Numerical Analysis MTH614

Spring 2012, Korea University

Boundary value problem

The differential equations in the previous lecture are of first order and have one initial condition to satisfy, but all the specified conditions are on the same endpoint. These are initial value problems.

In this lecture we describe how to estimate the solution to boundary value problems, differential equations with conditions imposed at two different endpoints.

The two point boundary value problems in the lecture involve a second order differential equation of the form

$$y'' = f(x, y, y'), \ a \le x \le b, \tag{1}$$

with boundary conditions

$$y(a) = \alpha, \ \text{and } y(b) = \beta. \tag{2}$$

Linear shooting method

We denote that the equation has form such as

$$f(x, y, y') = p(x)y' + q(x)y + r(x), \qquad (1)$$

which is called linear.

The shooting method is based on transforming the boundary condition problem into an initial boundary problem.
Then a trial and error approach determines the solution of the initial value problem that satisfies the given boundary conditions.

The following theorem contains general conditions that guarantees that the problem is well-posed.

Suppose the function f in the boundary value problem

$$y'' = f(x, y, y'),\ a \le x \le b,\ y(a) = \alpha,\ y(b) = \beta, \qquad (1)$$

is continuous on the set

$$D = \{(x, y, y') | a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\},$$

and that f_y and $f_{y'}$ are also continuous on D.

If $f_{y'}(x, y, y') > 0$ for all $(x, y, y') \in D$, and a constant M with

$|f_{y'}(x, y, y')| \le M,$ for all $(x, y, y') \in D$,

then the boundary value problem has a unique solution.

If the linear boundary value problem

$$y'' = f(x, y, y') = p(x)y' + q(x)y + r(x),\ a \le x \le b,\ y(a) = \alpha,\ y(b) = \beta,$$

satisfies $p(x),\ q(x),\ \text{and}\ r(x)$ are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$,

then the boundary value problem has a unique solution.

Let us consider the two initial value problem,

$$y'' = p(x)y' + q(x)y + r(x),\ \ a \le x \le b,\ \ y(a) = \alpha,\ \ y'(a) = 0, \tag{1}$$

and

$$y'' = p(x)y' + q(x)y,\ \ a \le x \le b,\ \ y(a) = 0,\ \ y'(a) = 1, \tag{2}$$

Denote that $y_1(x)$ is the solution to (1), $y_2(x)$is also the solution to (2).

and let $y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x).$ (3)

Then $y'(x) = y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x)$ and $y''(x) = y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2''(x)$.

Therefore

$$y''(x) = p(x)y_1'(x) + q(x)y_1 + \frac{\beta - y_1(b)}{y_2(b)}(p(x)y_2' + q(x)y_2) \tag{4}$$

$$= p(x)\left(y_1' + \frac{\beta - y_1(b)}{y_2(b)} y_2'\right) + q(x)\left(y_1 + \frac{\beta - y_1(b)}{y_2(b)} y_2\right) + r(x) \tag{5}$$

$$= p(x)y'(x) + q(x)y(x) + r(x). \tag{6}$$

We can verify that

$$y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)} y_2(a) = \alpha + \frac{\beta - y_1(b)}{y_2(b)} \cdot 0 = \alpha,$$

$$y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)} y_2(b) = y_1(b) + \beta - y_1(b) = \beta.$$

The shooting method for linear equation is based on the replacement of the linear boundary value problem by the two initial value problems. To do this, we need numerical methods such as Euler and Runge-Kutta to determine the proper conditions.

For example, we approximate the solution of the boundary value problem

$$y'' = y' + y - 2(1+x)\cos(x) + (x-4)\sin(x),$$

$$0 \leq x \leq \pi,\ \ y(0) = 0,\ \ y(\pi) = -\pi,$$

and its exact solution is given by

$$y(x) = x\cos(x) + \sin(x).$$

We use Euler's method to approximate $y_1(x)$ and $y_2(x)$, and then iterate this until they reach to the proper solution under the shooting method.

