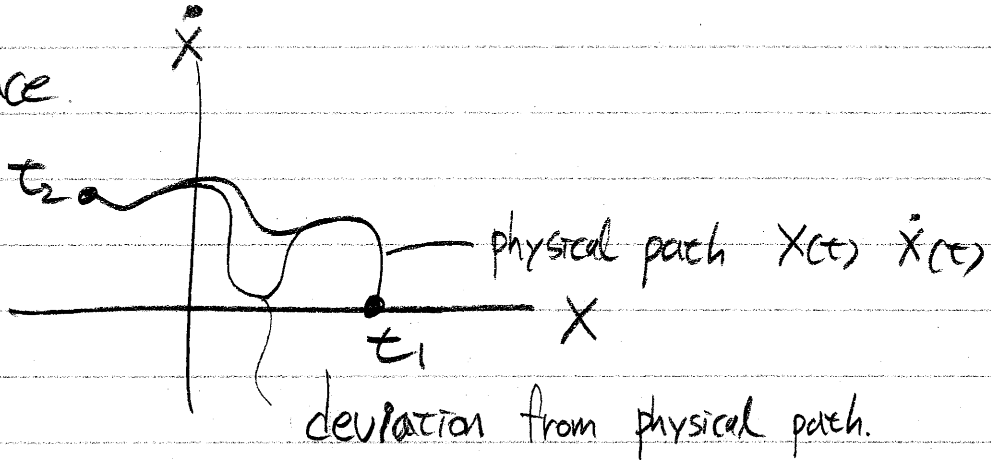
CHAPTER 6. Calculus of variation

$$S = \int_{t_1}^{t_2} dt\, L(x;\dot{x};t)$$

phase space



$$X(t) + \underbrace{\delta X(t)}_{\text{unphysical deviation.}}$$

$$S(X)$$

$$\hookrightarrow X_{phy} + \delta X$$

S is a function of the path $X(t)$.

"functional"

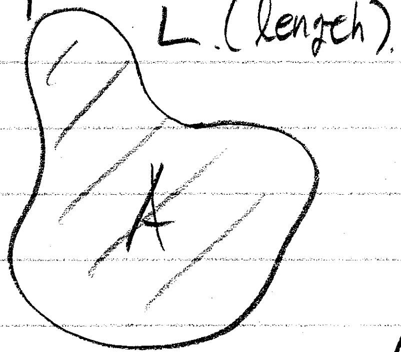
$$F(f) \quad , \quad \delta F[f]\Big|_{f_0} = F[f_0 + \delta f] - F[f_0]$$

$$\delta S = S[x + \delta x] - S[x]$$

$$\delta S = \int_{t_1}^{t_2} \delta L \, dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right] dt$$

Example

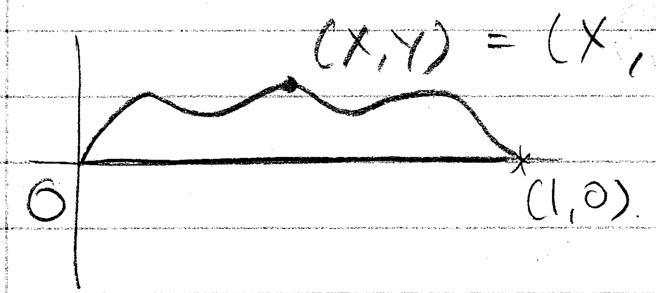


Shape $\Rightarrow f$

Area $\Rightarrow A$.

$A[f]$.

6.



f : minimized length shape of line.

$$\underset{\text{Length}}{\delta L} = \int dl = \int d\sqrt{x^2+y^2} = \int_0^1 dx\sqrt{1+(dy/dx)^2}$$

