Chapter 4 Electrostatics and Gravitation

4.1 Basic properties

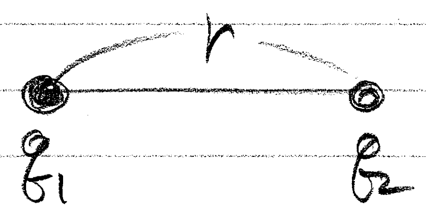
* electric charge q

* SI unit C (Coulomb)

$$e \simeq 1.6 \times 10^{-19}\ C. \quad \begin{cases} q > 0 & \text{positive} \\ q < 0 & \text{negative} \\ q = 0 & \text{neutral} \end{cases}$$

* electric charge is conserved

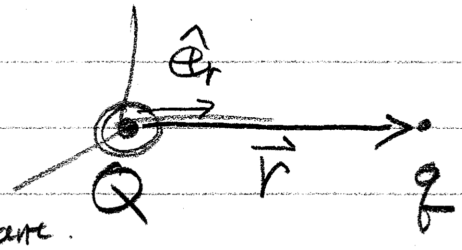
* $F = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_1 q_2}{r^2}$



4.1.2 Coulomb's law and Newton's law of gravity.

$$\begin{cases} \vec{F}_{Q\to q} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Qq}{r^2}\hat{e}_r \\ \vec{F} \propto \hat{e}_r \text{ : central force.} \end{cases}$$

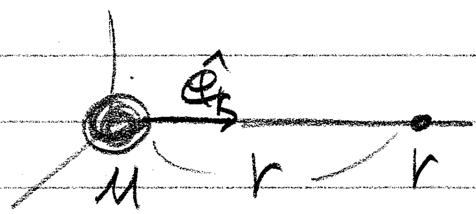
ε_0 → permitivity constant.



$$\begin{cases} \vec{F}_{q\to Q} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Qq}{r^2}(-\hat{e}_r) \\ \vec{F}_{Q\to q} + \vec{F}_{q\to Q} = \vec{0} \quad \text{(Newton's third law)} \end{cases}$$

$$\begin{cases} \vec{l}_q = \vec{r}\times\vec{p} \qquad \vec{\tau}_q = \dfrac{d\vec{l}_q}{dt} = \dfrac{d}{dt}(\vec{r}\times\vec{p}) \\ \text{(origin is on the Q)} \qquad = \cancel{\vec{v}\times m\vec{v}}^{\,0} + \cancel{\vec{r}\times\vec{F}}^{\,0} = 0 \end{cases}$$

$$\begin{cases} \vec{F} = -G\dfrac{Mm}{r^2}\hat{e}_r \\ \vec{F}_{m\to M} + \vec{F}_{M\to m} = \vec{0} \end{cases}$$

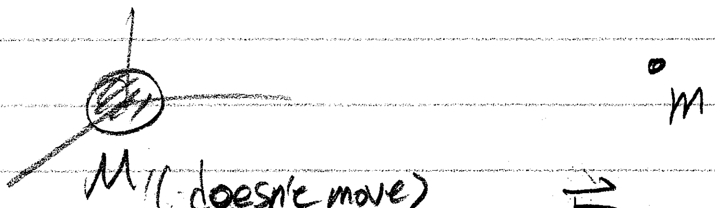


$F \propto \dfrac{1}{r^2}$: inverse square law.

4.2 Electric and Gravitational "Field"

Scalar field $\phi(t, \vec{x})$

vector field

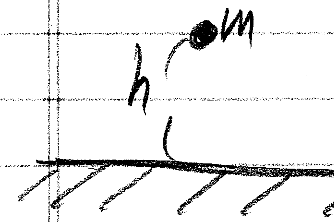


$$\vec{F}(t, \vec{x}) = -G\frac{Mm}{r^2}\hat{e}_r$$

no time dependence

$\Longrightarrow$ static vector field

$$\vec{x} = x\hat{e}_x + y\hat{e}_y + h\hat{e}_z$$



$U(\vec{x}) = mgh$ $\Longrightarrow$ static scalar field

example)

$$\left\{ \underbrace{\vec{F}}_{\text{vector}} = -\vec{\nabla}\underbrace{U}_{\text{scalar}} \right.$$

