned} G_\delta(f) &= f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) \\ &= f_s G(f) + f_s \sum_{n=-\infty, n\neq 0}^{\infty} G(f - nf_s) \end{aligned}$$

Three Different Cases

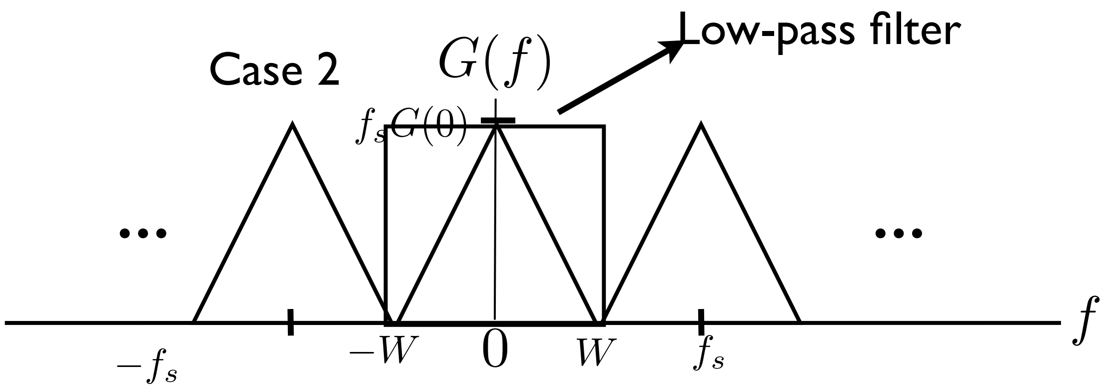
- Let us consider the strictly band-limited signal such as

$$G(f) = 0, \quad \text{for } |f| \geq W$$

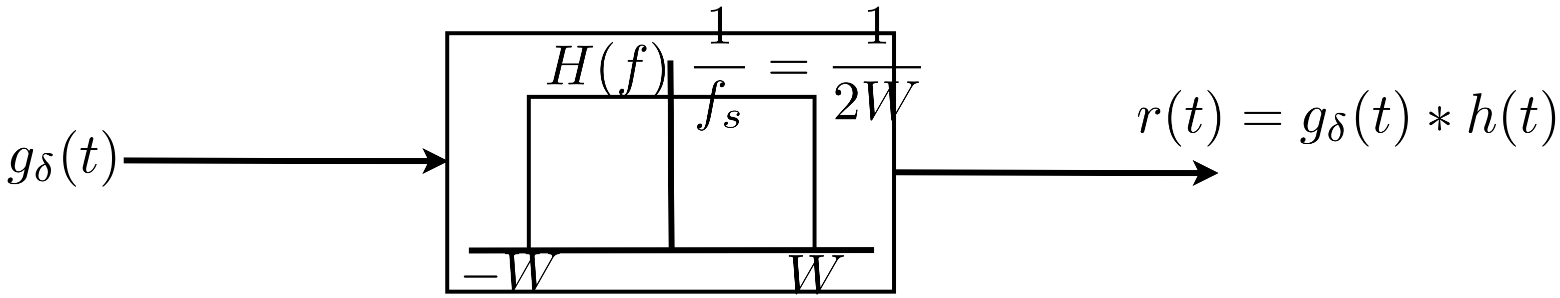
- Depending on the sampling rate we have three different cases
  - Case 1 $f_s > 2W$
  - Case 2 $f_s = 2W$
  - Case 3 $f_s < 2W$


$G(0)$
$G(f)$
$-W$
0
W
f
Case 3
$G(f)$
Aliasing phenomenon
$f_s G(0)$
...
...
$-f_s$
$-W$
0
W
f_s
f
Case 1
$G(f)$
$f_s G(0)$
...
...
$-f_s$
$-W$
0
W
f_s
f
$f_s - W$
Case 2
$G(f)$
$f_s G(0)$
...
...
$-f_s$
$-W$
0
W
f_s
f

- Our goal is to reconstruct the sampled signal into the original signal
  - Case 3: Impossible to reconstruct
  - Case 1 and 2: Low-pass filtering the sampled signal gives the original analog signal.
- Consider Case 2, i.e., $f_s = 2W$



- We can show $r(t) = g(t)$



$$
\begin{aligned}
r(t) &= g_\delta(t) * h(t) \\
&= \left(\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \right) * \operatorname{sinc}(2Wt) \\
&= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \left(\delta(t - nT_s) * \operatorname{sinc}(2Wt)\right) \\
&= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \operatorname{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] \\
&= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \operatorname{sinc}\left[2Wt - n\right]
\end{aligned}
$$

