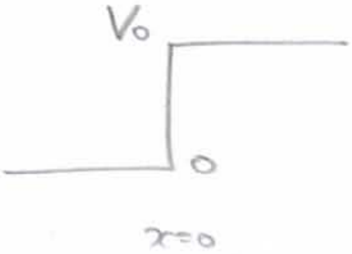
1. As claimed in class, finish the computation to obtain T, which is the coefficient of the outgoing wave to ∞ in the problem of the potential step.

⇒ i)



$$\psi(x) = \begin{cases} e^{ikx} + R e^{-ikx} & (x<0) \\ T e^{ik'x} & (x>0) \end{cases}$$

ii) Boundary conditions.

① $x=0$ 에서 $\psi(x)$ 연속

② $x=0$ 에서 $\psi'(x)$ 연속.

B.C. ① → $1 + R = T$

B.C. ② → $ik - ikR = ik' T$

$k(1-R) = k'T$

$\therefore 1-R = \frac{k'}{k} T$

합 : $2 = \left(1+\frac{k'}{k}\right) T$

$$T = \frac{2}{1+\frac{k'}{k}} = \frac{2k}{k+k'}.$$

iii) k'과 k의 관계는?

wave 가 가진 에너지를 E 라고 하자. 그러면 time-independent Schrödinger equation 에 의해..

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) & (x<0) \\ \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\psi(x) = E\psi(x) & (x>0) \end{cases}$$

$\Downarrow$

$$\therefore \frac{\hbar^2 k^2}{2m} = E \qquad \frac{\hbar^2 k'^2}{2m} + V_0 = E$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad k' = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

iv)

$$\therefore T = \frac{2}{1+\frac{k'}{k}} = \frac{2}{1+\sqrt{\frac{2m(E-V_0)}{2mE}}}$$

$$= \frac{2}{1+\sqrt{\frac{E-V_0}{E}}}$$

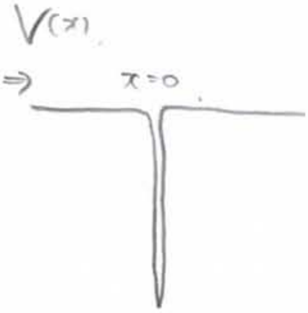
$\rightarrow$ $V_0 = 0$ 이면.. $T = 1$

2. Consider the Dirac delta potential given by.

$$V(x) = -\frac{\hbar^2 \lambda}{2ma}\delta(x),$$

where λ is a dimensionless positive constant. a is an arbitrary positive constant. When particles are incident from the left, compute the reflection and transmission coefficients.

⇒ i)



왼쪽에서 입자가 들어온다.

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & (x<0) \\ Te^{ikx} & (x>0) \end{cases}$$

이때.. $x \neq 0$ 에서 $V(x)=0$ 이므로.. $x>0$ 일때와 $x<0$ 일때 모두 wave number k는 동일하고..

$$\frac{\hbar^2 k^2}{2m} = E \rightarrow k = \sqrt{\frac{2mE}{\hbar^2}} \text{ 이다}$$

ii) Boundary conditions

B.C ① $x=0$ 에서 $\psi(x)$ 연속

B.C ② $x=0$ 에서 $\psi'(x)$ 불연속.

⇓

왜냐하면.. Schrödinger 방정식에 의해 δ function potential은 $\psi'(x)$의 불연속을 주기 때문이다.

Schrödinger eqn.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - \frac{\hbar^2\lambda}{2ma}\delta(x)\,\psi(x) = E\psi(x)$$

위 방정식을 $-\varepsilon < x < \varepsilon$ 에서 적분하자. $(\varepsilon \ll 1)$

$\Downarrow$

$$-\frac{\hbar^2}{2m}\left(\frac{d\psi(x)}{dx}\bigg|_{x=\varepsilon} - \frac{d\psi(x)}{dx}\bigg|_{x=-\varepsilon}\right)$$

$$-\frac{\hbar^2\lambda}{2ma}\psi(0) = 0.$$

$$\therefore \frac{d\psi(x)}{dx}\bigg|_{x=0^-} - \frac{d\psi(x)}{dx}\bigg|_{x=0^+} = \frac{\lambda}{a}\psi(0)$$

iii)

B.C. ① $\rightarrow$ $1+R=T$

B.C. ② $\rightarrow$ $(ik - ikR) - ikT = \frac{\lambda}{a}\cdot T$

$$1-R-T = \frac{\lambda}{ika}T$$

$$1-R = \left(1-\frac{i\lambda}{ka}\right)T$$

iv)

$$\therefore \quad 2 = \left(2 - \frac{i\lambda}{ka}\right) T$$

$$T = \frac{2}{2 - \frac{i\lambda}{ka}} . \qquad (\lambda = 0 \text{ 이면 } T = 1)$$

$$R = T - 1 = \frac{2}{2 - \frac{i\lambda}{ka}} - 1$$

$$= \frac{2 - 2 + \frac{i\lambda}{ka}}{2 - \frac{i\lambda}{ka}} = \frac{\frac{i\lambda}{ka}}{2 - \frac{i\lambda}{ka}}$$

