Chap. 7 Generaliged Coordinates

7.1 Inertial Cartesian Coordinate System

Some time Cartesian Coordinate incovenient

→ We have to choose coordinate system.

$\delta S = 0$; action should be minimized

* Same origin, do not move the frame :

express coordinate in terms of other coordinate ⇒ shape of Lagrangian

1) 1-dim.

Action : $\delta S = \int_{t_1}^{t_2} dt\, \delta L(x, \dot{x} ; t)$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right)$$

Since $\frac{\partial L}{\partial \dot{x}} \delta \dot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \delta x$,

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x + \cancel{\frac{\partial L}{\partial \dot{x}} \delta x \Big|_{t_1}^{t_2}}^{\,0}$$

ex) $x = ay + b$, a.b are const. ⇒ Should it

$x = \tan y$ different from the origin eq.?

⇒ new parameter detemines x 1-1 correspondent.

⇒ result must be the same!

2) 2-dim. → x and y are Independent variable

Action ; $\delta S = \int_{t_1}^{t_2} dt\, \delta L(x, y, \dot{x}, \dot{y} ; t)$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} \right)$$

$$= \int_{t_1}^{t_2} dt \left\{ \underline{\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]} \delta x + \underline{\left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\right]} \delta y \right\}$$

ex) change variable $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow$ Can change the eq. about angle θ and r ?

Suppose you have potential energy :

$$U = -G\frac{Mm}{r} \quad \text{: gravity}$$

In this case, pola coordinate system is better then Cartesian

Then, in pola coordinate system, what is eq. of motion?

Is it possible?

$$\delta S = \int_{t_1}^{t_2} dt\, \delta L(r, \theta, \dot{r}, \dot{\theta}\, ; t)$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial r}\delta r + \frac{\partial L}{\partial \theta}\delta\theta + \frac{\partial L}{\partial \dot{r}}\delta\dot{r} + \frac{\partial L}{\partial \dot{\theta}}\delta\dot{\theta} \right]$$

$$= \int_{t_1}^{t_2} dt \left\{ \left[\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)\right]\delta r + \left[\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\right]\delta\theta \right\}$$

If it is can, then 2nd one is much better then 1st one to desc
with this potential energy ($U = -GMm/r$).

Lagrangian : $L = T - U$

$$= \frac{1}{2}m\dot{\vec{x}}^2 - U(\vec{x})$$

If the potential energy is function of r only, not dependence θ,

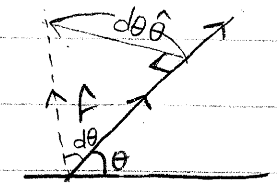
$$L = \underbrace{\ ?\ }_{\text{want to rewrite in terms of } r \text{ and } \theta} - U(r)$$

Then what is $\vec{x}$?

$$\vec{x} = r\hat{r} \ ; \quad \frac{d\vec{x}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$$\dot{\hat{r}} = \frac{d}{dt}\hat{r} = \frac{\partial \hat{r}}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{r}}{\partial \theta}\frac{d\theta}{dt}$$

Let us think about it.



If you change the radius, but $\hat{r}$ does not change.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\partial \vec{r}}{\partial r} \Rightarrow \frac{\partial \hat{r}}{\partial r} = 0 .$$

If you change the angle, then direction is change.

($\hat{r}, \hat{\theta}$ are perpendicural. $\Rightarrow \hat{r}\cdot\hat{\theta} = 0$)

$$d\hat{r} = \hat{r}(\theta + d\theta) - \hat{r}(\theta) = d\theta\,\hat{\theta}$$

$$\Rightarrow \hat{\theta} = \frac{d\hat{r}}{d\theta}$$

That means,

$$\dot{\hat{r}} = \hat{\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\theta}$$

So, final result is

$$\frac{d\vec{x}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Now we find out $\dot{\vec{x}}^2$; $L = \boxed{\frac{1}{2}m\dot{\vec{x}}^2} - U(\vec{x})$

in Cartesian coordinate system, Kinetic energy does not have any coordinate dependence. Depending on only velocit

$$\dot{\vec{x}}^2 = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Therefore, the kinetic energy can rewrite :

$$T = \frac{1}{2}m\dot{\vec{x}}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Therefore, the Lagrangian should be

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r),}$$ in pola coordinate system

Equation of motion In new coordinate system Is same In Cartesian coordinate system, If the new variable are Independent.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

Therefore, the equation of motion for this problem are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \quad \text{and} \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

Real force ;

$$m r\dot{\theta}^2 - \frac{\partial U}{\partial r} - \frac{d}{dt}(m\dot{r}) = 0, \qquad 0 - \frac{d}{dt}(m r^2 \dot{\theta}) = 0$$

rotation Inertia

$$\Rightarrow \frac{d}{dt}(I\dot{\theta}) = I\ddot{\theta} = I\alpha = 0$$

$\Rightarrow$ Angular momentum Is conserved.