Summary of Sampling Theorem

- The sampling theorem for strictly band-limited signals of finite energy in two equivalent parts
  - Analysis: A band-limited signal of finite energy that has no frequency components higher than W herts is completely described by specifying the values of the signal at instants of time separated by 1/2W seconds.
  - Synthesis: A band-limited signal of finite energy that has no frequency components higher than W hertz is completely recovered from knowledge of its samples taken at the rate of 2W samples per second
  - Nyquist rate
    - The sampling rate of 2W samples per seconds for a signal bandwidth of W herts
  - Nyquitst interval: 1/2W (measured in seconds)

Aliasing Phenomenon

- Aliasing Phenomenon
  - The phenomenon of a high-frequency component in the spectrum of the signal seemingly taking on the identify of a lower frequency in the spectrum of its sampled version.
  - To combat the effects of aliasing in practices
    - *Prior to sampling : a low-pass anti-alias filter is used to attenuate those high-frequency components of a message signal that are not essential to the information being conveyed by the signal*
    - *The filtered signal is sampled at a rate slightly higher than the Nyquist rate.*
  - Physically realizable reconstruction filter
    - *The reconstruction filter is of a low-pass kind with a passband extending from –W to W*
    - *The filter has a non-zero transition band extending form W to* f_s*-W*