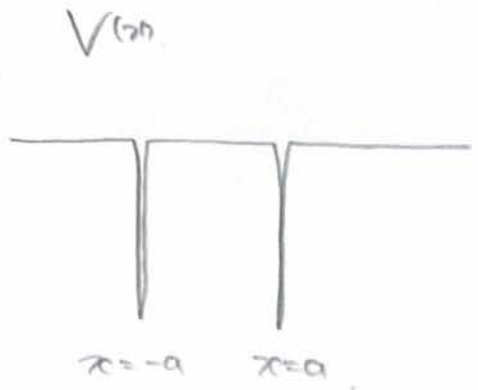
$$\left[\begin{aligned} |T|^2 &= \frac{2}{2 - \frac{i\lambda}{ka}} \cdot \frac{2}{2 + \frac{i\lambda}{ka}} = \frac{4}{4 + \left(\frac{\lambda}{ka}\right)^2} \\ |R|^2 &= \frac{\left(\frac{\lambda}{ka}\right)^2}{4 + \left(\frac{\lambda}{ka}\right)^2} \end{aligned}\right.$$

3. Consider the two Dirac delta potentials given by.

$$V(x) = -\frac{\hbar^2 \lambda}{2ma}\left[\delta(x-a) + \delta(x+a)\right],$$

where λ and a are described in the previous problem. Here, also, compute the reflection and transmission coefficients.

$\Rightarrow$. i).



$$\Rightarrow \psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & (x<-a) \\ Ae^{ikx} + Be^{-ikx} & (-a<x<a) \\ Te^{ikx} & (x>a) \end{cases}$$

ii). B.C. ① $x=-a,\ a$ 에서 $\psi(x)$ 연속

B.C ② $x=-a,\ a$ 에서 $\psi'(x)$ 불연속.

iii) B.C. ①

$$\begin{cases} e^{ik(-a)} + Re^{-ik(-a)} = Ae^{ik(-a)} + Be^{-ik(-a)} \\ Ae^{ika} + Be^{-ika} = Te^{ika} \end{cases}$$

정리

$$\begin{cases} 1 + Re^{2ika} = A + Be^{+2ika} \\ A + Be^{-2ika} = T \end{cases}$$

B.C. ②

$$\begin{cases} \left.\dfrac{d\psi(x)}{dx}\right|_{x=-a^-} - \left.\dfrac{d\psi(x)}{dx}\right|_{x=-a^+} = \dfrac{\lambda}{a}\psi(-a) \\ \left.\dfrac{d\psi(x)}{dx}\right|_{x=a^-} - \left.\dfrac{d\psi(x)}{dx}\right|_{x=a^+} = \dfrac{\lambda}{a}\psi(a) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} ik\left(e^{ik(-a)} - Re^{-ik(-a)}\right) - ik\left(Ae^{ik(-a)} - Be^{-ik(-a)}\right) \\ \quad = \dfrac{\lambda}{a}\cdot\left(e^{ik(-a)} + Re^{-ik(-a)}\right) \\ ik\left(Ae^{ika} - Be^{-ika}\right) - ikTe^{ika} \\ \quad = \dfrac{\lambda}{a}\cdot\left(Te^{ika}\right) \end{cases}$$

정리

$$\begin{cases} 1 - Re^{+2ika} - A + Be^{2ika} = \dfrac{\lambda}{ika}\left(1 + Re^{2ika}\right) \\ A - Be^{-2ika} - T = \dfrac{\lambda}{ika}T \end{cases}$$

위) 4개의 식

① $1+Re^{2ika} = A+Be^{2ika}$

② $A+Be^{-2ika} = T$

③ $1-Re^{+2ika} - A+Be^{2ika} = \frac{\lambda}{ika}\left(1+Re^{2ika}\right)$

④ $A-Be^{-2ika} = \left(1+\frac{\lambda}{ika}\right)T$

②+④

$$2A = \left(2+\frac{\lambda}{ika}\right)T$$

$$A = \left(1+\frac{\lambda}{2ika}\right)T \quad \cdots ⑤$$

②−④

$$2Be^{-2ika} = -\frac{\lambda}{ika}T$$

$$B = -\frac{\lambda}{2ika}\cdot e^{2ika}\cdot T \quad \cdots ⑥$$

$$\xrightarrow{① + ③} \quad 2 - 2A = \frac{\lambda}{ika}\left(1 + R e^{2ika}\right).$$

