Fourier Spectral Method

Introduction

Spectral Method

- Spectral method is closely related to the finite difference method which we have studied.

- In the finite difference method, we use a small number of points to estimate the each value of a next step.

- For example, the heat equation is approximated by

$$u_i^{n+1} = u_i^n + \frac{a\Delta t}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Spectral Method

- From the Taylor's expansion, we notice that the order of the finite difference method in the last slide is 2nd order accuracy in space, i.e.,

$$error \sim O(N^{-2})$$

  N is the number of grid points.

- Actually, if we use more points to approximate, the order of error reduces algebraically, i.e., for α number of points stencil,

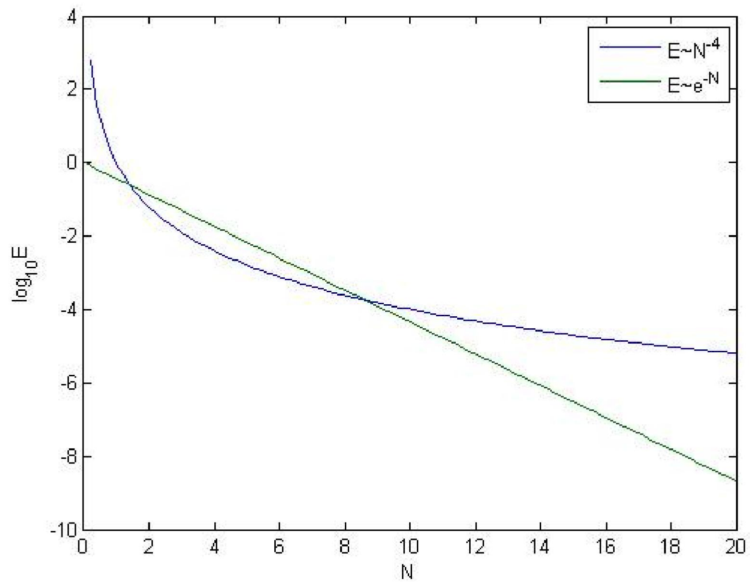
$$error \sim O(N^{-\alpha})$$

Spectral Method

- In contrast, we use all grid points for the spectral method. This is the reason the spectral method is considered as a limit of the finite difference method.

- In spectral method, the error reduces exponentially, i.e., for N number of grid points,

$$error \sim e^{-N}$$



Fourier Spectral Method

- One of the common spectral method is the Fourier spectral method which uses the discrete Fourier transformation.

- There are both even and odd cases depending on the number of grid points, however, we just focus on the even case which is more commonly used.

- At first, we need an assumption that the given functions are periodic.

Fourier Spectral Method

- Let g be a complex-valued, periodic known function in [0, L] and f be a function which we want to find so that

$$\Delta f(x) = g(x)$$

on the interval

- So, $f_m^n \approx f(m\Delta x, n\Delta t)$

Fourier Spectral Method

- Now, we apply the Fourier inverse transform to f and g

$$f_m^n = \frac{1}{M} \sum_{p=1}^{M} \hat{f}_p^n e^{i x_m \xi_p}$$

$$g_m^n = \frac{1}{M} \sum_{p=1}^{M} \hat{g}_p^n e^{i x_m \xi_p}$$

where $\xi_p = \frac{2\pi(p-1)}{L}$

- It is a kind of projection to the Fourier space.

Fourier Spectral Method

- The exponential terms are basis of the Fourier space.

- We can prove this by defining an inner product in the space such as

$$\langle a, b \rangle = \int a\bar{b}dx$$

  where a and b are functions of x.

- From the definition of an inner product, each exponential terms are orthogonal.

Fourier Spectral Method

- So,

$$\frac{\partial^2}{\partial x^2} f(x) = g(x)$$

$$\frac{\partial^2}{\partial x^2} \frac{1}{M} \sum_{p=1}^{M} \hat{f}_p^n e^{ix_m \xi_p} = \frac{1}{M} \sum_{p=1}^{M} \hat{g}_p^n e^{ix_m \xi_p}$$

Since exponential terms are orthogonal in the Fourier space,

$$\hat{f}_p^n (i\xi_p)^2 = \hat{g}_p^n$$

$$\therefore \hat{f}_p^n = -\frac{\hat{g}_p^n}{\xi_p^2}$$

Fourier Spectral Method

- To the result, we apply the inverse Fourier transform again:

$$f_m^n = \frac{1}{M}\sum_{p=1}^{M} \hat{f}_p^n e^{ix_m \xi_p}$$

$$= \frac{1}{M}\sum_{p=1}^{M} -\frac{\hat{g}_p^n}{\xi_p^2} e^{ix_m \xi_p}$$

- From the Fourier transform, we know the coefficient of g. So, we can get the value of f by plugging into above equation.

$$\hat{g}_p^n = \sum_{m=1}^{M} g_m^n e^{-ix_m \xi_p}$$