4.3 Gauss' law and inverse square law

$$\vec{V} = \frac{\hat{e}_r}{r^2} \qquad \hat{e}_r = \frac{\vec{r}}{r} \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_i^2}$$

$$\left(= \sqrt{(x^2+y^2+z^2)}\right)$$

(Einstein notation)

$$\vec{\nabla} \circ \vec{V} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right) \circ \frac{\vec{r}}{r^3}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{x_i}{r^3}\right) = \frac{1}{r^3}\frac{\partial x_i}{\partial x_i} + x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r^3}\right)$$

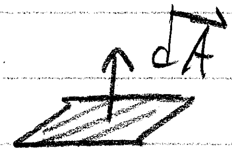
$$= \frac{3}{r^3} + x_i\left(-3r^{-4}\right) \cdot \left(\frac{1}{2}\frac{1}{r} \cdot 2x_i\right)$$

$$= \frac{1}{r^3} \cdot \left(3 - \frac{3x_i^2}{r^2}\right) = 0, \quad r \neq 0$$

$$\Longrightarrow \vec{\nabla} \circ \left(\frac{\vec{r}}{r^3}\right) = 0 \quad (\text{if } r \neq \vec{0})$$

4.3 $\vec{\nabla}\circ\left(\vec{r}/r^3\right) = 0$ if $\vec{r} \neq \vec{0}$

$$\int \vec{\nabla}\circ\left(\frac{\vec{r}}{r^3}\right)\cdot d^3\vec{x} = \oint_{S=\partial V} \frac{\vec{r}}{r^3}\circ d\vec{A}$$



Consider a sphere with radius R.

$$\int \vec{\nabla}\circ\left(\frac{\vec{r}}{r^3}\right) d^3\vec{x} = \oint_{S=\partial V} \frac{\vec{r}}{r^3}\circ d\vec{A} = \oint \frac{\vec{r}\circ\hat{e}_r}{r^3}\, dA$$

$$= \oint \frac{R}{R^3}\, dA = \frac{1}{R^2}\oint dA = 4\pi.$$

$$\longrightarrow \int \vec{\nabla}\circ\left(\frac{\vec{r}}{r^3}\right) d^3\vec{x} = 4\pi.$$

$$\int \left[\vec{\nabla}\circ\left(\frac{1}{4\pi}\cdot\frac{\hat{e}_r}{r^2}\right) d^3\vec{x} = 1\right.$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1, \quad \delta(x) = 0 \text{ if } x \neq 0.$$

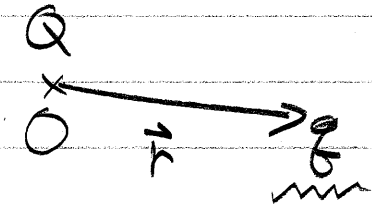
$\downarrow$ 3-D

$$\int \delta^{(3)}(x)\, d^3\vec{x} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \cdot \delta^{(3)}(x,y,z) = 1$$

$$\Longrightarrow \nabla\circ\left(\frac{1}{4\pi}\cdot\frac{\hat{e}_r}{r^2}\right) = \delta^{(3)}(\vec{x})$$

4.3 $\vec{\nabla}\cdot\left(\frac{1}{4\pi}\frac{\hat{e}_r}{r^2}\right) = \delta^{(3)}(\vec{x})$, $r=|\vec{x}|$

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{\hat{e}_r}{r^2}\, q\, Q.$$



$$q\vec{E} = \vec{F},$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\hat{e}_r}{r^2}Q. \Longrightarrow \vec{\nabla}\cdot\vec{E} = \delta^{(3)}(\vec{x})\cdot\frac{Q}{\varepsilon_0}.$$

$$\left\{\begin{array}{l} * \int \delta(x)\,dx = 1 \\ \int \delta(x) f(x)\,dx = f(0) \\ \int \delta(x) f(y-x)\,dx = f(y) \end{array}\right\} \int \delta^3(x) f(\vec{y}-\vec{x})\,d^3\vec{x} = f(\vec{y})$$