$$\xrightarrow{① - ③} \quad 2R e^{2ika} - 2B e^{2ika} = -\frac{\lambda}{ika}\left(1 + R e^{2ika}\right)$$

$$\Downarrow$$

$$\begin{cases} 1 - A = \dfrac{\lambda}{2ika}\left(1 + R e^{2ika}\right) \\[2ex] R \cdot e^{2ika} + \dfrac{\lambda}{2ika} \cdot R e^{2ika} = -\dfrac{\lambda}{2ika} + B e^{2ika}. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{\lambda}{2ika} \cdot R e^{2ika} = 1 - \dfrac{\lambda}{2ika} - A. & \cdots ⑦ \\[2ex] \left(1 + \dfrac{\lambda}{2ika}\right) R e^{2ika} = -\dfrac{\lambda}{2ika} + B e^{2ika} & \cdots ⑧. \end{cases}$$

⑤를 ⑦에 대입

$$\frac{\lambda}{2ika} R e^{2ika} = 1 - \frac{\lambda}{2ika} - \left(1 + \frac{\lambda}{2ika}\right) T$$

⑥을 ⑧에 대입

$$\left(1 + \frac{\lambda}{2ika}\right) R e^{2ika} = -\frac{\lambda}{2ika} + \left(-\frac{\lambda}{2ika} e^{2ika} T\right) \times e^{2ika}$$

$$= -\frac{\lambda}{2ika}\left[1 + T e^{4ika}\right]$$

$\Downarrow$

$$\begin{cases} \dfrac{\lambda}{2ika} e^{2ika} \cdot R + \left(1 + \dfrac{\lambda}{2ika}\right) T = 1 - \dfrac{\lambda}{2ika} \\ \left(1 + \dfrac{\lambda}{2ika}\right) e^{2ika} R + \dfrac{\lambda}{2ika} e^{4ika} T = -\dfrac{\lambda}{2ika} \end{cases}$$

$$\begin{pmatrix} \dfrac{\lambda}{2ika} e^{2ika} & 1 + \dfrac{\lambda}{2ika} \\ \left(1 + \dfrac{\lambda}{2ika}\right) e^{2ika} & \dfrac{\lambda}{2ika} e^{4ika} \end{pmatrix} \begin{pmatrix} R \\ T \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\lambda}{2ika} \\ -\dfrac{\lambda}{2ika} \end{pmatrix}$$

$$\therefore \begin{pmatrix} R \\ T \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{2ika} e^{2ika} & 1 + \frac{\lambda}{2ika} \\ \left(1 + \frac{\lambda}{2ika}\right) e^{2ika} & \frac{\lambda}{2ika} e^{4ika} \end{pmatrix}^{-1} \begin{pmatrix} 1 - \frac{\lambda}{2ika} \\ -\frac{\lambda}{2ika} \end{pmatrix}.$$

$\frac{\lambda}{2ika} = \Gamma$ 라 하자.

$$\therefore \begin{pmatrix} R \\ T \end{pmatrix} = \begin{pmatrix} \Gamma e^{2ika} & 1 + \Gamma \\ (1+\Gamma) e^{2ika} & \Gamma e^{4ika} \end{pmatrix}^{-1} \begin{pmatrix} 1 - \Gamma \\ -\Gamma \end{pmatrix}$$

$$= \frac{1}{\Gamma^2 e^{6ika} - (1+\Gamma)^2 e^{2ika}} \begin{pmatrix} \Gamma e^{4ika} & -(1+\Gamma) \\ -(1+\Gamma) e^{2ika} & \Gamma e^{2ika} \end{pmatrix} \times \begin{pmatrix} 1 - \Gamma \\ -\Gamma \end{pmatrix}$$

$$= \frac{e^{-4ika}}{\Gamma^2 e^{2ika} - (1+\Gamma)^2 e^{-2ika}} \cdot \begin{pmatrix} \Gamma(1-\Gamma) e^{4ika} + \Gamma(1+\Gamma) \\ -(1+\Gamma)(1-\Gamma) e^{2ika} - \Gamma^2 e^{2ika} \end{pmatrix}$$

$$\therefore R = \frac{e^{-4ika}\,\Gamma\left[(1-\Gamma)e^{4ika} + (1+\Gamma)\right]}{\Gamma^2 e^{2ika} - (1+\Gamma)^2 e^{-2ika}}$$

$$T = \frac{e^{-4ika}\cdot e^{2ika}\cdot(-1)}{\Gamma^2 e^{2ika} - (1+\Gamma)^2\cdot e^{-2ika}}$$

$$= -\frac{e^{-2ika}}{\Gamma^2 e^{2ika} - (1+\Gamma)^2 e^{-2ika}}$$

$$R = \frac{e^{-2ika}\,\Gamma\left[(1-\Gamma)e^{2ika} + (1+\Gamma)e^{-2ika}\right]}{\Gamma^2 e^{2ika} - (1+\Gamma)^2 e^{-2ika}}$$

v) 정리.