Then, we can derive the same result at Cartesian coordinate system?

$$L = \frac{1}{2}m\dot{\vec{x}}^2 - U, \qquad \text{where potential energy is function of } r \text{ only}$$

Eq. of motion ;

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \Rightarrow \frac{\partial r}{\partial x}\frac{\partial U}{\partial r} + \frac{d}{dt}(m\dot{x}) = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \Rightarrow \frac{\partial r}{\partial y}\frac{\partial U}{\partial r} + \frac{d}{dt}(m\dot{y}) = 0$$

Since,

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \qquad -\frac{\partial U}{\partial r} = F \text{ (central force)}$$

Then, the eq. of motions are

$$\left.\begin{aligned} \frac{x}{r}F(r) &= \frac{d}{dt}(m\dot{x}) \\ \frac{y}{r}F(r) &= \frac{d}{dt}(m\dot{y}) \end{aligned}\right\} \Rightarrow \frac{\vec{r}}{r}F(r) = \frac{d}{dt}(m\dot{\vec{r}})$$

Can't derive angular momentum conservation directly.

Chap. 7 Generalized Coordinates.

7.2 Time-Independent Rotation

$X' = RX$, where R is transformation matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

Before cosider about ↑ that, think about action, S, is the Scalar.

After rotation, the scalar function must be the same to original.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow x_i' = \sum_{j=1}^{3} R_{ij} x_j$$

varies covariantly.

$X'^2 = x_i' x_i' = x_i' \delta_{ij} x_j'$, where δ_{ij} is metric tensor

(and also)

$$= x_j x_j = x_i \delta_{ij} x_j$$

Let check it.

$$x_i' x_i' = x_i' \delta_{ij} x_j' = R_{ia} x_a \delta_{ij} R_{jb} x_b$$
$$= R_{ia} x_a R_{ib} x_b$$
$$= x_a \boxed{R_{ia} R_{ib}} x_b$$
$$= x_a \delta_{ab} x_b$$

$$\Rightarrow \sum_{i=1}^{3} R_{ia} R_{ib} = \delta_{ab} \Leftrightarrow \sum_{i=1}^{3} R^T_{ai} R_{ib} = (\mathbb{1})_{ab}$$

$\Leftrightarrow (R^T R)_{ab} = (\mathbb{1})_{ab}$, for any a and b.

$$\Rightarrow R^T R = \mathbb{1}.$$

And if RR^T also Identity matrix, then $R^T = R^{-1}$.

$$x_a x_a = x_a \delta_{ab} x_b = R^T_{ai} x_i' \delta_{ab} R^T_{bj} x_j' \quad (\because X' = RX \Leftrightarrow R^T X' = R^T R X$$
$$= R^T_{ai} x_i' R^T_{aj} x_j' \qquad \Leftrightarrow (R^T X')_a = x_a$$
$$= x_i' \boxed{R_{ia} R^T_{aj}} x_j' \qquad \Leftrightarrow R^T_{ai} x_i' = x_a$$
$$= x_i' \delta_{ij} x_j'$$

$$\Rightarrow \sum_{a=1}^{3} R_{ia}R^T_{aj} = \delta_{ij} \Leftrightarrow \sum_{a=1}^{3} R_{ia}R^T_{aj} = (\mathbb{1})_{ij}$$

$$\Leftrightarrow (RR^T)_{ij} = (\mathbb{1})_{ij}, \text{ for } \forall i, j$$

$$\Rightarrow RR^T = \mathbb{1}.$$

Therefore, $R^T = R^{-1}$ and we said that R is orthogonal matrix.

Let's see what does means. ; $RR^T = \mathbb{1}$.

