Chapter 3

Oscillations

3.1 Simple Harmonic Oscillator

1. The force by spring which has Modulus of elasticity k is

$$F = -kx, \tag{3.1}$$

and by Newton's Second law,

$$F = ma = m\ddot{x}. \tag{3.2}$$

Hence,

$$m\ddot{x} = -kx \quad \rightarrow \quad m\ddot{x} + kx = 0. \tag{3.3}$$

Let us define that $w_0^2 = k/m$. Then, above equation of motion becomes

$$\ddot{x} + w_0^2 x = 0. \tag{3.4}$$

2. The general solution of the above equation of motion is

$$x(t) = x_0 \cos w_0 t + \frac{v_0}{w_0} \sin w_0 t, \tag{3.5a}$$

$$\dot{x}(t) = v_0 \cos w_0 t - x_0 w_0 \sin w_0 t \tag{3.5b}$$

,where $x_0 = x(0)$ and $v_0 = \dot{x}(0)$.

3. Using following trigonometric relation,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta, \tag{3.6a}$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta, \tag{3.6b}$$

the solution can be rewritten as

$$x(t) = A\cos(w_0 - \delta) \tag{3.7a}$$

$$\dot{x}(t) = -Aw_0 \sin(w_0 t - \delta) \tag{3.7b}$$

,where

$$A = \sqrt{x_0^2 + \frac{v_0^2}{w_0^2}} \tag{3.8a}$$

$$\delta = \tan^{-1} \frac{v_0}{w_0 x_0} \tag{3.8b}$$

4. Then, the kinetic energy, potential energy and the total mechanical energy are of the form

$$T(t) = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}kA^2\sin^2(w_0t-\delta), \tag{3.9a}$$
$$U(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kA^2\cos^2(w_0t-\delta), \tag{3.9b}$$
$$E(t) = T(t)+U(t) = \frac{1}{2}kA^2. \tag{3.9c}$$

3.2 Phase Diagram

3.2.1 Phase Space

1. The phase space is the collection of points $[x(t), \dot{x}(t)]$. A single point $P(x, \dot{x})$ in the phase space is called a representative point.

3.2.2 Exercise

1. From previous results, the solution of one-dimensional simple harmonic oscillator is

$$x(t) = A\cos(w_0t-\delta) \tag{3.10a}$$
$$\dot{x}(t) = -Aw_0\sin(w_0t-\delta). \tag{3.10b}$$

Because $\cos^2 a + \sin^2 a = 1$, We can compute following relation.

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2w_0^2} = 1. \tag{3.11}$$

2. Likewise, the kinetic energy and potential energy, and total mechanical energy of simple harmonic oscillator are

$$T(t) = \frac{1}{2}mA^2w_0^2\sin^2(w_0t-\delta), \tag{3.12a}$$
$$U(t) = \frac{1}{2}kA^2\cos^2(w_0t-\delta), \tag{3.12b}$$
$$E = \frac{1}{2}kA^2. \tag{3.12c}$$

Above eqauations leads to

$$A^2 = 2E/k, \tag{3.13a}$$
$$A^2w_0^2 = 2E/m. \tag{3.13b}$$

Then, we can compute following relation.

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1. \tag{3.14}$$

3.2.3 Problem

1. Let us consider one-dimensional problem. A particle moves under a conservative force. $U(x)$ is the potential energy of particle. We have known that If $\boldsymbol{F}$ is conservative, $\boldsymbol{F} = -\frac{d}{dx}U(x)$. Hence, the equation of motion is

$$F = m\ddot{x} = -\frac{d}{dx}U \quad \rightarrow \quad \ddot{x} + \frac{du}{dx} = 0 \tag{3.15}$$

,where $u = U/m$.