$$\begin{cases} T = -\dfrac{e^{-2ika}}{\Gamma^2 e^{2ika} - (1+\Gamma)^2 e^{-2ika}} \\[2ex] R = \dfrac{e^{-2ika}\cdot\Gamma\left[(1-\Gamma)e^{2ika} + (1+\Gamma)e^{-2ika}\right]}{\Gamma^2 e^{2ika} - (1+\Gamma)^2 e^{-2ika}} \end{cases}$$

where $\Gamma = \dfrac{\lambda}{2ika}$

iii) $\therefore \langle p \rangle = \hbar k \langle \psi | \psi \rangle + \frac{\hbar}{2i} u^2(x) \Big]_{\text{boundary}}$.

만약.. $|\psi\rangle$ 가 normalized 되어 있다면..

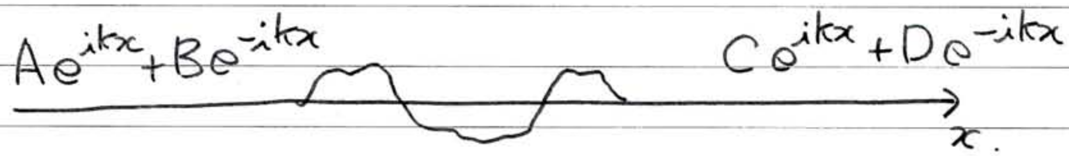
$\langle \psi | \psi \rangle = 1$ 로 들 수 있고.. 그럴 경우

$u^2(x) \longrightarrow 0$ at boundary.

$\therefore \langle p \rangle = \hbar k$

2. Gasiorowicz Problem 1. in Ch. 4.

Consider an arbitrary potential localized on a finite part of the x-axis. The solutions of the Schrödinger equation to the left and to the right of the potential region are given by..



respectively. Show that.. if we write

$$C = S_{11} A + S_{12} D$$

$$B = S_{21} A + S_{22} D$$

that is, relate the "outgoing" waves to the "ingoing" waves by..

$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

then the following relations hold

$$|S_{11}|^2 + |S_{21}|^2 = 1.$$

$$|S_{12}|^2 + |S_{22}|^2 = 1.$$

$$S_{11} S_{12}^* + S_{21} S_{22}^* = 0.$$

Use this to show that the matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

and its transpose are unitary.

⇒ i) 위치에 대한 potential 함수를 $V(x)$ 라 하자.

∴ Schrödinger equation..

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = i\hbar\frac{\partial\psi(x)}{\partial t}.$$

$\Downarrow$

이때..

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (\text{potential 왼쪽}) \\ Ce^{ikx} + De^{-ikx} & (\text{potential 오른쪽}) \end{cases}$$

ii) Flux.

$$j = \frac{\hbar}{2im}\left(\psi^*\frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right).$$

iii) time independent potential에 관한 문제 이므로.. steady state 일것!

즉.. $|j_{in}| = |j_{out}|$

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$$Ae^{ikx}$$

$$j_{in}\binom{\text{from}}{(\text{left})} = \frac{\hbar}{2im}\left[A^* \cancel{e^{-ikx}} \cdot A ik \cdot \cancel{e^{ikx}} + ikA^* \cancel{e^{-ikx}} \cdot A \cancel{e^{ikx}}\right]$$

$$= \frac{\hbar}{2im} \cdot 2ik\,|A|^2 = \frac{\hbar k}{m}|A|^2.$$

$$j_{in}\,{}^{\text{from}}(\text{right}) = \frac{\hbar}{2im}(-2ik)|D|^2 = -\frac{\hbar k}{m}|D|^2.$$

$$De^{-ikx}$$

$$j_{out}\binom{\text{to}}{\text{left}} = -\frac{\hbar k}{m}|B|^2$$

$$Be^{-ikx}$$

$$j_{out}\binom{\text{to}}{\text{right}} = \frac{\hbar k}{m}|C|^2.$$

$$Ce^{ikx}$$

Steady State

$$\therefore \frac{\hbar k}{m}\left(|A|^2+|D|^2\right) = \frac{\hbar k}{m}\left(|B|^2+|C|^2\right)$$

$$\downarrow$$

$$|A|^2+|D|^2 = |B|^2+|C|^2$$

iv) $C = S_{11}A + S_{12}D$

$\Downarrow$

$$|C|^2 = (S_{11}A + S_{12}D)(S_{11}^* A^* + S_{12}^* D^*)$$

$$= |S_{11}|^2 |A|^2 + (S_{11}A S_{12}^* D^* + S_{12}D S_{11}^* A^*)$$

$$+ |S_{12}|^2 |D|^2$$

$$|B|^2 = (S_{21}A + S_{22}D)(S_{21}^* A^* + S_{22}^* D^*)$$

$$= |S_{21}|^2 |A|^2 + (S_{21}A S_{22}^* D^* + S_{22}D S_{21}^* A^*)$$

$$+ |S_{22}|^2 |D|^2$$

v) $\therefore$
$$|A|^2 + |D|^2 = (|S_{11}|^2 + |S_{21}|^2)|A|^2$$
$$+ (|S_{12}|^2 + |S_{22}|^2)|D|^2$$
$$+ (S_{11}S_{12}^* + S_{21}S_{22}^*) AD^*$$
$$+ (S_{11}^* S_{12} + S_{21}^* S_{22}) A^* D$$