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}, \quad \text{where } R_i = (R_{i1}\ R_{i2}\ R_{i3})$$

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} = \begin{pmatrix} R_1R_1^T & R_1R_2^T & R_1R_3^T \\ R_2R_1^T & R_2R_2^T & R_2R_3^T \\ R_3R_1^T & R_3R_2^T & R_3R_3^T \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{R_iR_j^T = \delta_{ij}}$$

In short, this kind linear transformation ($X' = RX$), that keep,

$$\boxed{X'^2 = X^2} \Leftrightarrow R^TR = RR^T = \mathbb{1} \ (\text{or } R^T = R^{-1})$$

$$\downarrow$$

$A \cdot B = A' \cdot B'$; any scalar product is invariant under transformation

(pf) $A' \cdot B' = A'_i B'_i = (R_{ia}A_a)(R_{ib}B_b)$

$$= A_a R_{ia} R_{ib} B_b$$

$$= A_a R^T_{ai} R_{ib} B_b$$

$$= A_a (R^TR)_{ab} B_b = A_a \delta_{ab} B_b = A \cdot B \quad \square.$$

Let study variation of action.

$$\delta S = \int_{t_1}^{t_2} dt\, \delta L(x_i, \dot{x}_j; t)$$

$$= \int_{t_1}^{t_2} dt \sum_i \underbrace{\left[\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\right]}_{\text{Vector}} \cdot \underbrace{\delta x_i}_{\text{Vector}}$$

neglecting the surface ter

In classical mechanics, time is scalar.

Therefore, it might be rewrite,

$$\left[\frac{\partial L}{\partial x_j'} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j'}\right)\right] \delta x_j'$$

Since, the relation is

$$X' = RX \iff R^T X' = X \iff x_i = (R^T X')_i = R^T_{ia} x'_a = R_{ai} x'_a,$$

therefore,

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left[\quad ? \quad \right] R_{ai} x'_a$$

Find that how the derivative transform.

$$X' = RX \; ; \quad \boxed{\frac{\partial}{\partial x_i'} \overset{?}{=} R_{ij} \frac{\partial}{\partial x_j}}$$

$$\vec{r} = (x_1, x_2, x_3) \longrightarrow (x_1', x_2', x_3')$$

basis vector; $e_1 \quad e_2 \quad e_3$ new basis; $e_1' \quad e_2' \quad e_3'$

Any point can be expressed in terms of either e_i or e_i'.

⇒ That means we can make chain rule to replace this derivative in terms of different coordinate system.

$$\frac{\partial}{\partial x_i'} = \sum_{j=1}^{3} \left(\frac{\partial x_j}{\partial x_i'}\right) \frac{\partial}{\partial x_j}$$

We know that ; $X = R^T X' \iff x_j = R^T_{ja} x'_a = R_{aj} x'_a$

$$\Rightarrow \frac{\partial x_j}{\partial x'_a} = R_{aj}$$

Therefore, $\frac{\partial}{\partial x'_I} = \sum_{J=1}^{3} \left(\frac{\partial x_J}{\partial x'_I}\right) \frac{\partial}{\partial x_J} = \sum_{J=1}^{3} R_{IJ} \frac{\partial}{\partial x_J}$

$$\Rightarrow \boxed{\nabla' = R\nabla}$$

So, just keep about every things,

$$\delta S = \int_{t_1}^{t_2} dt \sum_I \left[\frac{\partial L}{\partial x_I} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_I}\right) \right] \delta x_I$$

$*\ A_I \delta x_I = A'_I \delta x'_I$, where $A_I = \frac{\partial L}{\partial x_I} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_I}\right)$

$$= \left[\frac{\partial L}{\partial x'_I} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'_I}\right) \right] \delta x'_I$$

$$= \left[\frac{\partial L}{\partial x_J} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_J}\right) \right] R_{IJ} R_{Ik} \delta x_k$$

$$\left(= R^T_{JI} R_{Ik} = (R^T R)_{Jk} = \delta_{Jk} \right)$$

$$= \left[\frac{\partial L}{\partial x_J} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_J}\right) \right] \delta x_J$$; exactly same!

Chap. 7 Generalized Coordinates

7.3 Fixed Curvilinear Coordinate System.

$$L = T(\dot{\vec{x}}) - U(\vec{x})$$

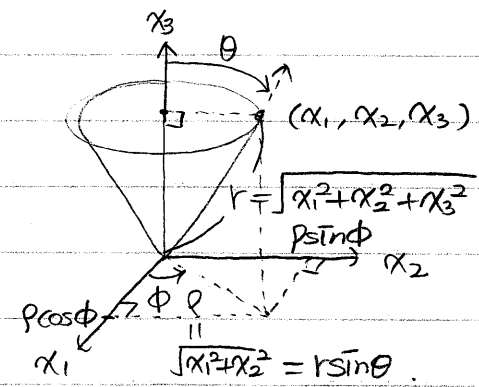
$$\begin{cases} x_1 = x_1(q_1, q_2, q_3) \\ x_2 = x_2(q_1, q_2, q_3) \\ x_3 = x_3(q_1, q_2, q_3) \end{cases}$$

(q_1, q_2, q_3) ; ex) Rotation (x_1', x_2', x_3')

Cylinderical coordinates (ρ, ϕ, z)