2. Because $dx = \dot{x}dt$, substituting this result into eq. (3.15),

$$\frac{d}{dt}\dot{x} + \frac{du}{dx} = 0 \quad \rightarrow \quad \dot{x}\frac{d}{dx}\dot{x} + \frac{du}{dx} = 0. \tag{3.16}$$

3. Integrating eq. (3.16),

$$\frac{1}{2}\frac{d}{dx}\dot{x}^2 + \frac{du}{dx} = 0 \quad \rightarrow \quad \frac{1}{2}\dot{x}^2 + u = \text{constant}. \tag{3.17}$$

   Hence, for any conservative force with the potetial energy $U(x)$,

$$\frac{1}{2}m\dot{x}^2 + U(x) = E = \text{constant}. \tag{3.18}$$

3.3 Damped Oscillation

1. In general physics, we have learned about that the resistance force like as air is proportional to velocity of object, approximately. In this section, we use this force as damping force. Let us apply damping force into harmonic oscillator. Then equation of motion becomes

$$F = m\ddot{x} = -kx - b\dot{x} \tag{3.19}$$

   , where b is proportionality constant.($b > 0$)

2. We can rewrite above eqation of motion of the form

$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = 0 \tag{3.20}$$

   , where

$$\beta = \frac{b}{2m}, \quad w_0^2 = \frac{k}{m}. \tag{3.21}$$

3. The solution can be obtained using a trial solution $x = e^{\lambda t}$. Substituting trial solution into eq. (3.20), we obtain

$$(\lambda^2 + 2\beta\lambda + w_0^2)e^{\lambda t} = 0. \tag{3.22}$$

   Because $e^{\lambda t}$ is not always zero, we can omit $e^{\lambda t}$.

$$\lambda^2 + 2\beta\lambda + w_0^2 = 0. \tag{3.23}$$

   We call eq. (3.23) characteristic equation.

4. Solving characteristic equation, we obtatin following solution about λ.

$$\lambda = \begin{cases} -\beta \pm i\sqrt{w_0^2 - \beta^2}, & (w_0 > \beta) \\ -\beta \pm \sqrt{\beta^2 - w_0^2}, & (\beta > w_0) \\ -\beta, & (\beta = w_0). \end{cases} \tag{3.24}$$

5. Let us consider $w_0 > \beta$ case. The general solution is linear combination of two trial solutions.

$$\begin{aligned} x(t) &= c_1 e^{(-\beta + iw')t} + c_2 e^{(-\beta - iw')t} \\ &= e^{-\beta t}(c_1 e^{iw't} + c_2 e^{-iw't}) \\ &= e^{-\beta t}(c_1' \cos w't + c_2' \sin w't) \\ &= Ae^{-\beta t} \cos(w't - \delta). \end{aligned} \tag{3.25}$$

   , where $w' = \sqrt{w_0^2 - \beta^2}$. We call this case overdamping oscillation.

6. Otherwise, if $\beta > w_0$, ther general solution is

$$\begin{aligned} x(t) &= c_1 e^{(-\beta+\sqrt{\beta^2-w_0^2})t} + c_2 e^{(-\beta-\sqrt{\beta^2-w_0^2})t} \\ &= e^{-\beta t}(c_1 e^{\sqrt{\beta^2-w_0^2}t} + c_2 e^{-\sqrt{\beta^2-w_0^2}t}) \\ &= e^{-\beta t}(c_1' \cosh\sqrt{\beta^2-w_0^2}t + c_2' \sinh\sqrt{\beta^2-w_0^2}t) \end{aligned} \tag{3.26}$$

   We call this case underdamping oscillation.

7. In the last case, if $\beta = w_0$, we only have one trial solution $e^{-\beta t}$. Therefore, we need another solution. this another solution is $te^{-\beta t}$. Using Wronskian, we will check that two solutions are linear independent. The general solution is

$$x(t) = e^{-\beta t}(c_1 + c_2 t). \tag{3.27}$$

   We call this case critical damping oscillation.