항등식 이므로..

$$\left(\begin{array}{l} |S_{11}|^2 + |S_{21}|^2 = 1 \\ |S_{12}|^2 + |S_{22}|^2 = 1 \\ S_{11}S_{12}^* + S_{21}S_{22}^* = 0 \end{array}\right)$$

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③ (ii) $$S^\dagger S = \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

$$= \begin{pmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}^* S_{12} + S_{21}^* S_{22} \\ S_{12}^* S_{11} + S_{22}^* S_{21} & |S_{12}|^2 + |S_{22}|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$\therefore$ unitary !

(iii) $$(S^T)^\dagger S^T = S^* S^T$$

$$= \begin{pmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{pmatrix} \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}$$

$$= \begin{pmatrix} |S_{11}|^2 + |S_{12}|^2 & S_{11}^* S_{21} + S_{12}^* S_{22} \\ S_{21}^* S_{11} + S_{22}^* S_{12} & |S_{21}|^2 + |S_{22}|^2 \end{pmatrix}$$

(iiii) S가 unitary 라는 사실을 이용하자.

$\therefore$ $\det S = e^{i\delta}$.

$\therefore$ $S_{11} S_{22} - S_{12} S_{21} = e^{i\delta}$.

양변에 S_{12}^* 곱.

$$\therefore \underbrace{S_{11} S_{12}^*}_{= -S_{21} S_{22}^*} S_{22} - |S_{12}|^2 S_{21} = e^{i\delta} S_{12}$$

$$\therefore\ -S_{21}\big(\underbrace{|S_{22}|^2 + |S_{12}|^2}_{=1}\big) = e^{i\delta} S_{12}$$

$$\therefore\ -S_{21} = e^{i\delta} S_{12} \rightarrow \underline{S_{21} = -e^{i\delta} S_{12}}$$

ix)

- $|S_{11}|^2 + |S_{12}|^2 = |S_{11}|^2 + |S_{21}|^2 = 1.$
- $|S_{21}|^2 + |S_{22}|^2 = |S_{12}|^2 + |S_{22}|^2 = 1.$
- $S_{11}^* S_{21} + S_{12}^* S_{22} = S_{11}^*(-e^{i\delta} S_{12}) + (-e^{i\delta} S_{21}^*) S_{22} = e^{-i\delta}(S_{11}^* S_{12} + S_{21}^* S_{22}) = 0.$
- $S_{21}^* S_{11} + S_{22}^* S_{12} = (S_{11}^* S_{21} + S_{12}^* S_{22})^* = 0.$

$$\therefore\ (S^T)^\dagger S^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$\therefore$ $\underline{S^T \text{ is unitary.}}$

3. Gasiorowicz Problem 2 in Ch. 4.

Calculate the elements of the scattering matrix, S_{11}, S_{12}, S_{21}, and S_{22} for the potential

$$V(x) = 0 \qquad x<-a$$
$$= V_0 \qquad -a<x<a$$
$$= 0 \qquad x<a.$$

and show that the general conditions proved in Problem 1 are indeed satisfied.

$\Rightarrow$ i) $k^2 = \frac{2mE}{\hbar^2}$, $q^2 = \frac{2m(E-V_0)}{\hbar^2}$

The solution.

$$u(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<-a \\ Ee^{iqx} + Fe^{-iqx} & |x|<a \\ Ce^{ikx} + De^{-ikx} & x>a \end{cases}$$

ii) $x=a$ 와 $x=-a$ 에서.. $u(x)$ 와 $\frac{du}{dx}$ 연속.