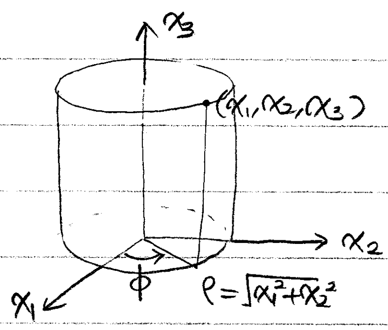
Spherical pola Coordinates (r, θ, ϕ)

For example, can represent x_i to spherical pola coordinates,



$$x_1 = r\sin\theta\cos\phi$$
$$x_2 = r\sin\theta\sin\phi$$
$$x_3 = r\cos\theta .$$
$$\Rightarrow q_1 = r, \quad q_2 = \theta, \quad q_3 = \phi .$$
$$\Rightarrow U(x_1, x_2, x_3)$$
$$= U(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

Represent x_i to culiderical coordinates,

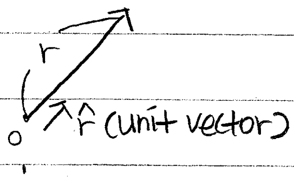


$$x_1 = \rho\cos\phi$$
$$x_2 = \rho\sin\phi$$
$$x_3 = z$$
$$\Rightarrow q_1 = \rho, \quad q_2 = \phi, \quad q_3 = z.$$
$$\Rightarrow U(x_1, x_2, x_3) = U(\rho\cos\phi, \rho\sin\phi, z)$$

Now, derive express of the kinetic energy.

1) Spherical pola coordinates

$$\begin{cases} x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\sin\phi \\ x_3 = r\cos\theta \end{cases} \xrightarrow[\text{directly}]{\text{Another way: can be obtain}} \vec{x} = r\hat{r}.$$



The kinetic energy just $\dot{\vec{x}}$:

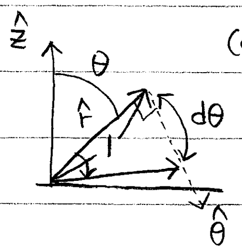
$$\dot{\vec{x}} = \frac{d}{dt}\vec{x}(r,\theta,\phi) = \frac{d}{dt} r(r)\hat{r}(\theta,\phi)$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + \ ?$$

Because $\hat{r}$ is the function of θ and ϕ, using the chain rule, then

$$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi}$$



(ϕ is fixed)

Distance $= d\theta$.

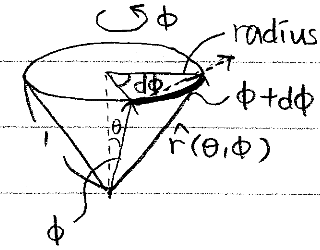
Direction $= \hat{\theta}$ ($\hat{r}$ and $\hat{\theta}$ are make right angle!)

$d\hat{r} = \hat{r}(\theta+d\theta) - \hat{r}(\theta) = d\theta\,\hat{\theta}$

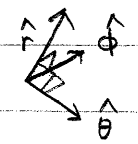
$$\Rightarrow \boxed{\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta}}$$

Therefore, $\displaystyle \frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi} = \dot{\theta}\hat{\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi}$

Let consider a cone.



radius $= \sin\theta$ $\Rightarrow$ Length of bold line $= d\phi \sin\theta$



Direction $= \hat{\phi}$ ($\hat{r}, \hat{\theta}$, and $\hat{\phi}$ are make right angle)

$$d\hat{r} = \hat{r}(\theta, \phi + d\phi) - \hat{r}(\theta, \phi) = d\phi \sin\theta\, \hat{\phi}$$

$$\Rightarrow \boxed{\sin\theta\, \hat{\phi} = \frac{\partial \hat{r}}{\partial \phi}}$$

Therefore, $\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \frac{d\phi}{dt}\frac{\partial \hat{r}}{\partial \phi} = \dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\, \hat{\phi}$

Let substitute this into eq. of $\dot{\vec{x}}$,

$$\dot{\vec{x}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\, \dot{\phi}\hat{\phi}$$

※ Important ; $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ make triad

$$\Rightarrow \hat{r} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{r}, \quad \hat{\phi} \times \hat{r} = \hat{\theta}.$$

Because they are make triad, scalar product is just sum of their own squal.

$$(\dot{\vec{x}})^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\phi}^2$$

Therefore, the kinetic energy

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\phi}^2),$$

and the potential energy must be

$$U = U(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$$

So, Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U$$

Then, what is eq. of motion?