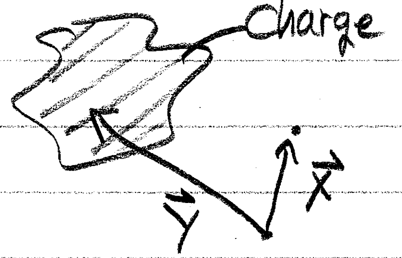
$$\int \delta^{(3)}(\vec{x})\, \rho(\vec{y}-\vec{x})\, d^3\vec{x} = \rho(\vec{y})$$

$$\vec{\nabla}\cdot\vec{E} = \delta^3(\vec{x})\frac{Q}{\varepsilon_0} = \frac{\rho(\vec{x})}{\varepsilon_0}.$$

$$\int \vec{\nabla}\cdot\vec{E}\cdot d^3\vec{x} = \int \frac{\rho(\vec{x})}{\varepsilon_0}\cdot d^3\vec{x} = \frac{Q}{\varepsilon_0}.$$ Gauss law.

4.4 Electric and Gravitational Potential

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\,\frac{\hat{e}_r}{r^2}$$



$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int dq \cdot \frac{\hat{e}_r}{r^2}, \quad \vec{r} \equiv \vec{x} - \vec{y}$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{y})}{|\vec{y}-\vec{x}|^2}\cdot \hat{e}_r \, d^3\vec{y}$$

$$U(\vec{x}) = U(\vec{r}_{ref}) - \int_{\vec{r}_{ref}}^{\vec{x}} \vec{F} \cdot d\vec{r}$$

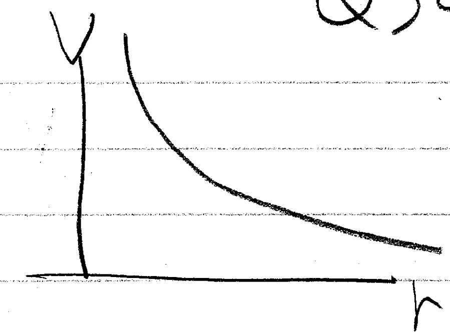
$$U(\vec{r}_{ref}) \equiv 0, \text{ as } |\vec{r}| \to \infty.$$

$$V = \frac{U}{q} = -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{r}$$ (one charged particle : simple case)

$$= -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{\vec{r}} \frac{\vec{r}}{r^3} \cdot dr = \frac{-Q}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{dr}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r}$$

$Q > 0$



(distributed charge)

$$dV(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r}$$

$$V(\vec{x}) = \int dV = \frac{1}{4\pi\varepsilon_0}\int\frac{dQ}{r} = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

4.4

$$V(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{y}\ \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

$$\vec{F} = -\vec{\nabla}U \qquad \vec{E} = -\vec{\nabla}V$$

$$\vec{E} = -\vec{\nabla}V = -\vec{\nabla}_{(\vec{x})} \cdot \frac{1}{4\pi\varepsilon_0}\int d^3\vec{y}\ \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

$$= -\left(\hat{e}_i \frac{\partial}{\partial x_i}\right)\frac{1}{4\pi\varepsilon_0}\int d^3\vec{y}\ \frac{\rho(\vec{y})}{\sqrt{(x_j-y_j)^2}}$$

$$= \int d^3\vec{y} \cdot \frac{1}{4\pi\varepsilon_0} \cdot \rho(\vec{y}) \cdot \hat{e}_i\,(-1) \cdot \frac{(x-y)}{|\vec{x}-\vec{y}|^3} \cdot (-1)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3\vec{y} \cdot \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|^3} \cdot (\vec{x}-\vec{y})$$

4.4

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{e}_r}{r^2} = \frac{Q}{\varepsilon_0}\cdot\frac{1}{4\pi}\frac{\hat{e}_r}{r^2}$$

$$\vec{\nabla}\cdot\vec{E} = Q/_{\varepsilon_0}\,\delta^{(3)}(\vec{x}) \Rightarrow \frac{\rho(\vec{x})}{\varepsilon_0}$$