$x=-a$

$$Ae^{-ika} + Be^{ika} = Ee^{-iqa} + Fe^{iqa} \qquad ①$$
$$ik(Ae^{-ika} - Be^{ika}) = iq(Ee^{-iqa} - Fe^{iqa}) \qquad ②$$

$$Ce^{ika} + De^{-ika} = Ee^{iqa} + Fe^{-iqa} \qquad ③$$
$$ik(Ce^{ika} - De^{-ika}) = iq(Ee^{iqa} - Fe^{-iqa}) \qquad ④$$

$x=a$

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① $Ae^{-ika} + Be^{ika} = Ee^{-iga} + Fe^{iga}$

② $\frac{k}{g}(Ae^{-ika} - Be^{ika}) = Ee^{-iga} - Fe^{iga}$

- $2Ee^{-iga} = A\left(1+\frac{k}{g}\right)e^{-ika} + B\left(1-\frac{k}{g}\right)e^{ika}$ ⋯ Ⓐ

- $2Fe^{iga} = A\left(1-\frac{k}{g}\right)e^{-ika} + B\left(1+\frac{k}{g}\right)e^{ika}$ ⋯ Ⓑ

③ $Ce^{ika} + De^{-ika} = Ee^{iga} + Fe^{-iga}$

④ $\frac{k}{g}(Ce^{ika} - De^{-ika}) = Ee^{iga} - Fe^{-iga}$.

- $2Ee^{iga} = \left(1+\frac{k}{g}\right)Ce^{ika} + \left(1-\frac{k}{g}\right)De^{-ika}$ ⋯ Ⓒ

- $2Fe^{-iga} = \left(1-\frac{k}{g}\right)Ce^{ika} + \left(1+\frac{k}{g}\right)De^{-ika}$ ⋯ Ⓓ

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Ⓐ $\times \left(1+\frac{k}{q}\right)$ − Ⓑ $\times \left(1-\frac{k}{q}\right)$

$\Downarrow$

$$2E \cdot e^{-iqa}\left(1+\frac{k}{q}\right) - 2F \cdot e^{iqa}\left(1-\frac{k}{q}\right)$$

$$= A\left(1+\frac{k}{q}\right)^2 e^{-ika} - A\left(1-\frac{k}{q}\right)^2 e^{-ika}$$

$$= e^{-2iqa}\left(1+\frac{k}{q}\right)\cdot\left[\left(1+\frac{k}{q}\right)C\cdot e^{ika} + \left(1-\frac{k}{q}\right)D\cdot e^{-ika}\right]$$

$$- e^{2iqa}\left(1-\frac{k}{q}\right)\cdot\left[\left(1-\frac{k}{q}\right)Ce^{ika} + \left(1+\frac{k}{q}\right)De^{-ika}\right]$$

$$Ce^{ika}\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]$$

$$= Ae^{-ika}\left[\left(1+\frac{k}{q}\right)^2 - \left(1-\frac{k}{q}\right)^2\right]$$

$$+ De^{-ika}\left[\left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)e^{2iqa} - \left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)e^{-2iqa}\right]$$

$$C = A\cdot\frac{e^{-2ika}\left[\left(1+\frac{k}{q}\right)^2 + \left(1-\frac{k}{q}\right)^2\right]}{\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]}$$

$$+ D\cdot\frac{e^{-2ika}\left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)\left[e^{2iqa} - e^{-2iqa}\right]}{\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]}$$

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$$Ⓒ \times \left(1-\frac{k}{q}\right) - Ⓓ \times \left(1+\frac{k}{q}\right)$$

$$\Downarrow$$

$$2E e^{iqa}\left(1-\frac{k}{q}\right) - 2F e^{-iqa}\left(1+\frac{k}{q}\right)$$

$$= \left(1-\frac{k}{q}\right)^2 D e^{-ika} - \left(1+\frac{k}{q}\right)^2 D e^{-ika}$$

$$= e^{2iqa}\left(1-\frac{k}{q}\right)\left[A\left(1+\frac{k}{q}\right)e^{-ika} + B\left(1-\frac{k}{q}\right)e^{ika}\right]$$

$$- e^{-2iqa}\left(1+\frac{k}{q}\right)\left[A\left(1-\frac{k}{q}\right)e^{-ika} + B\left(1+\frac{k}{q}\right)e^{ika}\right]$$

$$B \cdot e^{ika}\left[\left(1-\frac{k}{q}\right)^2 e^{2iqa} - \left(1+\frac{k}{q}\right)^2 e^{-2iqa}\right]$$

$$= A e^{-ika}\left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)\left[e^{-2iqa} - e^{2iqa}\right]$$

$$+ D e^{-ika}\left[\left(1-\frac{k}{q}\right)^2 - \left(1+\frac{k}{q}\right)^2\right]$$

$$B = A \cdot \frac{e^{-2ika}\left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)\left[e^{-2iqa} - e^{2iqa}\right]}{\left[\left(1-\frac{k}{q}\right)^2 e^{2iqa} - \left(1+\frac{k}{q}\right)^2 e^{-2iqa}\right]}$$

$$+ D \cdot \frac{e^{-2ika}\left[\left(1-\frac{k}{q}\right)^2 - \left(1+\frac{k}{q}\right)^2\right]}{\left[\left(1-\frac{k}{q}\right)^2 e^{2iqa} - \left(1+\frac{k}{q}\right)^2 e^{-2iqa}\right]}$$