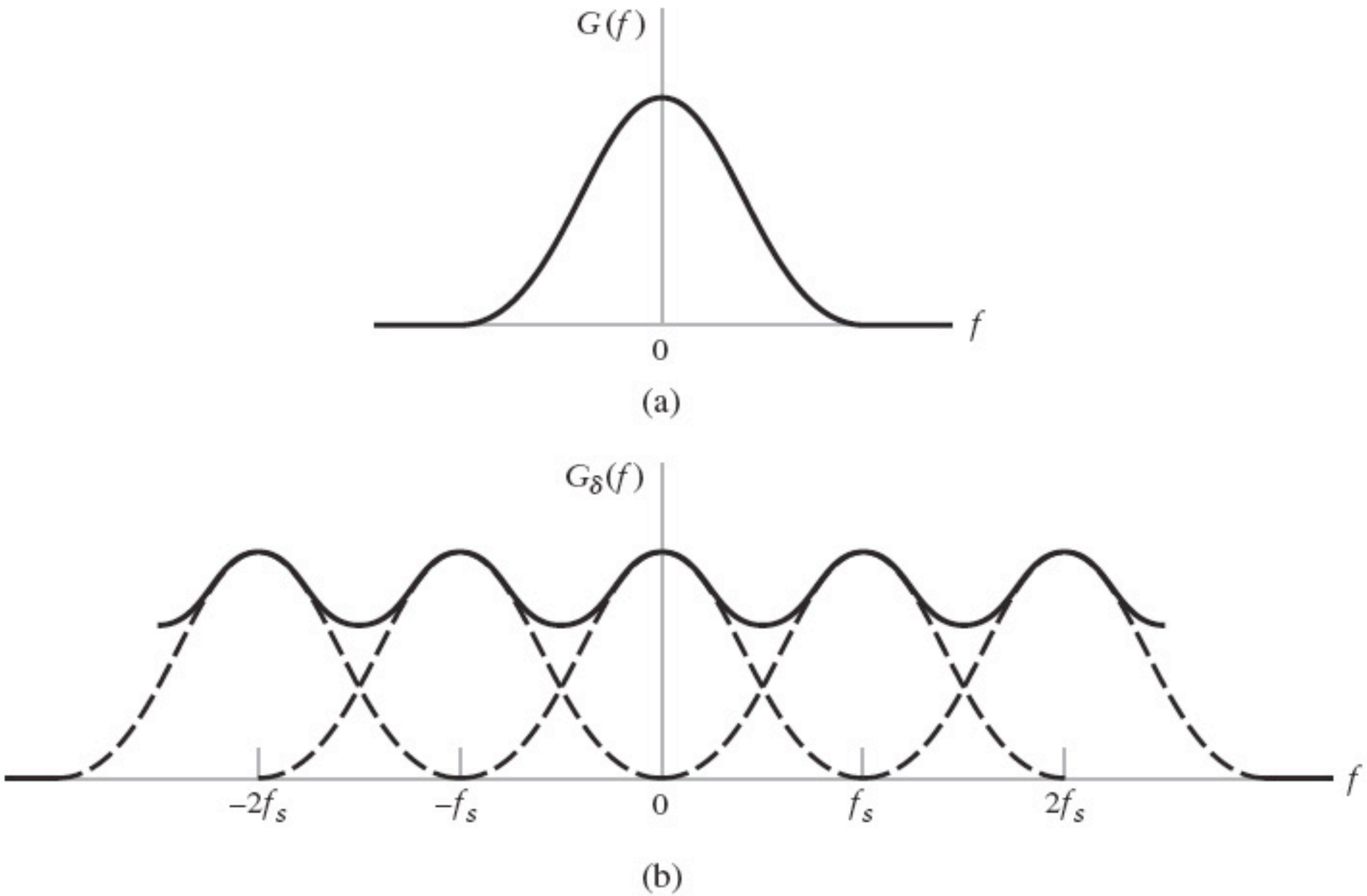


FIGURE 5.3 (*a*) Spectrum of a signal. (*b*) Spectrum of an undersampled version of the signal, exhibiting the aliasing phenomenon.

[Ref: Haykin & Moher, Textbook]