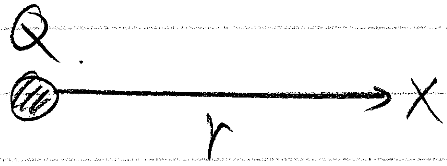
$$\updownarrow$$

$$\vec{g} = -\frac{GM}{r^2}\hat{e}_r$$

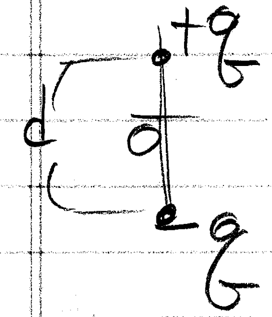
$$\vec{\nabla}\cdot\vec{g} = -4\pi GM\,\delta^{(3)}(\vec{x}) = -4\pi G\,\rho(\vec{x})$$

$$V = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r} \longleftrightarrow V = -GM\cdot\frac{1}{r}$$

4.5 Potential due to an Electric dipole.



$$V_{monopole} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

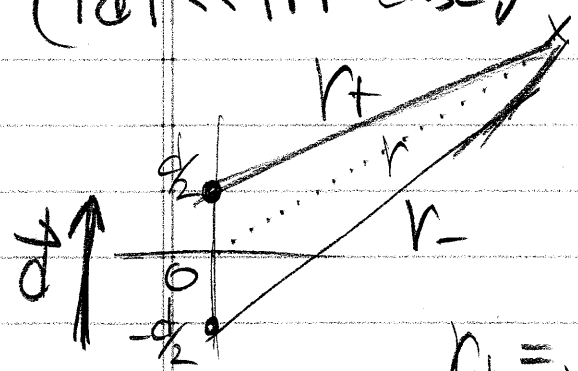


$d \ll r \longrightarrow r_{+q} \simeq r. \quad r_{-q} \simeq r$

$$V_{monopole} \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{r}\right) = 0$$

monopole contribution $\longrightarrow$ zero.

$(|\vec{d}| \ll |\vec{r}|$ case.)



$$\begin{cases} \vec{r}_+ = \vec{r} - \frac{1}{2}\vec{d} \\ \vec{r}_- = \vec{r} + \frac{1}{2}\vec{d} \end{cases}$$

$$r_+ = \sqrt{\vec{r}_+ \cdot \vec{r}_+} = \sqrt{r^2 - \vec{r}\cdot\vec{d} + d^2/4}$$

$$= r\sqrt{1 - \frac{\vec{d}\cdot\vec{r}}{r^2} + \frac{d^2}{4r^2}} = r\sqrt{\left(1 + \frac{d^2}{4r^2}\right) - \frac{d\cdot r}{r^2}}$$

$$r_- = r\sqrt{1 + \frac{\vec{d}\cdot\vec{r}}{r^2} + \frac{d^2}{4r^2}} = r\sqrt{\left(1 + \frac{d^2}{4r^2}\right) + \frac{d\cdot r}{r^2}}$$

$$r_\pm = r\sqrt{1+\frac{d^2}{4r^2}}\cdot\sqrt{1\mp\Delta} \qquad \Delta \equiv \frac{d\circ r}{r^2+d^2/4}$$

Since $\Delta \ll 1$ $\left(\frac{d}{r}\right)$

$$\frac{1}{\sqrt{1-\Delta}} = 1+\frac{1}{2}\Delta + \frac{3}{8}\Delta^2 + O(\Delta^3)$$

$$\frac{1}{\sqrt{1+\Delta}} = 1-\frac{1}{2}\Delta + \frac{3}{8}\Delta^2 + O(\Delta^3)$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r\sqrt{1+\frac{d^2}{4r^2}}}\cdot\left(\frac{1}{\sqrt{1-\Delta}} - \frac{1}{\sqrt{1+\Delta}}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\,\frac{1}{r\sqrt{1+d^2/4r^2}}\cdot\left(\Delta + O(\Delta^3)\right)$$

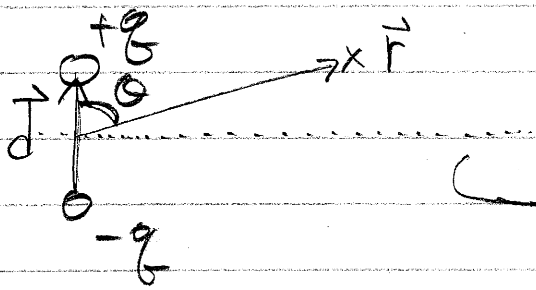
$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{r^2+d^2/4}}\cdot\left(\frac{\vec{d}\circ\vec{r}}{r^2+\frac{d^2}{4}}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r}\cdot\frac{\vec{d}\circ\vec{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q\vec{d}\cdot\hat{e}_r}{r^2}$$