$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} S_{11}(k) & S_{12}(k) \\ S_{21}(k) & S_{22}(k) \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

$$S_{11}(k) = \frac{e^{-2ika}\left[\left(1+\frac{k}{q}\right)^2 - \left(1-\frac{k}{q}\right)^2\right]}{\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]}$$

$$S_{12}(k) = \frac{e^{-2ika}\left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)\left[e^{2iqa} - e^{-2iqa}\right]}{\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]}$$

$$S_{21}(k) = + \frac{e^{-2ika}\left(1+\frac{k}{q}\right)\left(1-\frac{k}{q}\right)\left[e^{2iqa} - e^{-2iqa}\right]}{\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]}$$

$$S_{22}(k) = \frac{e^{-2ika}\left[\left(1+\frac{k}{q}\right)^2 - \left(1-\frac{k}{q}\right)^2\right]}{\left[\left(1+\frac{k}{q}\right)^2 e^{-2iqa} - \left(1-\frac{k}{q}\right)^2 e^{2iqa}\right]}$$

mathematica를 통해.. 위 matrix가
unitary 임을 알수 있다

NO.
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5. Gasiorowicz Problem 6. in Ch. 4.

Consider the scattering matrix for the potential

$$\frac{2m}{\hbar^2} V(x) = \frac{\lambda}{a}\delta(x-b) \longrightarrow \boxed{V(x) = \frac{\hbar^2 \lambda}{2ma}\delta(x-b)}$$

Show that it has the form

$$\begin{pmatrix} \frac{2ika}{2ika-\lambda} & \frac{\lambda}{2ika-\lambda}e^{-2ikb} \\ \frac{\lambda}{2ika-\lambda}e^{2ikb} & \frac{2ika}{2ika-\lambda} \end{pmatrix}$$

Prove that it is unitary, and that it will yield the condition for bound states when the elements of that matrix become infinite. (This will only occur for $\lambda<0$).

$\Rightarrow$ i)

$$u(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<b \\ Ce^{ikx} + De^{-ikx} & x>b \end{cases}$$

ii) $u(x)$는 $x=b$ 에서 연속.

$\frac{du}{dx}$ 는 $x=b$ 에서 불연속.

$$\left.\frac{du}{dx}\right|_{b+\varepsilon} - \left.\frac{du}{dx}\right|_{b-\varepsilon} = \frac{\hbar^2}{2ma}\cdot\frac{2m}{\hbar^2}\cdot\lambda u(b)$$

$$= \frac{\lambda}{a}u(b).$$

999) $u(b) = Ae^{ikb} + Be^{-ikb} = Ce^{ikb} + De^{-ikb}$ ··· ㄱ

$$\frac{du}{dx}\Big|_{x=b+\varepsilon} - \frac{du}{dx}\Big|_{x=b-\varepsilon} = ik\left(Ce^{ikb} - De^{-ikb}\right) - ik\left(Ae^{ikb} - Be^{-ikb}\right)$$

$\therefore$ $\cancel{Ae^{ikb} - Be^{-ikb} = Ce^{ikb} - De^{-ikb}}$ ~~··· ㄴ~~

~~ㄱ + ㄴ~~ $= \frac{\lambda}{a}\cdot\left(Ae^{ikb} + Be^{-ikb}\right)$

$\cancel{2Ae^{ikb} =}$

$$Ae^{ikb}\left(\frac{\lambda}{a} + ik\right) + Be^{-ikb}\left(\frac{\lambda}{a} - ik\right)$$ ··· ㄴ

$$= ikCe^{ikb} - ikDe^{-ikb}$$

ㄱ $\times \left(\frac{\lambda}{a} - ik\right)$ − ㄴ

$$= Ae^{ikb}\left[\cancel{\frac{\lambda}{a}} - ik - \cancel{\frac{\lambda}{a}} - ik\right] = Ce^{ikb}\left[\frac{\lambda}{a} - ik - ik\right] + De^{-ikb}\left[\frac{\lambda}{a} - \cancel{ik} + \cancel{ik}\right]$$

$$\therefore C = A\cdot\frac{-2ik}{\frac{\lambda}{a} - 2ik} + D\cdot\frac{e^{-2ikb}\left(-\frac{\lambda}{a}\right)}{\frac{\lambda}{a} - 2ik}$$

$$= A\cdot\frac{2ik}{2ik - \frac{\lambda}{a}} + D\cdot\frac{e^{-2ikb}\,\frac{\lambda}{a}}{2ik - \frac{\lambda}{a}}$$