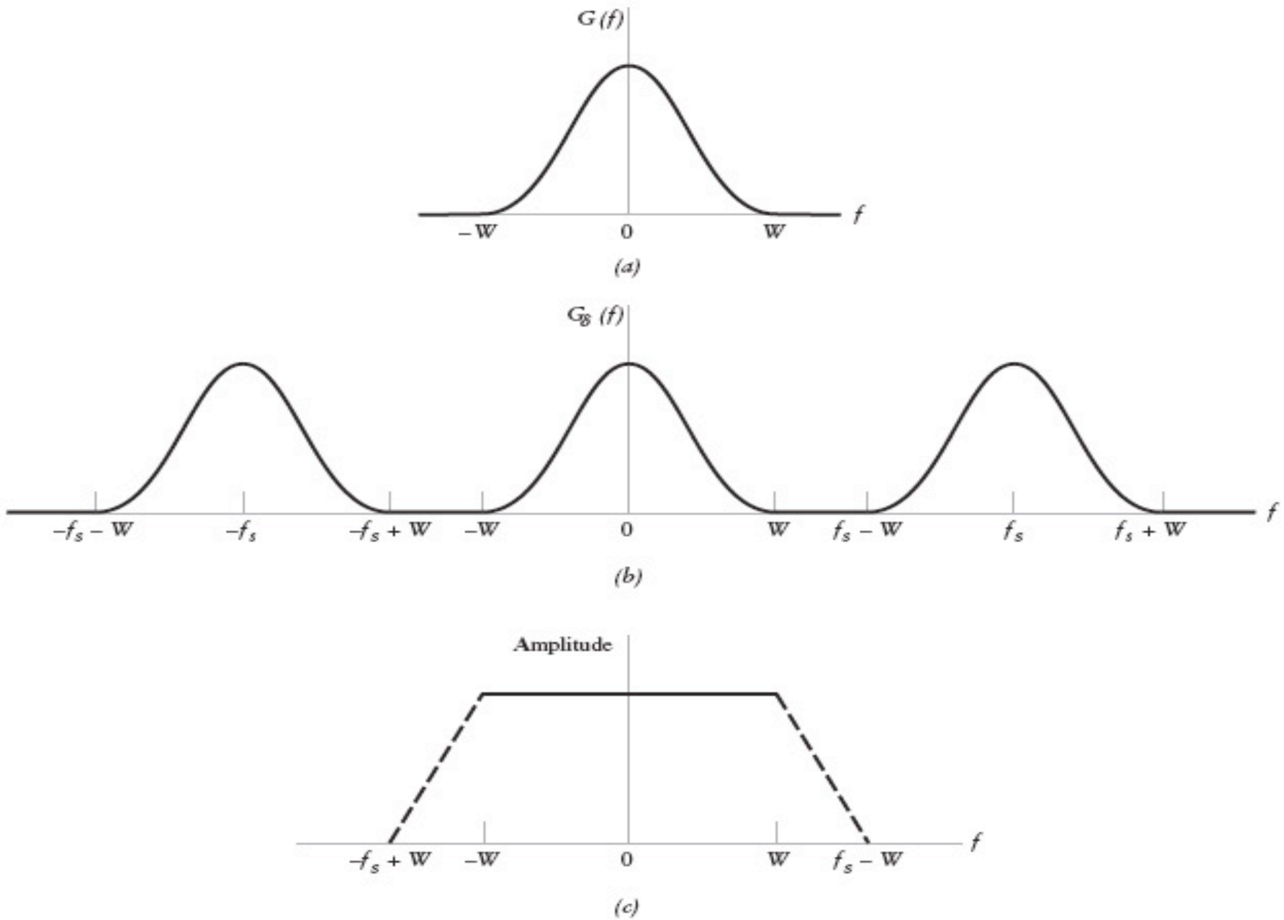


FIGURE 5.4 (*a*) Anti-alias filtered spectrum of an information-bearing signal. (*b*) Spectrum of instantaneously sampled version of the signal, assuming the use of a sampling rate greater than the Nyquist rate. (*c*) Idealized amplitude response of the reconstruction filter.

[Ref: Haykin & Moher, Textbook]

Pulse-Amplitude Modulation

- Pulse-Amplitude Modulation (PAM)
  - The amplitude of regularly spaced pulses are varied in proportion to the corresponding sample values of a continuous message signal.
  - Two operations involved in the generation of the PAM signal
    - *Instantaneous sampling of the message signal m(t) every Ts seconds,*
    - *Lengthening the duration of each sample, so that it occupies some finite value T.*

- Sample-and-Hold Filter : Analysis
  - The PAM signal is

$$s(t) = \sum (nT_s) h(t - nT_s)$$

  - The *h(t)* is a standard rectangular pulse of unit amplitude and duration

$$h(t) = rect\left(\frac{t - \frac{T}{2}}{T}\right) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases}$$

  - The instantaneously sampled version of *m(t)* is

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

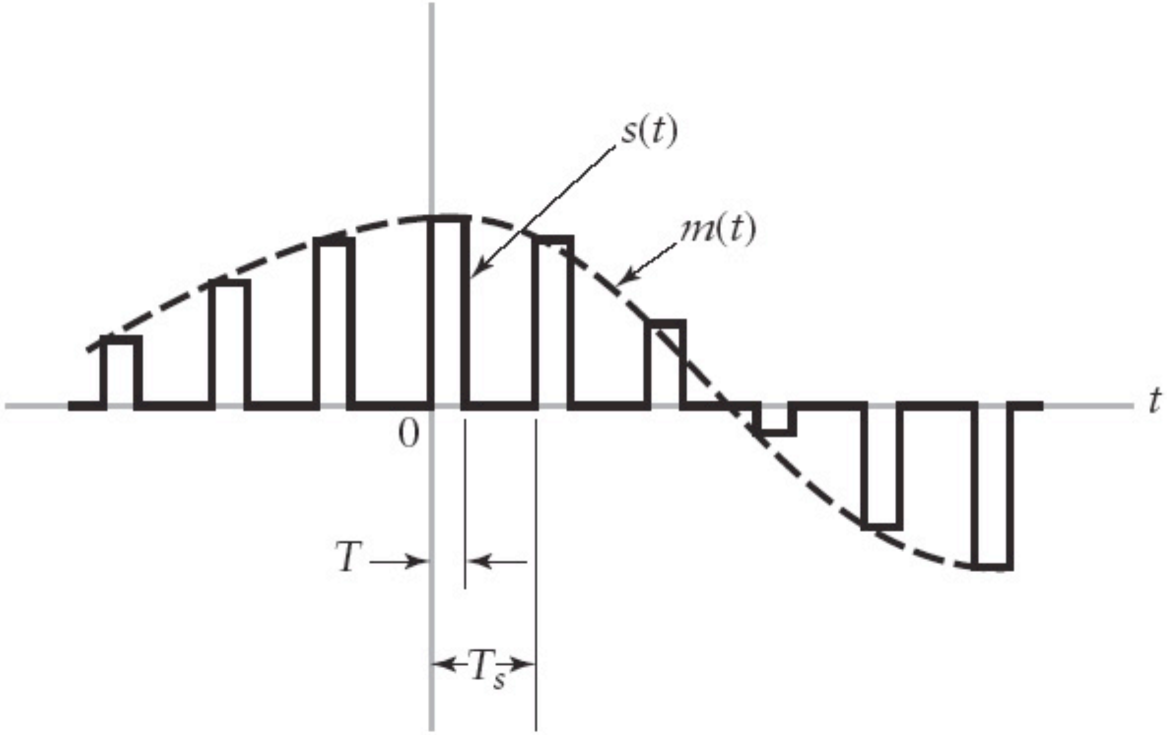


Figure 5.5 Flat-top sampling of a message signal.