(constraint $\delta f(0) = \delta f(1) = 0.$)

$$\begin{cases} \delta L = \int_0^1 \sqrt{(\delta f')^2+1}\, dx. \\ \delta S = \int_{t_1}^{t_2} \delta L\, dt \end{cases}$$

$$\delta L_{(\text{length})} = \int_0^1 dx.\, \delta G(f, f'; x), \quad G = \sqrt{f'^2+1}$$

$$= \int_0^1 \left(\frac{\partial G}{\partial f}\delta f + \frac{\partial G}{\partial f'}\delta f' \right) dx.$$

$$= \int_0^1 \left[\frac{\partial G}{\partial f}\delta f + \frac{\partial G}{\partial f'}\frac{d(\delta f)}{dx} \right] dx.$$

$$= \int_0^1 \left[\cancel{\frac{d}{dx}\left(\frac{\partial G}{\partial f'}\delta f\right)}^{\,0} + \left(\frac{\partial G}{\partial f} - \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right)\right) \right] dx.$$

$$\frac{\partial G}{\partial f} - \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right) = 0 \quad , G = \sqrt{(f')^2+1}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{f'}{\sqrt{(f')^2+1}}\right) = 0$$

$$\frac{f'}{\sqrt{f'^2+1}} = c \qquad f'^2 = c^2(f'^2+1)$$

$$f'^2 = \frac{c^2}{1-c^2} = \text{constant.}$$

$$f' = \text{constant.}$$

$$\Rightarrow f' = \text{constant} \quad f = ax+b.$$

$$f(0) = f(1) = 0 \longrightarrow f = 0.$$ shortest path.

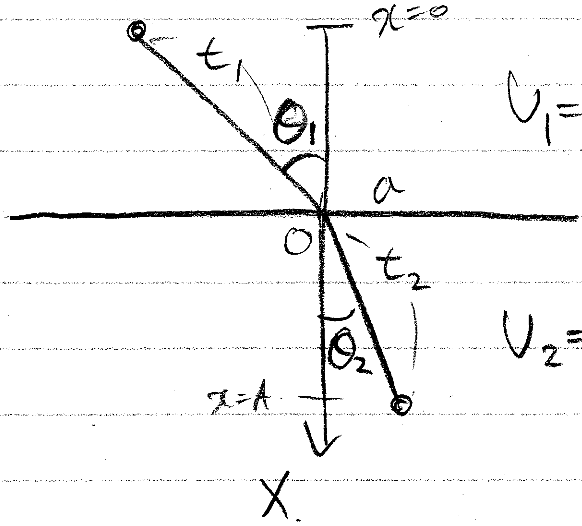
$$H = p\dot{x} - L\Big|_{(x,p)} \qquad \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$$\Updownarrow$$

$$\frac{d}{dx}(H') = -\frac{\partial G}{\partial x} = 0.$$

$$H' = \frac{\partial G}{\partial f'} f' - G = \frac{-1}{\sqrt{(f')^2+1}} = \text{Conserved}$$

Fermat's Principle and Snell's Law



$v_1 = \frac{c}{n_1}$

$v_2 = \frac{c}{n_2}$

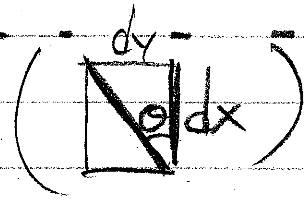
Snell's law

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}.$$

$$T = t_1 + t_2 = \int_1 dt + \int_2 dt.$$

$$\begin{cases} dl_1 = v_1 dt_1 \Rightarrow dt_1 = \frac{dl_1}{v_1} \\ dl_2 = v_2 dt_2 \Rightarrow dt_2 = \frac{dl_2}{v_2} \\ \sin\theta = \frac{dy}{\sqrt{dx^2+dy^2}} = \frac{(dy/dx)}{\sqrt{1+(dy/dx)^2}} \end{cases}$$

$$dl = d\sqrt{dx^2+dy^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$



$$T = \int \frac{dl_1}{v_1} + \int \frac{dl_2}{v_2}$$

$$= \int_0^A \left[\frac{\Theta(a-x)}{v_1} + \frac{\Theta(x-a)}{v_2} \right] \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

$$= \int_0^A dx \cdot \frac{1}{V(x)} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad V(x) = \begin{cases} V_1 & x < a \\ V_2 & x > a \end{cases}$$

Euler Lagrange $\Big\}$ $G(x) = \dfrac{1}{V(x)} \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