Thus, we iterate them using Euler's method as follows

$$\begin{aligned} y_1(t_{i+1}) &= y_1(t_i) + hy_1'(t_i), \\ y_1'(t_{i+1}) &= y_1'(t_i) + hy_1''(t_i) = y_1'(t_i) + h[p(t_i)y_1' + q(t_i)y_1(t_i) + r(t_i)], \\ y_2(t_{i+1}) &= y_2(t_i) + hy_2'(t_i), \\ y_2'(t_{i+1}) &= y_2'(t_i) + hy_2''(t_i) = y_2'(t_i) + h[p(t_i)y_2' + q(t_i)y_2(t_i)], \end{aligned} \tag{1}$$

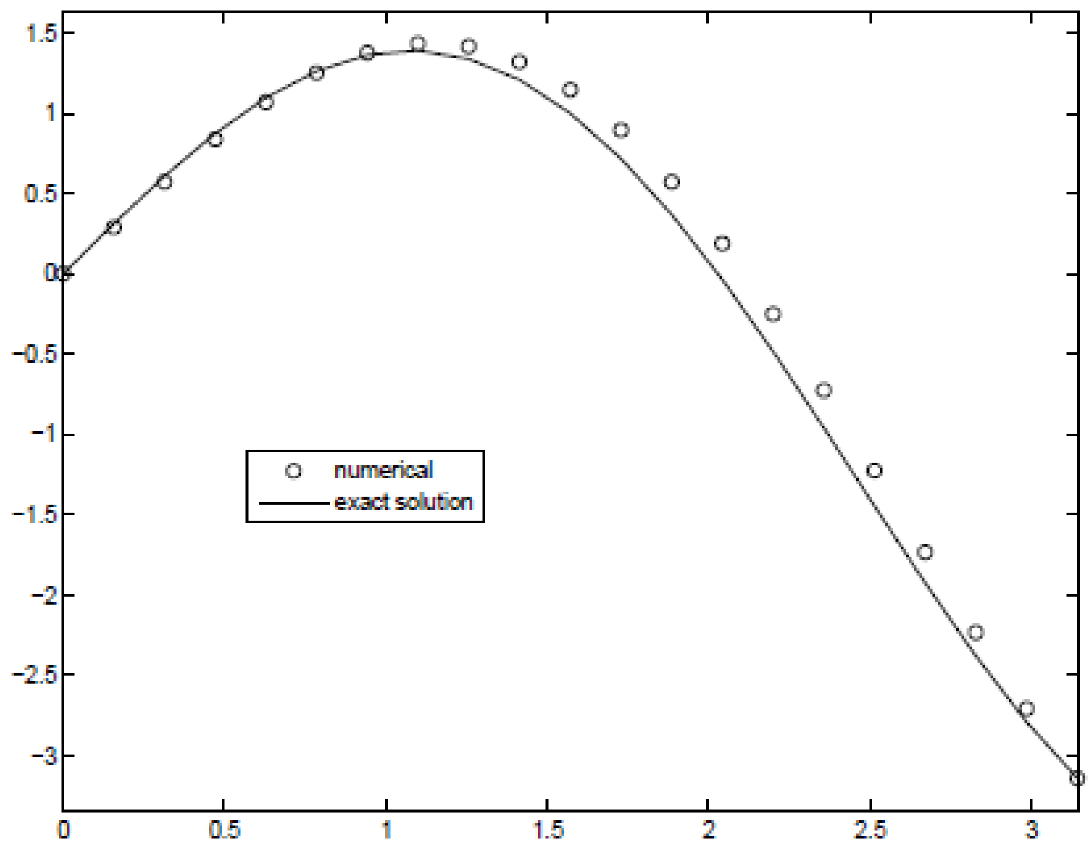
where $h = (b-a)/(N),\ t_i = a + hi,$

$$y_1(a) = \alpha,\ y_1'(a) = 0,\ y_2(a) = 0,\ y_2'(a) = 1. \tag{2}$$

Finally, we can obtain

$$y(t_i) = y_1(t_i) + \frac{\beta - y_1(t_N)}{y_2(t_N)} y_2(t_i),\ i = 0, 1, \ldots, N. \tag{3}$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%% shooting.m %%%%%%%%%%%%%%%%%%%%%%%%%%
clear; clc; clf; n=20;
x=linspace(0, pi, n+1); h=x(2)-x(1);
ex=x.*cos(x)+sin(x);
rx=inline('-2*(1+fx)*cos(fx)+(fx-4)*sin(fx)');
u1(1)=ex(1); v1(1)=0; u2(1)=0; v2(1)=1;
for i=1:n
 u1(i+1)=u1(i)+h*v1(i);
 v1(i+1)=v1(i)+h*(v1(i)+u1(i)+rx(x(i)));
 u2(i+1)=u2(i)+h*v2(i);
 v2(i+1)=v2(i)+h*(v2(i)+u2(i));
end
y=u1+u2*(ex(n+1)-u1(n+1))/u2(n+1);
plot(x,y,'ko',x,ex,'k-')
axis([x(1) x(n+1) min(y)-0.2 max(y)+0.2])
legend('numerical ','exact solution',2)
```



1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
0.5
1
1.5
2
2.5
3
numerical
exact solution

Nonlinear shooting method

The shooting method for the nonlinear second order boundary value problem is similar to the linear shooting method

$$y'' = f(x, y, y'),\ a \le x \le b,\ y(a) = \alpha,\ y(b) = \beta, \quad (1)$$

but the solution to a nonlinear problem cannot be represented as a linear combination of the solutions to two initial boundary problems.

Nonlinear shooting problem for the initial value problem has the form

$$y'' = f(x, y, y'),\ a \le x \le b,\ y(a) = \alpha,\ y'(a) = t. \quad (2)$$

To do this, we choose the parameters $t = t_k$ in a manner to ensure that

$$\lim_{k\to\infty} y(b, t_k) = y(b) = \beta,$$

where $y(x, t_k)$ denotes that the solution to the initial value problem (2) with $t = t_k$ and $y(x)$ is the solution of (1).

We use the Newton method to solve the problem

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{(dy/dt)(b, t_{k-1})}, \qquad (1)$$

which is necessary to determine $(dy/dt)(b, t_{k-1})$. (2)

To find this term, we differentiate $y'' = f(x, y, y')$. That is,