- To modify $m_\delta(t)$ so as to assume the same form as the PAM signal

$$
\begin{aligned}
m_\delta(t) * h(t) &= \int_{-\infty}^{\infty} m_\delta(\tau) h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m(nT_s)\delta(\tau - nT_s) h(t-\tau)\, d\tau \\
&= \sum_{n=-\infty}^{\infty} m(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s) h(t-\tau)\, d\tau \quad (5.11)
\end{aligned}
$$

- The PAM signal s(t) is mathematically equivalent to the convolution of $m_\delta(t)$, the instantaneously sampled version of $m(t)$, and the pulse $h(t)$

$$m_\delta(t) * h(t) = \sum_{n=-\infty}^{\infty} m(nT_s) h(t - nT_s) \quad (5.12)$$

$$s(t) = m_\delta(t) * h(t) \quad (5.13)$$

$$M_\delta(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) \quad (5.15)$$

$$S(f) = M_\delta(f) H(f) \quad (5.14)$$

$$S(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) H(f) \quad (5.16)$$

- Aperture Effect and its Equalization
  - Aperture effect
    - *The distortion caused by the use of pulse-amplitude modulation to transmit an analog information-bearing signal*
  - Equalizer
    - *Decreasing the in-band loss of the reconstruction filter as the frequency increases*
    - *The amplitude response of the equalizer is*

$$\frac{1}{|H(f)|} = \frac{1}{T \operatorname{sinc}(fT)} = \frac{\pi f}{\sin(\pi f T)}$$

  - The noise performance of a PAM system can never be better than direct transmission of the message signal

  - For transmission over long distances, PAM would be used only as a means of message processing for time-division multiplexing.