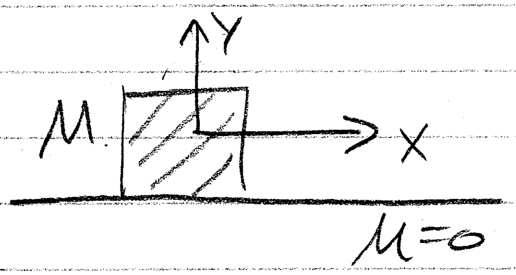
$$\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial (dy/dx)}\right) = 0.$$

$$\Longrightarrow \frac{\partial G}{\partial (dy/dx)} = \text{constant} \Longrightarrow \frac{1}{V(x)} \frac{dy/dx}{\sqrt{1+(dy/dx)^2}} = \text{constant}$$

$$* \quad \frac{dy/dx}{\sqrt{1+(dy/dx)^2}} = \sin\theta$$

$$\frac{\sin\theta}{V(x)} = \text{constant} = \frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}$$

6.4 Constraints and Lagrange's Undetermined multiplier



$$L = T - U$$

$$= \left(\frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2\right) - (Mgy)$$

constraint

$$\begin{cases} y = 0 \\ \dot{y} = 0 \end{cases}$$

$$L = \frac{1}{2}M\dot{x}^2 + U_{constant}$$

$$\left.\begin{aligned} &\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \\ &0 - \frac{d}{dt} P_x = 0 \end{aligned}\right\}$$

$\frac{\partial L}{\partial x} = 0 \Rightarrow$ x is a cyclic coordinates

$P_x = \frac{\partial L}{\partial \dot{x}}$ is conserved.

(conjugate momentum)

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) - MgY \quad , \quad \underline{\underline{Y=0}}$$

$$\left[\lambda f(X,Y) = 0 \Rightarrow \left(f(x,y) = Y \right) \right]$$

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) - MgY + \lambda f(X,Y)$$

$$\delta S = \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt \left\{ \left[\frac{\partial L}{\partial X} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) \right] \delta X + \left[\frac{\partial L}{\partial Y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) \right] \delta Y \right\}$$

$$\delta f = \delta(Y) = \underline{\delta Y + 0 \cdot \delta X = 0}$$

$\delta X, \delta Y \Rightarrow$ NOT Independent

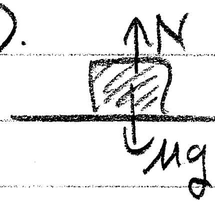
But, we introduce a additional free parameter "λ", $\delta X, \delta Y$ can be deviated independently

$$\frac{\partial L}{\partial X} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0 \Longrightarrow M\dot{X} = \text{constant.}$$

$$\frac{\partial L}{\partial Y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) = 0 \Longrightarrow -Mg + \lambda - \frac{d}{dt}(M\dot{Y}) = 0.$$

$Y = 0 \Rightarrow \dot{Y} = 0$. (return to physical system)

$0 = Mg - \lambda \Rightarrow \lambda = Mg$ (represents normal force).



$f(x,y)=0$

$\boxed{\text{Constraint}}$ — Undetermined multiplier

Apply $\boxed{f(x,y)=0}$

$L = \frac{1}{2}M(\dot{x}^2 + \cancel{\dot{y}^2}) - Mg\cancel{y}$

$L = \frac{1}{2}M\dot{x}^2$

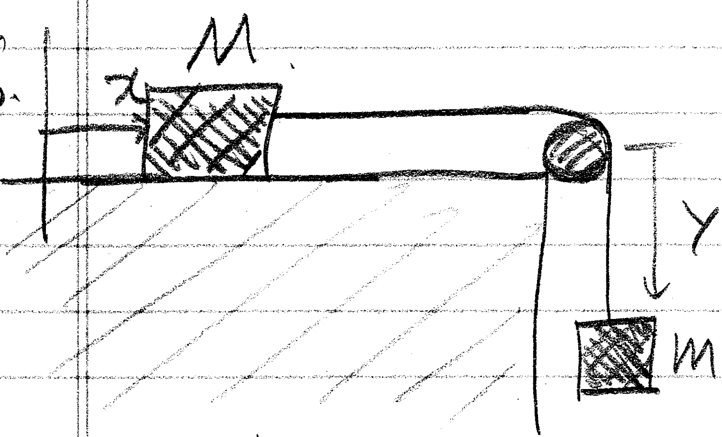
$$\left(L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) - Mgy + \lambda f(x,y) \right)$$

$\boxed{f(x,y)=0}$ Apply

λ is determined.

return to physical system.

4m.

6. 

