Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #25

2012. 11. 26

School of Electrical Engineering

Korea University

Prof. Young-Chai Ko

Outline

- PSK vs. QAM
- Frequency-modulated digital signals (Frequency shift keying: FSK)

SER of M-QAM

$$P_M = 1 - (1 - P_{\sqrt{M}})^2 = 1 - (1 - 2P_{\sqrt{M}} + P_{\sqrt{M}}^2)$$

$$\le 2P_{\sqrt{M}} \quad = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{\mathcal{E}_{av}}{N_0}}\right)$$

$$\le 4Q\left(\sqrt{\frac{3}{M-1}\frac{\mathcal{E}_{av}}{N_0}}\right)$$

Comparison with M-PSK

$$P_M \approx 2Q\left(\sqrt{2\rho_s}\sin\frac{\pi}{M}\right)$$

- Define the ratio of the arguments of Q function for the two signal format:

$$\mathcal{R}_M = \frac{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}{2\rho_s \sin^2 \frac{\pi}{M}} = \frac{3/(M-1)}{2\sin^2 \frac{\pi}{M}}$$

- M=4,

$$\mathcal{R}_4 = 1$$

  - which means the SER performances of QAM and PSK are the same.

- M>4,

$$\mathcal{R}_M > 1$$

  - which means the SER of QAM is better than the one of PSK.

Advantage of M-ary QAM over M-ary PSK

M	$10\log_{10}\mathcal{R}_M$
8	1.65
16	4.20
32	7.02
64	9.95

Frequency Shift Keying (FSK)

- Signal waveform of binary FSK

$$u_0(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos 2\pi f_0 t, \qquad 0 \le t \le T_b$$

$$u_1(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos 2\pi f_1 t, \qquad 0 \le t \le T_b$$

$\mathcal{E}_b$: signal energy/bit

T_b : duration of the bit interval

  - Frequency separation

$$f_1 = f_0 + \Delta f$$

- Signal waveforms of M-ary FSK

$$u_m(t) = \sqrt{\frac{2\mathcal{E}_s}{T}} \cos(2\pi f_c t + 2\pi m \Delta f t), \qquad m = 0, 1, \ldots, M-1, \quad 0 \le t \le T$$

where $\mathcal{E}_s = k\mathcal{E}_b$ is the energy per symbol
$T = kT_b$ is the symbol interval,
Δf is the frequency separation between frequencies, i.e., $\Delta f = f_m - f_{m-1}$
where $f_m = f_c + m\Delta f$

- Energy
  - M FSK waveforms have equal energy $\mathcal{E}_s$

Frequency separation

- Frequency separation Δf determines the degree to which we can discriminate among the M possible transmitted signals.

- Define the *correlation coefficients* as a measure of the similarity (or dissimilarity) between a pair of signal waveforms:

$$\gamma_{mn} = \frac{1}{\mathcal{E}_s}\int_0^T u_m(t)u_n(t)\,dt$$

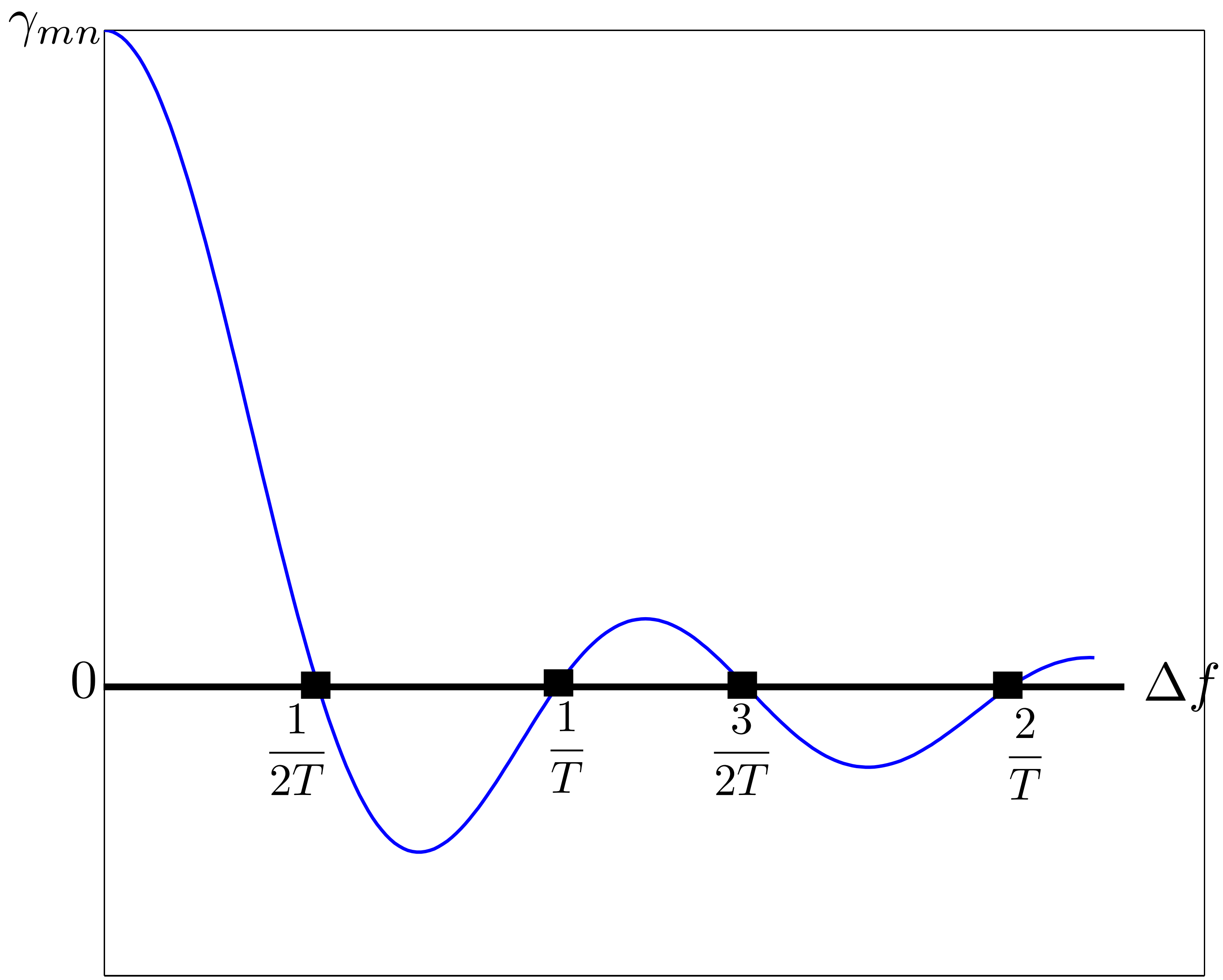
$$= \frac{1}{\mathcal{E}_s}\int_0^T \frac{2\mathcal{E}_s}{T}\cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi n\Delta f t)\,dt$$

$$= \frac{1}{T}\int_0^T \cos(2\pi(m-n)\Delta f t)\,dt + \int_0^T \cos[4\pi f_c t + 2\pi(m+n)\Delta f t]\,dt$$

=0

$$= \frac{\sin 2\pi(m-n)\Delta f T}{2\pi(m-n)\Delta f T}$$

- Correlation coefficient versus Δf



- Note that the signal waveforms are orthogonal when Δf is a multiple of $1/2T$.
- Hence, the minimum frequency separation between successive frequencies for orthogonality is $1/2T$.

Geometric representation of M-ary orthogonal FSK waveforms

$$\mathbf{s}_0 = (\sqrt{\mathcal{E}_s},\ 0,\ 0,\ \ldots,\ 0)$$

$$\mathbf{s}_1 = (0,\ \sqrt{\mathcal{E}_s},\ 0,\ \ldots,\ 0)$$

$$\vdots$$

$$\mathbf{s}_{M-1} = (0,\ ,0,\ \ldots,\ 0,\ \sqrt{\mathcal{E}_s})$$

⇒ $\text{dimensionality}: \ M$

- Orthonormal basis function

$$\psi_m(t) = \sqrt{2/T}\cos 2\pi(f_c + m\Delta f)t$$

- Distance between the pairs of signal vectors

$$d = \sqrt{2\mathcal{E}_s} \quad \text{for all } m, n$$ which is also the minimum distance.

