Chapter 5. Euler-Lagrange equation

- An alternative way

Lagrangian $L \equiv T - U$

$$T = \frac{1}{2} m \dot{x}_i^2 \quad (\text{Einstein notation}) = T(\dot{x})$$

conservative force $\Rightarrow U = U(\vec{x})$

$$L = T(\dot{\vec{x}}) - U(\vec{x}) \Rightarrow L(\dot{x}, x)$$

Euler-Lagrange equation

$$\Rightarrow \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial}{\partial x_i} L = 0 \iff \vec{F} = m\vec{a} \right]$$

$$\vec{F} = m\vec{a} \Rightarrow \frac{d}{dt}\vec{P} - \vec{F} = \vec{0}$$

$$\frac{d}{dt} P_i - F_i = 0 \quad \text{for } i = 1, 2, 3$$

$$P_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2} m \dot{x}_i^2\right) = m\dot{x}_i$$

$$-F_i = +(\vec{\nabla} U)_i = +\frac{\partial}{\partial x_i} U = -\frac{\partial}{\partial x_i} L.$$

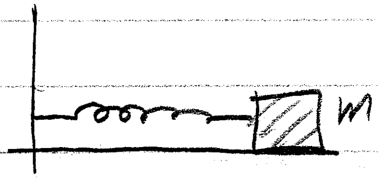
$$\Longrightarrow \frac{d}{dt} p_i - F_i = 0 \Longleftrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0.$$

$\vec{F} = m\vec{a}$: vector equation

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$: scalar equation.

5. $L = T - U$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0. \qquad p_i = \frac{\partial L}{\partial \dot{x}_i}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$F = -kx$$

$$U = -\int_0^x F \cdot dx = \frac{1}{2} k x^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$p_i = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \implies m\ddot{x} + kx = 0.$$

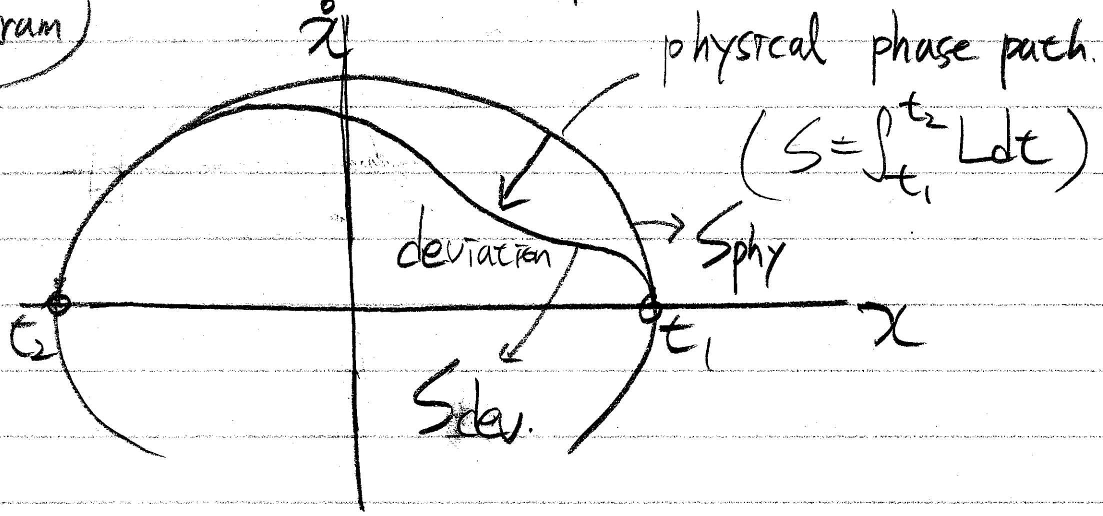
5.2 Toward calculation of variation

$$\underset{\text{action}}{S} \equiv \int_{t_1}^{t_2} dt\, L(\dot{\vec{x}}, \vec{x})$$

S is minimized $\Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0.$

phase diagram

Harmonic oscillator



$\underline{S_{phy} < S_{dev.}}$ $\Longrightarrow$ physical path $\Longleftrightarrow$ minimized action.

[lease action principle]

S is minimized

$\Downarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0.$$

$\vec{F} = m\vec{a}$ observation

5.2 $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$ (Simple harmonic oscillator)

$$\left.\begin{aligned} x(t) &= A\cos(\omega t - \delta), \quad \omega \equiv \sqrt{\tfrac{k}{m}} \\ \dot{x}(t) &= -A\omega \sin(\omega t - \delta) \end{aligned}\right\}$$

$$L = \frac{1}{2} m^2 A^2 \omega^2 \sin^2(\omega t - \delta) - \frac{1}{2} k A^2 \cos^2(\omega t - \delta)$$

$$= \frac{1}{2} k A^2 \left(\sin^2(\omega t - \delta) - \cos^2(\omega t - \delta) \right)$$

$$= \frac{1}{2} k A^2 \left[2 \sin^2(\omega t - \delta) - 1 \right]$$

$$S_{physical} = \int_{t_1}^{t_2} L(x_{phy}, \dot{x}_{phy})\, dt$$

$$= \frac{1}{2} k A^2 \int_{t_1 =}^{t_2 = t_1 + 2\pi/\omega = t_1 + T} \left[2 \sin^2(\omega t - \delta) - 1 \right] dt.$$

$$= \frac{1}{2} k A^2 \left[2 \times \frac{1}{2} \times T - T \right] = 0.$$

$\Rightarrow$ If there is some deviation on x, $\dot{x}$ then $S > 0$.