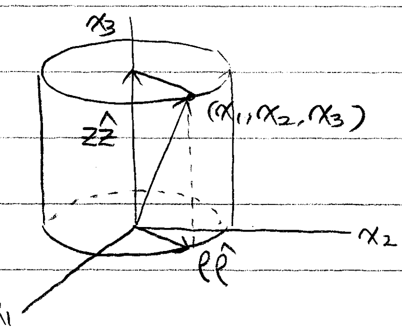
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \Rightarrow mr(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - \frac{\partial U}{\partial r} - m\ddot{r} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow mr^2\sin\theta\cos\theta\,\dot{\phi}^2 - \frac{\partial U}{\partial \theta} - mr^2\ddot{\theta} = 0$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \Rightarrow -\frac{\partial U}{\partial \phi} - mr^2\sin^2\theta\,\ddot{\phi} = 0$$

2) Cylinderical Coordinates

$$\begin{cases} x_1 = \rho\cos\phi \\ x_2 = \rho\sin\phi \\ x_3 = z \end{cases}$$



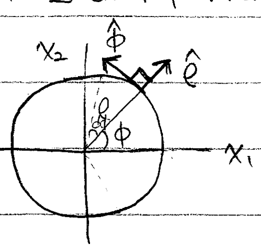
$$\vec{X} = \rho\hat{\rho} + z\hat{z} = \rho(\rho)\,\hat{\rho}(\phi) + z(z)\,\underbrace{\hat{z}}_{\text{const.}}$$

$$\dot{\vec{X}} = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{z}\hat{z}$$

Because $\hat{\rho}$ is function of ϕ only, so we Interest only ϕ dependence.

$$\frac{\partial\hat{\rho}}{\partial\phi} = ?$$

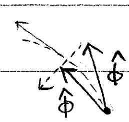
Just 2-dim. problem.



$$d\hat{\rho} = \hat{\rho}(\phi + d\phi) - \hat{\rho}(\phi) = d\phi\,\hat{\phi}$$

$$\Rightarrow \boxed{\hat{\phi} = \frac{\partial\hat{\rho}}{\partial\phi}}$$

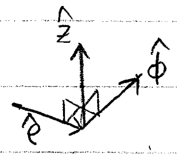
$$\boxed{\frac{\partial\hat{\phi}}{\partial\phi} = -\rho}$$



Because rotate $\hat{\phi}$, It is toward to center.

So, time derivative,

$$\dot{\vec{x}} = \dot{\rho}\hat{\rho} + \rho\frac{\partial\hat{\rho}}{\partial\phi}\frac{d\phi}{dt} + \dot{z}\hat{z} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$



$\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ are make triad.

$\Rightarrow \hat{\rho}\times\hat{\phi} = \hat{z}$, $\hat{\phi}\times\hat{z} = \hat{\rho}$, $\hat{z}\times\hat{\rho} = \hat{\phi}$.

Therefore, the velocity squal is just sum of own squar.

$$(\dot{\vec{x}})^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2$$

So, Lagrangian is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - U(\rho\cos\phi, \rho\sin\phi, z),$$

and the Euler-Lagrange eq. are

$$\frac{\partial L}{\partial \rho} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} = 0 \Rightarrow m\dot{\phi}^2 - \frac{\partial U}{\partial \rho} - m\ddot{\rho} = 0$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \Rightarrow -\frac{\partial U}{\partial \rho} - m\rho^2\ddot{\phi} = 0$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0 \Rightarrow -\frac{\partial U}{\partial \rho} - m\ddot{z} = 0$$

Chap. 7 Gereralized Coordinates

7.3 Fixed Curvilinear Coordinate system

Eq. of motion in Cartesian coordinate system is

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{x}_i}} = 0$$

$p_i = \frac{\partial L}{\partial \dot{x}_i}$; conjugate momentum to the coordinate x_i.

$F_i = \frac{d}{dt} p_i$; ~~i-th component of the componente~~

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \underbrace{\frac{\partial L}{\partial x_i}}$$ also force.

If Lagrangian is not depending on the i-th variable coordinate,

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow x_i \text{ is a cyclic coordinate}$$

$$\Rightarrow p_i = \frac{\partial L}{\partial \dot{x}_i} \text{ is conserved.}$$

Eq. of motion in generalized coordinate system is

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{q}_i}}\right) = \frac{\partial L}{\partial q_i}$$

Also,

$\frac{\partial L}{\partial \dot{q}_i} = p_i$; conjugate momentum to the generalized coordinate q_i

$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$; effective force

$$= \underbrace{\frac{\partial T}{\partial q_i}}_{\text{fictitious force}} - \underbrace{\frac{\partial U}{\partial q_i}}_{\text{generalized force}}$$

Let consider simple case ; 2-dim. pola coordinate system.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$\Rightarrow \begin{cases} P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} & \text{; linear momentum along the radial direction} \\ P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = I\dot{\theta} = l & \text{; angular momentum} \end{cases}$$

Conjugate momentum with respect to a coordinate, which is an angle variable, is an angular momentum.