3.4 Driven Oscillation

3.4.1 Definition

1. Let us consider a forced oscillation. We just apply $F(t)$ on eq. (3.19). Then the equation of motion becomes

$$F = m\ddot{x} = -kx - b\dot{x} + F(t). \tag{3.28}$$

   Then eq. (3.20) becomes

$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = f(t) \tag{3.29}$$

   , where $f(t) = F(t)/m$.

2. The solution of above equation of motion has two components.

$$x(t) = x_h(t) + x_p(t). \tag{3.30}$$

   , where $x_h(t)$ is the genoeral solution of the homogeneous equation

$$\ddot{x}_h + 2\beta\dot{x}_h + w_0^2 x_h = 0 \tag{3.31}$$

   and $x_p(t)$ is a particular solution. We can already solve $x_h(t)$.

3.4.2 Exercise

1. Let us consider a sinusoidal driving force

$$f(t) = f_0 \cos wt. \tag{3.32}$$

   Using Euler's equation, we can rewrite $\cos wt$ as $Re[e^{-iwt}]$.

$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = f(t) = f_0 \cos wt = f_0 Re[e^{-iwt}]. \tag{3.33}$$

2. Next, we deal with above equation of motion as complex funtion. Then, x becomes $z = x + iy$.

$$\ddot{z} + 2\beta\dot{z} + w_0^2 z = f_0 e^{-iwt}. \tag{3.34}$$

   The real part of eq. (3.34) is same as eq. (3.33).

3. By using the trial solution $z = ce^{\lambda t}$, we obtain

$$c(\lambda^2 + 2\beta\lambda + w_0^2)e^{\lambda t} = f_0 e^{-iwt}. \tag{3.35}$$

Therefore,

$$\lambda = -iw. \tag{3.36}$$

Substituting eq. (3.36) into eq. (3.35), we can obtain c.

$$c(-w^2 - 2\beta iw + w_0^2) = f_0 \tag{3.37a}$$

or

$$c = \frac{f_0}{w_0^2 - w^2 - i2\beta w}. \tag{3.37b}$$

We multiply eq. (3.37b) and $(w_0^2 - w^2 + i2\beta w)/(w_0^2 - w^2 + i2\beta w)$. Then,

$$c = \frac{f_0(w_0^2 - w^2 + i2\beta w)}{(w_0^2 - w^2)^2 + 4\beta^2 w^2} \tag{3.38a}$$

or

$$c = \frac{f_0 e^{i\delta}}{\sqrt{(w_0^2 - w^2)^2 + 4\beta^2 w^2}} \tag{3.38b}$$

, where

$$\delta = \tan^{-1}\frac{2\beta w}{w_0^2 - w^2}. \tag{3.38c}$$

4. Hence, the complex-valued particular solution $z(t)$ is

$$z(t) = \frac{f_0 e^{-i(wt-\delta)}}{\sqrt{(w_0^2 - w^2)^2 + 4\beta^2 w^2}}. \tag{3.39}$$

Then, by taking real part of $z(t)$, we obtain particular solution $x_p(t)$.

$$x_p(t) = \frac{f_0 \cos(wt - \delta)}{\sqrt{(w_0^2 - w^2)^2 + 4\beta^2 w^2}}. \tag{3.40}$$

5. The amplitude of the particular solution is of the form

$$A(w) = \frac{f_0}{\sqrt{(w_0^2 - w^2)^2 + 4\beta^2 w^2}}. \tag{3.41}$$

By taking derivative of $A(w)$ with respect to w, we can obtain resonance frequncy which maximizes amplitude.

$$w_R = \sqrt{w_0^2 - 2\beta^2} \tag{3.42}$$

6. The quality factor is defined by i

$$Q = \frac{w_R}{2\beta}. \tag{3.43}$$

3.5 Reponse to Impulse Forcing Functions

3.5.1 Heaviside step function

1. The Heaviside step function $\theta(x)$ is defined by

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{3.44}$$

3.5.2 Dirac delta function

1. The Diran delta function $\delta(x)$ is defined by

$$\delta(t) = \begin{cases} \infty, & \text{if } \ t = 0, \\ 0, & \text{if } \ t \neq 0. \end{cases} \tag{3.45a}$$