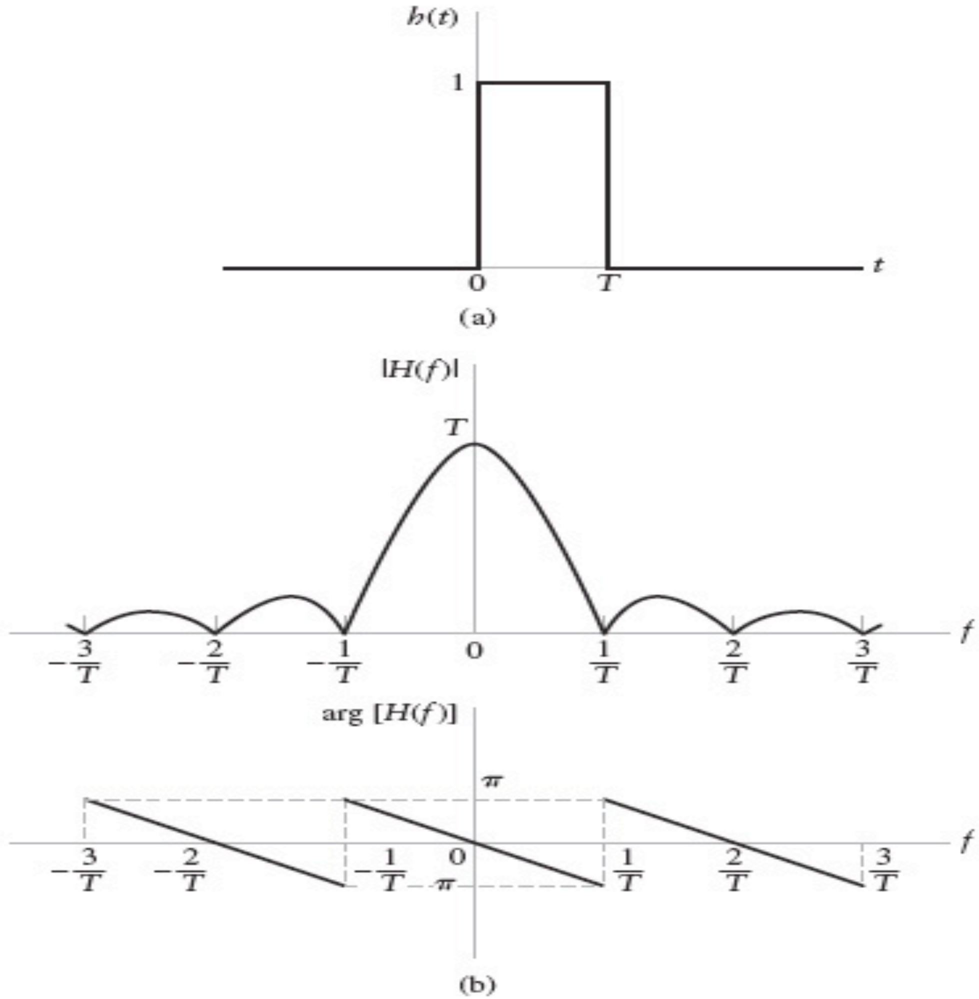


FIGURE 5.6 (*a*) Rectangular pulse $h(t)$. (*b*) Spectrum $H(f)$, defined in terms of its magnitude and phase.

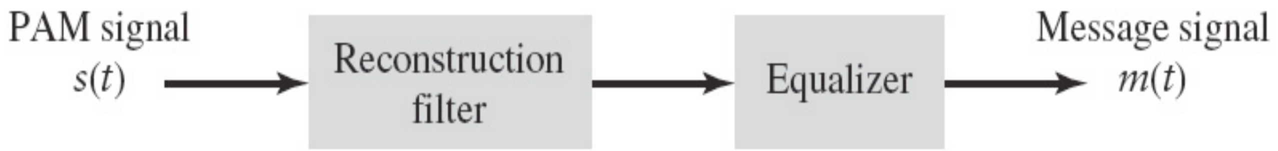


FIGURE 5.7 Recovering the message signal $m(t)$ from the PAM signal $s(t)$.

Pulse-Position Modulation

KEEE343 Communication Theory

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Pulse-Position Modulation

- PDM (Pulse-Duration Modulation)
  - Pulse-width or Pulse-length modulation
  - The samples of the message signal are used to vary the duration of the individual pulses.
  - PDM is wasteful of power
- PPM (Pulse-position modulation)
  - The position of a pulse relative to its unmodulated time of occurrence is varied in accordance with the message signal

$$s(t) = \sum_{n=-\infty}^{\infty} g(t - nT_s - k_p m(nT_s))$$

$$g(t) = 0, \ \ |t| > (T_s/2) - k_p |m(t)|_{\max}, \qquad \Rightarrow \ \ k_p |m(t)|_{\max} < T_2/2$$