1) Radial eq.

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \quad ; \quad \frac{\partial L}{\partial r} = \frac{\partial T}{\partial r} - \underbrace{\frac{\partial U}{\partial r}}_{\text{generalized force along the radial direction.}}$$

$$= \underbrace{\underbrace{mr\dot{\theta}^2}_{\text{Centrifugal fictitious force}} - \underbrace{\frac{\partial U}{\partial r}}_{\text{Real force}}}_{\text{Effective force.}}$$

2) Theta.

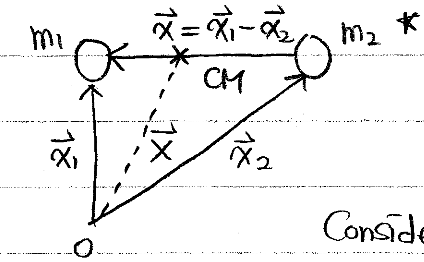
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \theta$ is cyclic coordinate

$\Rightarrow P_\theta = l$ is conserved.

Chap. 7 Generalized Coordinates

7.4 Generalized Coordinates for a Two-Body System



* $\boxed{T_1 + T_2 = T_{CM} + T_{Rotating}}$

Consider that what is center of mass ;

$$\vec{X} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$

It can be rewrite as ; $(m_1+m_2)\vec{X} = m_1\vec{x}_1 + m_2\vec{x}_2$

$$\underbrace{\vec{P}}_{\text{Total lineal momentum}} = (m_1+m_2)\dot{\vec{X}} = m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2 = \vec{P}_1 + \vec{P}_2 \qquad \text{— (a)}$$

And define that the relative coordinate,

$$\vec{x} = \vec{x}_1 - \vec{x}_2 \quad \Rightarrow \quad \dot{\vec{x}} = \dot{\vec{x}}_1 - \dot{\vec{x}}_2 = \frac{\vec{P}_1}{m_1} - \frac{\vec{P}_2}{m_2} \qquad \text{— (b)}$$

We can rewrite the result (a) and (b), using matrix,

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{m_1} & -\frac{1}{m_2} \end{pmatrix}\begin{pmatrix} \vec{P}_1 \\ \vec{P}_2 \end{pmatrix} = \begin{pmatrix} \vec{P} \\ \dot{\vec{x}} \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{P}_1 \\ \vec{P}_2 \end{pmatrix} = \frac{1}{\frac{1}{m_2} + \frac{1}{m_1}}\begin{pmatrix} \frac{1}{m_2} & 1 \\ \frac{1}{m_1} & -1 \end{pmatrix}\begin{pmatrix} \vec{P} \\ \dot{\vec{x}} \end{pmatrix}$$

Therefore,

$$\vec{P}_1 = \frac{m_2 m_1}{m_2+m_1}\left(\frac{\vec{P}}{m_2} + \dot{\vec{x}}\right), \qquad \vec{P}_2 = \frac{m_2 m_1}{m_2+m_1}\left(\frac{\vec{P}}{m_1} - \dot{\vec{x}}\right)$$

Let define $\frac{m_1 m_2}{m_1+m_2} = \mu$ is called reduced mass. So,

$$\frac{\vec{P}_1^{\,2}}{m_1} = \frac{\mu^2}{m_1}\left(\frac{\vec{P}}{m_2} + \dot{\vec{x}}\right)^2, \qquad \frac{\vec{P}_2^{\,2}}{m_2} = \frac{\mu^2}{m_2}\left(\frac{\vec{P}}{m_1} - \dot{\vec{x}}\right)^2$$

$$= \frac{\mu^2}{m_1}\left(\frac{\vec{P}^2}{m_2^2} + \cancel{\frac{2}{m_2}\vec{P}}\dot{\vec{x}} + \dot{\vec{x}}^2\right), \qquad = \frac{\mu^2}{m_2}\left(\frac{\vec{P}^2}{m_1^2} - \cancel{\frac{2}{m_1}\vec{P}}\dot{\vec{x}} + \dot{\vec{x}}^2\right)$$