Consider deviation on $x, \dot{x}$

$$\delta x_{(t)} = \lambda \cdot \sin \omega t .$$

$$\left.\begin{aligned} x_{(t)} &= x_{phy}(t) + \delta x_{(t)} \\ &= A\cos(\omega t - \delta) + (\lambda \sin \omega t) \\ \dot{x}_{(t)} &= -A\omega \sin(\omega t - \delta) + \omega\lambda \cos\omega t \end{aligned}\right\}$$

$$L = L_{phy} + L_1 + L_2$$

$$L_1 = \frac{1}{2} k 2A\lambda \cos\omega t - \delta \sin \omega t$$

$$+ \frac{1}{2} m(-A\lambda\omega^2) \sin\omega t - \delta \cos\omega t$$

$$L_2 = \frac{1}{2} m\lambda^2\omega^2 \cos^2\omega t - \frac{1}{2} k\lambda^2 \sin^2 \omega t .$$

$$S = S_{phy} + \underbrace{\int_{t_1}^{t_2} (L_1 + L_2)\, dt}_{\text{positive}} .$$

5.2 $L = L(x, \mathring{x}; t)$

$$X = X(t)_{phy} \longrightarrow X(t) + \delta X(t)$$

$$\mathring{X} = \mathring{X}(t)_{phy} \longrightarrow \mathring{X}(t) + \delta \mathring{X}(t)$$

$$L = L_{physical} + \delta L.$$

$$\delta L = \frac{\partial L}{\partial X}\delta X + \frac{\partial L}{\partial \mathring{X}}\delta \mathring{X} + \cancel{\frac{\partial L}{\partial t}\delta t}^{\,0} + O(\delta^2)$$

$$\begin{cases} \delta X(t) \quad \text{constraint.} \quad \delta X(t_1) = \delta X(t_2) = 0 \\ \delta \mathring{X} = \frac{d}{dt}\delta X. \end{cases}$$

$$\frac{\partial L}{\partial \mathring{x}}\delta \mathring{x} = \frac{\partial L}{\partial \mathring{x}}\frac{d}{dt}(\delta X) = \frac{d}{dt}\left(\frac{\partial L}{\partial \mathring{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \mathring{x}}\right)\delta x$$

$$\delta L = \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \mathring{x}}\right)\right]\delta x + \frac{d}{dt}\left(\frac{\partial L}{\partial \mathring{x}}\delta x\right)$$

$$S = \int_{t_1}^{t_2} L\,dt$$

$$\delta S = \int_{t_1}^{t_2} \delta L \cdot dt$$

$$= \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) + \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x \quad dt$$

$$= \cancel{\left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{t_1}^{t_2}}^{=0} + \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x\, dt$$

$= 0$ only if $\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0\right]$

5.3 Extension to 3 dimensions.

$$L(\vec{x}, \dot{\vec{x}}; t) = L(x_i, \dot{x}_j; t)$$

$$\delta f(x_1, x_2, \cdots x_n)$$

$$= \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 \cdots \frac{\partial f}{\partial x_n}\delta x_n = \vec{\nabla} f \circ \vec{\delta x}$$

$$f(\vec{x}+\vec{a}) = f(\vec{x}) + (\vec{a}\circ\vec{\nabla})f + \frac{1}{2!}(\vec{a}\circ\vec{\nabla})^2 f \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(\vec{a}\circ\vec{\nabla})^k}{k!} f = \exp[\vec{a}\circ\vec{\nabla}]\, f(\vec{x})$$

$$\vec{a} = \vec{\delta x} \longrightarrow 0.$$

$$f(\vec{x}+\delta\vec{x}) - f(\vec{x}) \simeq (\delta\vec{x}\circ\vec{\nabla}) f(\vec{x})$$

$$\delta L = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right) + \left[\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\right]\delta x_i$$

$$\delta S = \int_{t_1}^{t_2} \delta L \, dt = 0 \Longrightarrow \frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

$$i = 1, 2, 3 \cdots$$

$\underbrace{\delta X_1 \quad \delta X_2 \quad \delta X_3}$

All independent.

$$\Rightarrow \left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \frac{\partial L}{\partial X_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_2}\right) - \frac{\partial L}{\partial X_2} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_3}\right) - \frac{\partial L}{\partial X_3} = 0 \end{aligned}\right\}$$

5.5 Second form of Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\dot{x}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \dot{x}\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial \dot{x}}\frac{d\dot{x}}{dt} - \dot{x}\frac{\partial L}{\partial x} = 0$$

$$\left[\frac{d}{dt} f(x,\dot{x};t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial \dot{x}}\frac{d\dot{x}}{dt}\right]$$

$$\frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial \dot{x}}\frac{d\dot{x}}{dt} + \frac{\partial L}{\partial x}\frac{dx}{dt}\right) = 0$$

$$\frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}}\right) - \frac{dL}{dt} + \frac{\partial L}{\partial t} = 0$$

$$\frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} - L\right) = -\frac{\partial L}{\partial t}$$

$$\left.\begin{aligned}\dot{x}\frac{\partial L}{\partial \dot{x}} - L &\equiv H \quad \text{Hamiltonian}\\ &= \dot{x}p - L\end{aligned}\right\} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

if L does not have explicit time dependence

then $\frac{\partial L}{\partial t} = 0 = \frac{dH}{dt}$.

Hamiltonian is conserved