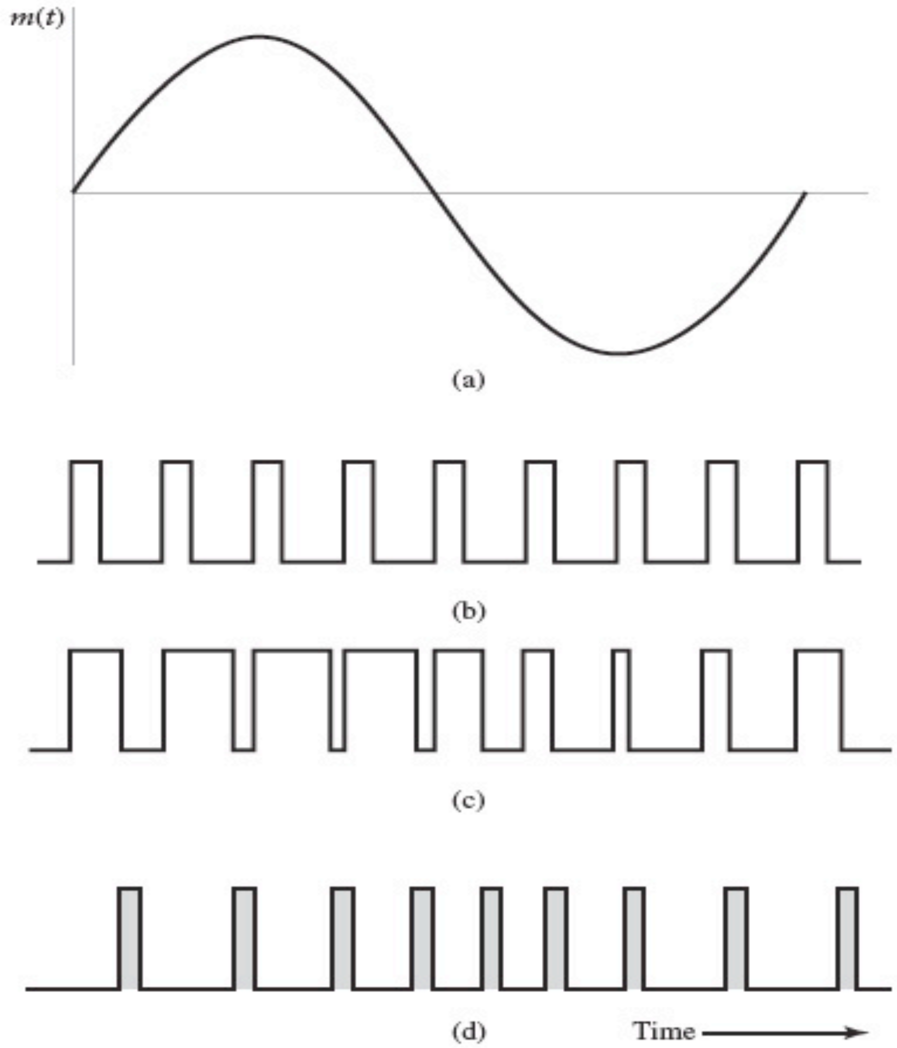

FIGURE 5.8 Illustration of two different forms of pulse-time modulation for the case of a sinusoidal modulating wave. (*a*) Modulating wave. (*b*) Pulse carrier. (*c*) PDM wave. (*d*) PPM wave.

[Ref: Haykin & Moher, Textbook]

Completing the Transition from Analog to Digital

- The advantages offered by digital pulse modulation
  - Performance
    - *Digital pulse modulation permits the use of regenerative repeaters, when placed along the transmission path at short enough distances, can practically eliminate the degrading effects of channel noise and signal distortion.*
  - Ruggedness
    - *A digital communication system can be designed to withstand the effects of channel noise and signal distortion*
  - Reliability
    - *Can be made highly reliable by exploiting powerful error-control coding techniques.*
  - Security
    - *Can be made highly secure by exploiting powerful encryption algorithms*
  - Efficiency
    - *Inherently more efficient than analog communication system in the tradeoff between transmission bandwidth and signal-to-noise ratio*
  - System integration
    - *To integrate digitized analog signals with digital computer data*

Quantization Process

- Amplitude quantization
  - The process of transforming the sample amplitude $m(nT_s)$ of a baseband signal $m(t)$ at time $t=nT_s$ into a discrete amplitude $v(nT_s)$ taken from a finite set of possible levels.

$$I_k : \{m_k < m \le m_{k+1}\}, \quad k = 1, 2, ..., L$$

  - Representation level (or Reconstruction level)
    - *The amplitudes* v_k *,* $k=1,2,3,\ldots\ldots,L$
  - Quantum (or step-size)
    - *The spacing between two adjacent representation levels*

$$v = g(m)$$

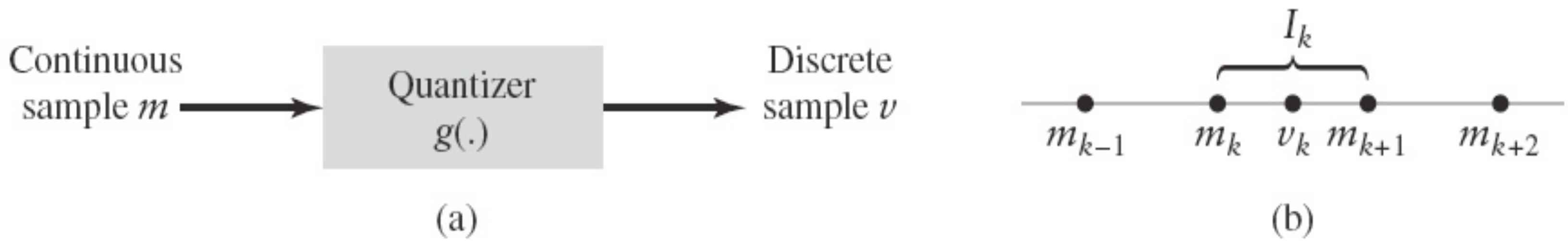

FIGURE 5.9 Description of a memoryless quantizer.