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㉠ $\times ik$ − ㉡

$$\Rightarrow \; Aik\, e^{ikb} + Bik\, e^{-ikb}$$

$$- Ae^{ikb}\left(\frac{\lambda}{a} + ik\right) - Be^{-ikb}\left(\frac{\lambda}{a} - ik\right)$$

$$= De^{-ikb}\, ik + ik\, De^{-ikb}$$

$$Be^{-ikb}\left(2ik - \frac{\lambda}{a}\right) = Ae^{ikb}\left(\frac{\lambda}{a}\right)$$

$$+ De^{-ikb}\, 2ik.$$

$$\therefore B = A \cdot \frac{e^{2ikb}\,\frac{\lambda}{a}}{2ik - \frac{\lambda}{a}} + D \cdot \frac{2ik}{2ik - \frac{\lambda}{a}}$$

$$\therefore \begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} \dfrac{2ika}{2ika - \lambda} & \dfrac{\lambda}{2ika - \lambda} e^{-2ikb} \\ \dfrac{\lambda}{2ika - \lambda} e^{2ikb} & \dfrac{2ika}{2ika - \lambda} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}.$$

Unitary?

NO.
DATE

$2ika = \lambda$ 일때.. scattering matrix 존재 X.

즉.. 이때 bound state!

$ik = \frac{\lambda}{2a} < 0$ $\boxed{\text{Eq. } (4-69)}$ //

Quantum Mechanics 1

Assignment 6

Due: May 9 (Thursday), 2013

1. If $|n\rangle$ is the nth harmonic oscillator eigenstate, evaluate:

   (a) $\langle n|a^{\dagger s}|n\rangle$, $\langle n|a^s|n\rangle$

   (b) $\langle n|x|n\rangle$, $\langle n|x^2|n\rangle$, $\langle n|x^4|n\rangle$

   (c) $\langle n|p|n\rangle$, $\langle n|p^2|n\rangle$, $\langle n|p^4|n\rangle$

   (d) $\langle m|a^{\dagger s}|n\rangle$, $\langle m|a^s|n\rangle$

   (e) $\langle m|x|n\rangle$, $\langle m|x^2|n\rangle$

   (f) $\langle m|p|n\rangle$, $\langle m|p^2|n\rangle$.

   Hints: (1) Work in the creation space representation and use the known orthonormality of the harmonic oscillator states. (2) Express x and p in terms of a and $a^\dagger$.

   Remarks: This problem is not hard if you know and understand what you are doing. By brute force methods, it's a mess!

2. Coherent states

   As shown in class, only the ground state of the harmonic oscillator has the minimum uncertainty $\Delta x \Delta p = \hbar/2$. However, we can construct the minimum uncertainty wave functions in the following way. That state is called the "coherent state" and it is defined as

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{1}$$

   that is, it is an eigenstate of an annihilation operator. Since a is not hermitian, its eigenvalue α is in general complex.

   (a) Compute $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$ in the state $|\alpha\rangle$, and show that $\Delta x \Delta p = \hbar/2$.

   (b) Show that the state $|\alpha\rangle$ can be written in the form

$$|\alpha\rangle = Ce^{\alpha a^\dagger}|0\rangle. \tag{2}$$

Hint: Recall the definition of the exponential of the operator given in class.

(c) Prove that if $f(a^\dagger)$ is any polynomial in $a^\dagger$, then

$$af(a^\dagger)|0\rangle = \frac{df(a^\dagger)}{da^\dagger}|0\rangle. \tag{3}$$

Using this fact, compute C.

(d) On the other hand, since the set of the energy eigenstates $\{|n\rangle\}$ forms a complete set, the state $|\alpha\rangle$ can be expanded as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{4}$$

Show that the coefficients c_n are given by

$$c_n = \frac{\alpha^n}{\sqrt{n!}}c_0. \tag{5}$$

(e) By normalizing $|\alpha\rangle$, show that $c_0 = \exp(-|\alpha|^2/2)$.

(f) From parts (d) and (e), you can find the probability for the state $|\alpha\rangle$ to contain n quanta. Find it, and it is called the Poisson distribution.

(g) Finally, compute the average number of quanta in the coherent state. That is, compute $\langle\alpha|a^\dagger a|\alpha\rangle$.

3. The Hamiltonian of a particle can be expressed in the form

$$H = \epsilon_1 a^\dagger a + \epsilon_2(a + a^\dagger), \quad [a, a^\dagger] = 1, \tag{6}$$

where ϵ_1 and ϵ_2 are constants.

(a) Find the energies of the eigenstates. (You are not required to find the corresponding state functions.)

(b) The same exept that the commutator of a and $a^\dagger$ is $[a, a^\dagger] = q^2$, where q is a pure number.

(Hint: Keeping the harmonic oscillator in mind, introduce new annihi9lation and creation operators b and $b^\dagger$ by writing

$$b = \alpha a + \beta, \quad b^\dagger = \alpha a^\dagger + \beta, \tag{7}$$

and choose the constants α and β wisely.