$\vec{p} \equiv q\vec{d}$: electric dipole momentum.

$$V = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{e}_r}{r^2}$$

4.5 $V = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{e}_r}{r^2}$, $\vec{p} = q\vec{d}$



$\vec{p} \cdot e_r = p\cos\theta = \frac{pz}{r}$

(Symmetry line ⇒ $V=0$)

$$V = \frac{p}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2} = \frac{p}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

$$\vec{E} = -\vec{\nabla} V = -\left(e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z}\right) \underbrace{\frac{p}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)}_{?}$$

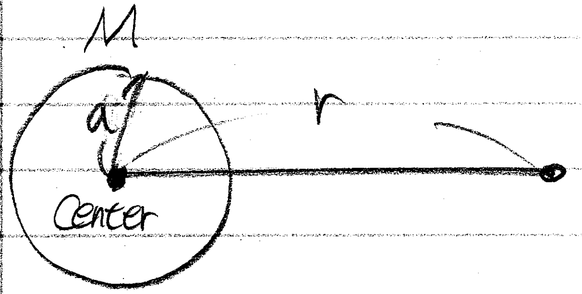
* professor had a mistake on this calculation.

$$\vec{E} = -\vec{\nabla} V = -\left(e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + e_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \frac{p}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2}$$

$$= \frac{p}{2\pi\varepsilon_0} \frac{1}{r^3} \left(\cos\theta\, \hat{e}_r + \frac{1}{2}\sin\theta\, \hat{e}_\theta\right).$$