$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m \dot{Y}^2$$

$$U = -mgY$$

$$L = T - U = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m \dot{Y}^2 + mgY$$

$$f(X,Y) = X - Y = 0 \qquad \text{(constraint)}$$

If we neglect a constraint,

$$M\ddot{X} = 0 \qquad M\ddot{Y} = -Mg \implies \text{Not the solution}$$

Case I. substitute in advance

$$L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2 + mgY \qquad Y = X$$

$$= \frac{1}{2}(M+m)\dot{X}^2 + mgX, \quad \ddot{X} = \frac{mg}{M+m}$$

Case II   Lagrange's Undetermined multiplier

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy + \lambda(x-y)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

$$\lambda - \frac{d}{dt}(M\dot{x}) = 0. \qquad -\lambda + mg - \frac{d}{dt}(m\dot{y}) = 0$$

Apply. $x = y$.

$$\lambda - \frac{d}{dt}(M\dot{x}) = 0.$$
$$-\lambda + mg - \frac{d}{dt}(m\dot{x}) = 0$$

$$mg = (M+m)\ddot{x} \qquad \ddot{x} = \frac{mg}{M+m}.$$

$$\lambda = M\ddot{x} = \frac{Mm}{M+m}g. \longrightarrow$$ represent the tension.

6.

$$L = L(x, \dot{x}, y, \dot{y}; t) + \lambda f(x, y)$$

$$\begin{cases} f(x; y) = 0 \\ \delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = 0 \\ \lambda\left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right) = 0 \end{cases}$$

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial \dot{y}}\delta \dot{y} + \lambda\left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right)$$

$$= \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial f}{\partial x}\right]\delta x + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) + \frac{\partial f}{\partial y}\right]\delta y + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\partial x + \frac{\partial L}{\partial \dot{y}}\delta y\right]$$

$$\delta S = \int_{t_1}^{t_2} \delta L \, dt$$

$$= \int_{t_1}^{t_2} dt \left\{\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial f}{\partial x}\right]\delta x + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) + \frac{\partial f}{\partial y}\right] \times \delta y\right\}$$

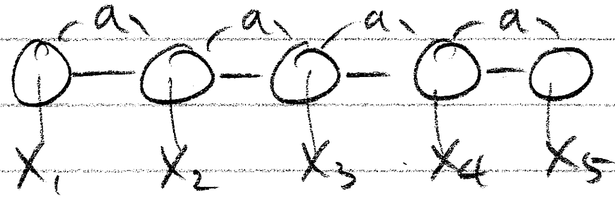
For 3 Dimensional case,

$L = L(x, \dot{x}, y, \dot{y}, z, \dot{z}; t)$

$f(x, y, z) = 0$

$$\Longrightarrow \quad \frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \lambda \frac{\partial f}{\partial x_i} = 0.$$

6



$$L = L(x_1, x_2 \cdots x_5, \dot{x}_1, \dot{x}_2 \cdots \dot{x}_5 ; t)$$

Constraints
$$\left.\begin{aligned} x_2 &= x_1 + a \\ x_3 &= x_2 + a \\ x_4 &= x_3 + a \\ x_5 &= x_4 + a \end{aligned}\right\}$$

$$\left.\begin{aligned} f_1 &= x_2 - x_1 - a = 0. \\ f_2 &= x_3 - x_2 - a = 0 \\ f_3 &= x_4 - x_3 - a = 0 \\ f_4 &= x_5 - x_4 - a = 0. \end{aligned}\right\} \begin{aligned} \lambda_1 \delta f_1 &= 0 \\ \lambda_2 \delta f_2 &= 0 \\ \lambda_3 \delta f_3 &= 0 \\ \lambda_4 \delta f_4 &= 0. \end{aligned}$$

$$\lambda_1 \sum_i \frac{\partial f_1}{\partial x_i} \delta x_i = 0.$$

$$\lambda_2 \sum_i \frac{\partial f_2}{\partial x_i} \delta x_i = 0$$

$$\vdots$$

$*(n > m)$

for n- degree of freedom, with m constraints.

$$L = L(x_1, x_2 \cdots x_n, \dot{x}_1, \dot{x}_2, \cdots \dot{x}_n ; t) + \sum_{k=1}^{m} \lambda_k f_k$$

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial x_i} = 0. \quad i = 1, 2, 3 \cdots n.$$