Therefore,

$$T_1+T_2 = \frac{1}{2}m_1\dot{\vec{x}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{x}}_2^{\,2}$$

$$= \frac{1}{2}\left(\frac{\vec{P}_1^{\,2}}{m_1}+\frac{\vec{P}_2^{\,2}}{m_2}\right)$$

$$= \frac{\mu^2}{2m_1m_2}\left(\frac{\vec{P}^{\,2}}{m_2}+m_2\dot{\vec{x}}^{\,2}+\frac{\vec{P}^{\,2}}{m_1}+m_1\dot{\vec{x}}^{\,2}\right)$$

$$= \frac{\mu^2}{2m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\vec{P}^{\,2}+\frac{\mu^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\dot{\vec{x}}^{\,2}$$

Because,

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2} \Rightarrow \mu\left(\frac{1}{m_1}+\frac{1}{m_2}\right)=1$$

$$\therefore T_1+T_2 = \frac{\mu}{2m_1m_2}\vec{P}^{\,2}+\frac{\mu}{2}\dot{\vec{x}}^{\,2} = \frac{\vec{P}^{\,2}}{2M}+\frac{1}{2}\mu\dot{\vec{x}}^{\,2}$$

where $M=m_1+m_2$; total mass. Define that $\boxed{\vec{p}=\mu\dot{\vec{x}}, \text{ and } \vec{P}=M\dot{\vec{X}}}$

$$\therefore T_1+T_2 = \boxed{\frac{\vec{P}^{\,2}}{2M}+\frac{\vec{p}^{\,2}}{2\mu}}$$

Let write down the eq. of motion. First, the original Lagrangian is

$$L=\frac{1}{2}m_1\dot{\vec{x}}_1^{\,2}+\frac{1}{2}m_2\dot{\vec{x}}_2^{\,2}-U(\vec{x}_1-\vec{x}_2)$$

$$=\frac{1}{2}M\dot{\vec{X}}^{\,2}+\frac{1}{2}\mu\dot{\vec{x}}^{\,2}-U(\vec{x})$$

Then the eq. of motions are

$$\frac{\partial L}{\partial X_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_i}\right)=0 \Rightarrow$$ X_i's are cyclic coordinates. $\vec{P}$ is conserved.

$$\frac{\partial L}{\partial x_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)=0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)=\frac{\partial L}{\partial x_i} \Rightarrow \boxed{\frac{d}{dt}(\mu\dot{x}_i)=-\frac{\partial U}{\partial x_i}}$$

If we choose the frame where center of mass fixed, then this two-body problem is reduced into one-body problem. That involve $\frac{1}{2}M\dot{\vec{X}}^2$ vanishing :

$$L = \frac{1}{2}\mu\dot{\vec{x}}^2 - U(\vec{x}) \quad ; \text{ single one-body problem.}$$

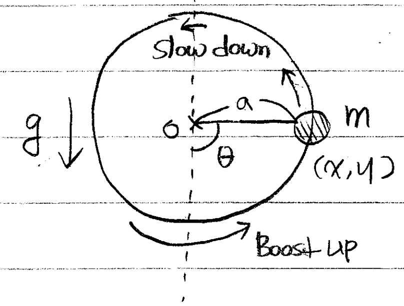
Chap. 7

Generalized Coordinates

Time-Dependent Potential

Let studing the system under a conservative force but as the special case, we introduce strenge system in which energy is not absolutly describe by sum of potential energy due to conservative force and kinetic energys.

Let consider a circle. There's no other force, except for gravity.



∴ Total mecanical energy is conserved.

1st, consider Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy$$

$x = a\sin\theta$, $y = a\cos\theta$, where $a = \text{const}$

Using this, Can make jus 1-dimansional probl

But, why we don't be introduce Lagrange undetermined multiplier to find constrain force as well.

∴ Don't consider Lagrange multiplier, 1st.

$$x = a\sin\theta \rightarrow \dot{x} = a\dot{\theta}\cos\theta$$
$$y = a\cos\theta \rightarrow \dot{y} = -a\dot{\theta}\sin\theta$$
$$\Rightarrow \dot{x}^2+\dot{y}^2 = a^2\dot{\theta}^2$$

Using this, then Lagrangian becomes

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta$$; sing 1-dimensional eq.

Or we can allow this a to be varied.

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta$$

Then we can consider Lagrange multiplier.

$$L' = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta + \lambda f(r,\theta)$$

where $f(r,\theta) = r - a$, a is a const.