, with integral definition

$$\int_{-\infty}^{\infty} \delta(t) dt = 1, \tag{3.45b}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0). \tag{3.45c}$$

2. Moreover, Dirac delta function is first-order derivative of Heaviside step function

$$\delta(t) = \lim_{\triangle \to 0} \frac{1}{\triangle} \left[\theta\left(t + \frac{\triangle}{2}\right) - \theta\left(t - \frac{\triangle}{2}\right) \right] = \theta'(t) \tag{3.46}$$

3.5.3 Excercise 1

1. Let us consider following driven oscillation.

$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = a_0 \theta(t) \tag{3.47}$$

, where $x(t) = 0$ for $t \leq 0$. We consider underdamping oscillation condition.

2. For $t > 0$, above equation of motion becomes

$$\ddot{x} + 2\beta\dot{x} + w_0^2 = a_0. \tag{3.48}$$

Then the particular solution can be obatined by taking a trial solution $x_p = c$.

$$w_0^2 c = a_0 \to c = \frac{a_0}{w_0^2}. \tag{3.49}$$

From previous results, we have learned about that the solution of under damping osciilation is

$$x_h = e^{-\beta t}(c_1 \cos w't + c_2 \sin w't) \tag{3.50}$$

, where $w' = \sqrt{w_0^2 - \beta^2}$.

3. Then the general solution for $t > 0$ is

$$\begin{aligned} x(t) &= x_p(t) + x_h(t) && (3.51a) \\ &= \frac{a_0}{w_0^2} + e^{-\beta t}(c_1 \cos w't + c_2 \sin w't). && (3.51b) \end{aligned}$$

4. Let us determine c_1 and c_2. Using initial condition, we obtain

$$x(t = 0) = \frac{a_0}{w_0^2} + c_1 = 0, \tag{3.52a}$$

$$\dot{x}(t = 0) = -\beta c_1 + c_2 w' = 0. \tag{3.52b}$$

Therefore,

$$c_1 = -\frac{a_0}{w_0^2}, \quad c_2 = \frac{\beta c_1}{w'} = -\frac{\beta a_0}{w_0^2 w'}. \tag{3.53}$$

5. Substituting eq. (3.53) into eq. (3.51b), we find the solution satisfying initial conditions

$$x(t) = \frac{a_0}{w_0^2} \left[1 - e^{-\beta t} \left(\cos w't + \frac{\beta}{w'} \sin w't \right) \right]. \tag{3.54}$$

3.5.4 Exercise 2

1. Next, we consider driven oscillation substituting $\theta(t)$ into $\delta(t)$ in eq. (3.47).

$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = a_0\delta(t). \tag{3.55}$$

2. We can solve above equation of motion like as 'exercise 1'. However, In this case, we use the relation between step and delta funtion.

$$\delta(t) = \theta'(t) = \dot{\theta}(t), \tag{3.56}$$

   with the general solution of 'exercise 1'.

$$x(t) = \frac{a_0}{w_0^2}\left[1 - e^{-\beta t}\left(\cos w't + \frac{\beta}{w'}\sin w't\right)\right]. \tag{3.57}$$

3. The first-order time derivative of eq. (3.47) is

$$\dddot{x} + 2\beta\ddot{x} + w_0^2\dot{x} = a_0\dot{\theta}(t) = a_0\delta(t). \tag{3.58}$$

   Let $y = \dot{x}$. then, eq. (3.58) can be rewritten as

$$\ddot{y} + 2\beta\dot{y} + w_0^2 y = a_0\delta(t). \tag{3.59}$$

   Hence, eq. (3.59) is eqivalent to eq. (3.55).

4. Therefore, using relation $y = \dot{x}$, we can easily obtain the general soution of eq. (3.55) by differentiating eq. (3.57) with respect to t.