[Ref: Haykin & Moher, Textbook]

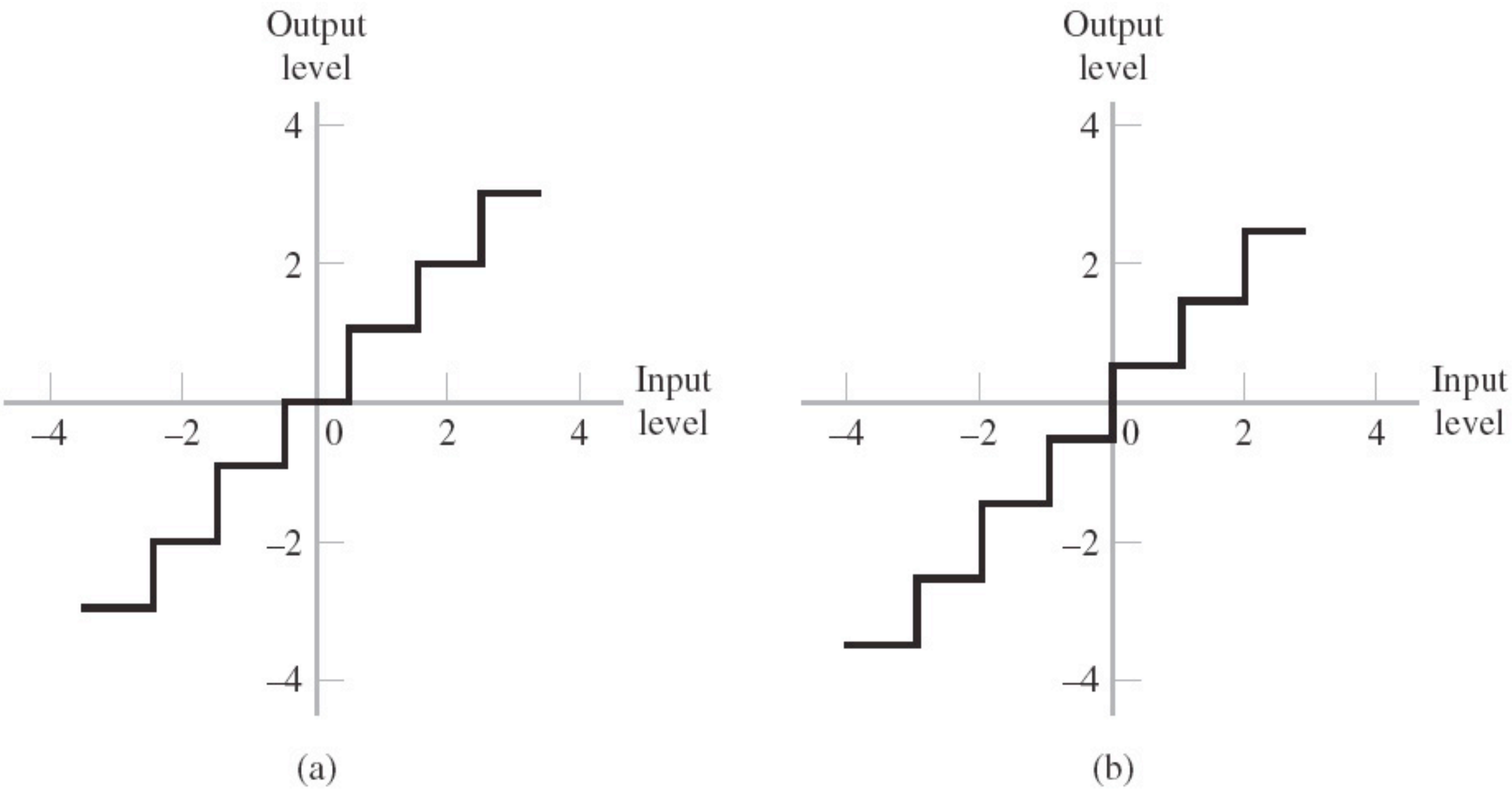

FIGURE 5.10 Two types of quantization: (*a*) midtread and (*b*) midrise.

[Ref: Haykin & Moher, Textbook]

Pulse-Code Modulation

- PCM (Pulse-Code Modulation)
  - A message signal is represented by a sequence of coded pulses, which is accomplished by representing the signal in discrete form in both time and amplitude
  - The basic operation
    - *Transmitter : sampling, quantization, encoding*
    - *Receiver : regeneration, decoding, reconstruction*

- Operation in the Transmitter
  1. Sampling
     1. *The incoming message signal is sampled with a train of rectangular pulses*
     2. *The reduction of the continuously varying message signal to a limited number of discrete values per second*
  2. Nonuniform Quantization
     1. *The step size increases as the separation from the origin of the input-output amplitude characteristic is increased, the large end-step of the quantizer can take care of possible excursions of the voice signal into the large amplitude ranges that occur relatively infrequently.*