Then, EOM (eq. of motion) for radius

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda \underbrace{\frac{\partial f}{\partial r}}_{1} = 0 \quad \Rightarrow mr\dot{\theta}^2 + mg\cos\theta - \frac{d}{dt}\cancel{(m\dot{r})}^{0} + \lambda = 0$$

and for theta $\quad \oplus\ r = a \Rightarrow \dot{r} = 0$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda \underbrace{\frac{\partial f}{\partial \theta}}_{0} = 0 \quad \Rightarrow -mgr\sin\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\Rightarrow \underbrace{ma^2\ddot{\theta} = -mga\sin\theta}_{\text{Torque.}}$$

From these result,

$$\lambda = -mr\dot{\theta}^2 - mg\cos\theta$$

and

$$\boxed{\ddot{\theta} + \frac{g}{a}\sin\theta = 0} \quad \Rightarrow \text{If } \theta \text{ is small enough,} \quad \ddot{\theta} + \frac{g}{a}\theta \approx 0$$

Let us, now, extend this idea to consider the Hamiltonian of this syst

1st, write down Lagrangian,

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta$$

then EOM is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \quad \Rightarrow \quad -mga\sin\theta - \frac{d}{dt}(ma^2\dot{\theta}) = 0$$

$$ma^2\ddot{\theta} + mga\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{a}\sin\theta = 0}$$ → We got the sam eq. of motion.

We know that because of the angle dependence of the potential energ theta, θ, is not a cyclic coordinate. Therefore, angular momentum is n conserved.

Let consider 2nd form of Euler-Lagrange eq.

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} =$$

Because there's no explicit time dependence in the Lagrangian. That means, Hamiltonian is conserved. What is Hamiltonian?

$$H = P_\theta \dot{\theta} - L = \frac{P_\theta^2}{ma^2} - \frac{P_\theta^2}{2ma^2} - mga\cos\theta$$

$$= \frac{P_\theta^2}{2ma^2} - mga\cos\theta$$

That is conserved quantity with respected time. Since,

$$\frac{P_\theta^2}{2ma^2} = \frac{l^2}{2I} \text{ ; Rotational kinetic energy.}$$

$$-mga\cos\theta = U,$$

therefore,

$$H = T+U \quad \text{(Total mechanical energy)}$$

Next, consider the strange problem. Every thing are same but $\theta = \omega t$, where ω is fixed.

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta, \quad \text{where } \theta = \omega t, \ \omega \text{ is fixed}$$

$$= \frac{1}{2}ma^2\omega^2 + mga\cos\omega t$$

Then Lagrangian has explicit time dependence.

$L = \frac{1}{2}ma^2\omega^2 + mga\cos\omega t$, where ω is fixed.

In this case, what is the EOM? In other to make the EOM, we have at least one variable.

$$L = \frac{1}{2}m(r'^2 + r^2\theta'^2) + mgr\cos\theta$$

$$= \frac{1}{2}m(r'^2 + r^2\omega^2) + mgr\cos\omega t$$

$$L' = \frac{1}{2}m(r'^2 + r^2\omega^2) + mgr\cos\omega t + \lambda(r-a)$$

Therefore, the EOM is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial r'}\right) + \lambda\frac{\partial}{\partial r}(r-a) = 0$$

$$mr\omega^2 + mg\cos\omega t - \frac{d}{dt}(mr') + \lambda = 0$$

Put $r=a$ into the EOM, then

$$ma\omega^2 + mg\cos\omega t + \lambda = 0$$

$$\therefore \lambda = -ma\omega^2 - mg\cos\omega t \quad \rightsquigarrow ?$$

Untill now, we don't know that what is it. Let consider Hamiltonian.

$$H = P_r r' - L = -L = -\frac{1}{2}ma^2\omega^2 - mga\cos\omega t$$

In this case, Hamiltonian does not a same as total mechanical energy.

$$H = -T + U$$

As another example, rotate same angular frequency but allow radius motion. First, consider Lagrangian.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$\dot{\theta} = \omega = \text{const.} \quad \boxed{\theta = \omega t}$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - U(r)$$

But in this problem, there's no explicit time dependence. Therefore,

$$\frac{\partial L}{\partial t} = 0, \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \Rightarrow H \text{ is conserved.}$$

Then, let compare the Hamiltonian and the total mechanical energy.

$$H = P_r\dot{r} - L = P_r\frac{P_r}{m} - \frac{1}{2}m\left[\left(\frac{P_r}{m}\right)^2 + r^2\omega^2\right] + U(r)$$

$$= \frac{P_r^2}{2m} - \frac{1}{2}mr^2\omega^2 + U(r)$$

$$E_{tot} = \frac{P_r^2}{2m} + \frac{1}{2}mr^2\omega^2 + U(r)$$

$$= \underbrace{\boxed{H}}_{\text{const.}} + \underbrace{mr^2\omega^2}_{\text{varied}} \quad \text{where } H \text{ is conserved.}$$

So, the total mechanical energy is not conserved because somebo
doing some work (make torque)