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial t}(x, y(x,t), y'(x,t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial t}. \qquad (3)$$

Since $\partial x/\partial t = 0$, $\frac{\partial y''}{\partial t} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial t}$.

Let $\frac{\partial y}{\partial t} = z$ then it follows

$$z''(x,t) = \frac{\partial f}{\partial y} z(x,t) + \frac{\partial f}{\partial y'} z'(x,t), \ a \le x \le b, \ z(a,t) = 0, \ z'(a,t) = 1, \qquad (4)$$

and $$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{z(b, t_{k-1})}, \qquad (5)$$

Thus, we can solve the boundary value problem with

$$
\begin{aligned}
y_{i+1} &= y_i + h y_i' \\
y_{i+1}' &= y_i' + h y_i'' = y_i' + h f(x_i, y_i, y_i') \\
z_{i+1} &= z_i + h z_i' \\
z_{i+1}' &= z_i' + h z_i'' = z_i' + h \left(\frac{\partial f}{\partial y} z_i + \frac{\partial f}{\partial y'} z_i' \right) \\
t_k &= t_{k-1} - \frac{y_{N,t_{k-1}} - \beta}{z_{N,t_{k-1}}}
\end{aligned}
\tag{1}
$$

where $t_0 = (\beta - \alpha)/(b - a),\ y_{0,t_k} = \alpha,\ y_{0,t_k}',\ z_{0,t_k} = 0,\ z_{0,t_k}' = 1,$

$$n = 0, 1, \ldots, N,\ \ k = 0, 1, \ldots, M.$$

Then we iterate it, when it approaches $|y(N, t_k)| \leq tol$.

For example, consider the boundary value problem

$$y'' = \frac{y^3 - y}{\epsilon^2},\ 0 \leq x \leq 1,\ y(0) = 0,\ y(1) = \tanh\left(\frac{1}{\sqrt{2}\epsilon}\right), \tag{2}$$

This problem has the exact solution $y(x) = \tanh\left(\frac{x}{\sqrt{2}\epsilon}\right).$ (1)

```
%%%%%%%%%%%%%%%%%%%% nonlinear_shooting.m %%%%%%%%%%%%%%%%%%%%%%%
clear; clf; clc; epsilon=0.5; n=20; flag=1; k=1;
x=linspace(0, 1, n+1); h=x(2)-x(1); ex=tanh(x/(sqrt(2)*epsilon));
tol=1.0e-5; t(k)=(ex(n+1)-ex(1))/(x(n+1)-x(1));
y(1)=0; z(1)=0; zp(1)=1; plot(x,ex,'k-'); hold
while (flag==1)
    yp(1)=t(k);
    for i=1:n
        y(i+1)=y(i)+h*yp(i);
        yp(i+1)=yp(i)+h*(y(i)^3-y(i))/epsilon^2;
        z(i+1)=z(i)+h*zp(i);
        zp(i+1)=zp(i)+h*(3*y(i)^2-1)*z(i)/epsilon^2;
    end
if (abs(y(end)-ex(end))<tol || k>2)
flag=0;
end
k=k+1; t(k)=t(k-1)-(y(end)-ex(end))/z(end);
if k==2
plot(x,y,'ko')
elseif k==3
    plot(x,y,'kd')
```

```
else
    plot(x,y,'ks')
end
end
legend('exact solution','first iteration', ...
    'second iteration','third iteration')
set(gca, 'fontsize',18);
```