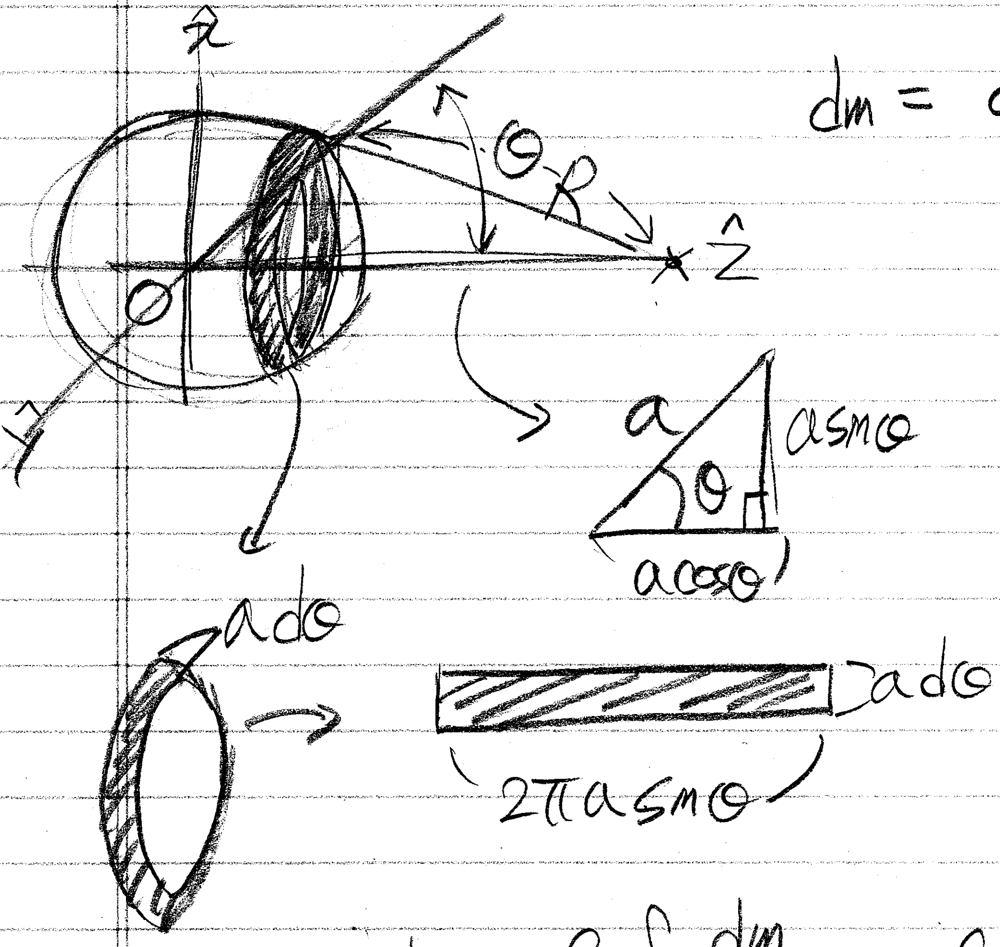
$V \sim r^{-2}$. $E \sim r^{-3}$

4.6 Shell theorem



$$V = -\frac{GM}{r} \quad (r>a)$$

$$V = \text{constant} \quad (r<a)$$



$$dm = \sigma\, dA, \quad \sigma = M/4\pi a^2$$

$$dA = 2\pi a^2 \sin\theta\, d\theta$$

$$= -G\int \frac{dm}{R} = -G\int \frac{\sigma\, 2\pi a^2 \sin\theta\, d\theta}{R}$$

$$\vec{R} = \vec{a} - r\hat{z} = \hat{x}\, a\sin\theta + \hat{z}\,(a\cos\theta - r)$$

$$R = \sqrt{(a\sin\theta)^2 + (a\cos\theta - r)^2}$$

$$= \sqrt{a^2 + r^2 - 2ar\cos\theta}$$

$$V = -G \cdot \int \frac{\sigma \cdot 2\pi a^2 \cdot \sin\theta\, d\theta}{R}$$

$$= -2\pi a^2 \sigma G \int \frac{\sin\theta\, d\theta}{\sqrt{a^2 + r^2 - 2ar\cos\theta}}$$

$$= -2\pi a^2 \sigma G \int_{-1}^{1} \frac{dt}{\sqrt{a^2 + r^2 - 2art}} \qquad t = \cos\theta$$

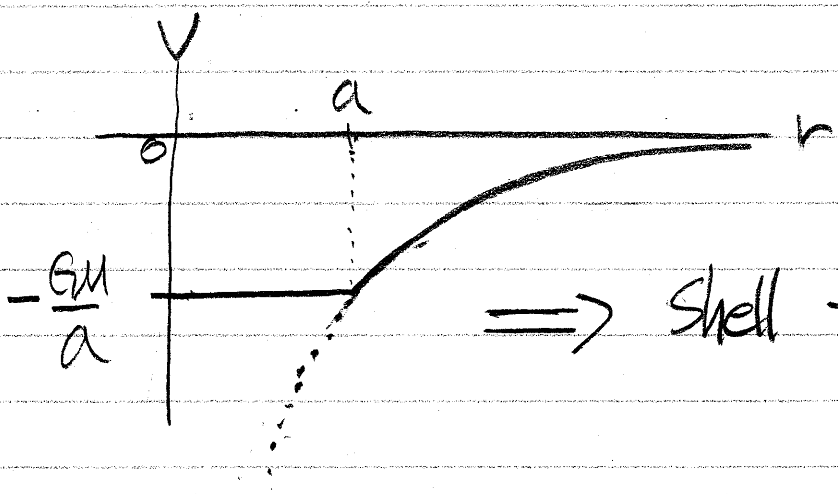
$$\int_{-1}^{1} \frac{dt}{\sqrt{a^2 + r^2 - 2art}} = \frac{2}{-2ar}\sqrt{a^2 + r^2 - 2art}\,\Big|_{-1}^{1}$$

$$= +\frac{1}{ar}\left(|r+a| - |r-a|\right)$$

$$\left(|r+a| - |r-a|\right) = \begin{cases} 2a & r > a \\ 2r & a > r \end{cases}$$

$$V = -2\pi a^2 \sigma G \cdot \frac{1}{ar} \begin{cases} \times 2a & r > a \\ \times 2r & a > r \end{cases}$$

$$\begin{cases} -\dfrac{4\pi a^2 \sigma G}{r} = -\dfrac{GM}{r} & \left(* \ \sigma = \dfrac{M}{4\pi a^2} \right) \\ -\dfrac{4\pi a \sigma G}{a} = -\dfrac{GM}{a} \end{cases}$$



$\Longrightarrow$ Shell theorem.