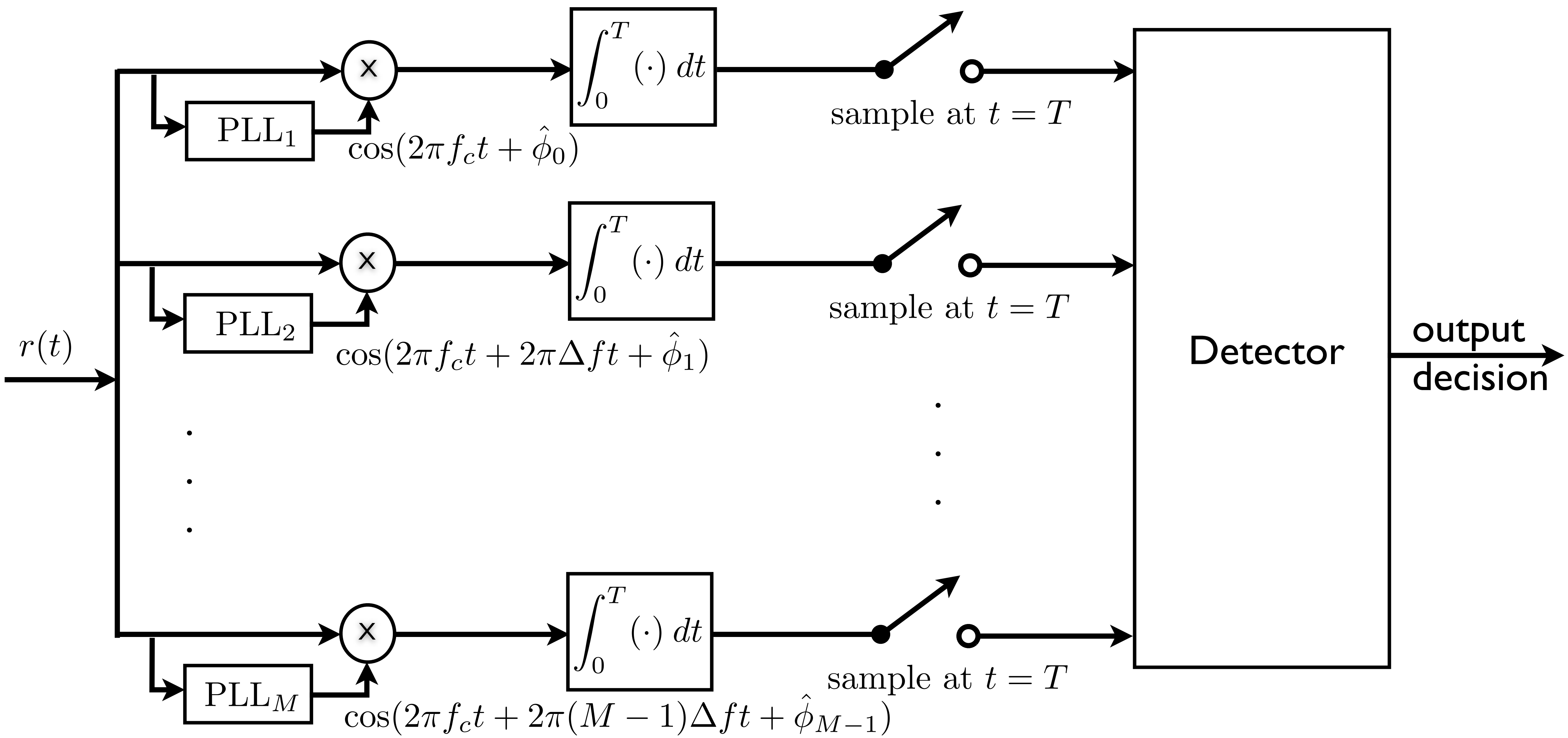
Demodulation and Detection of FSK

- Received signal

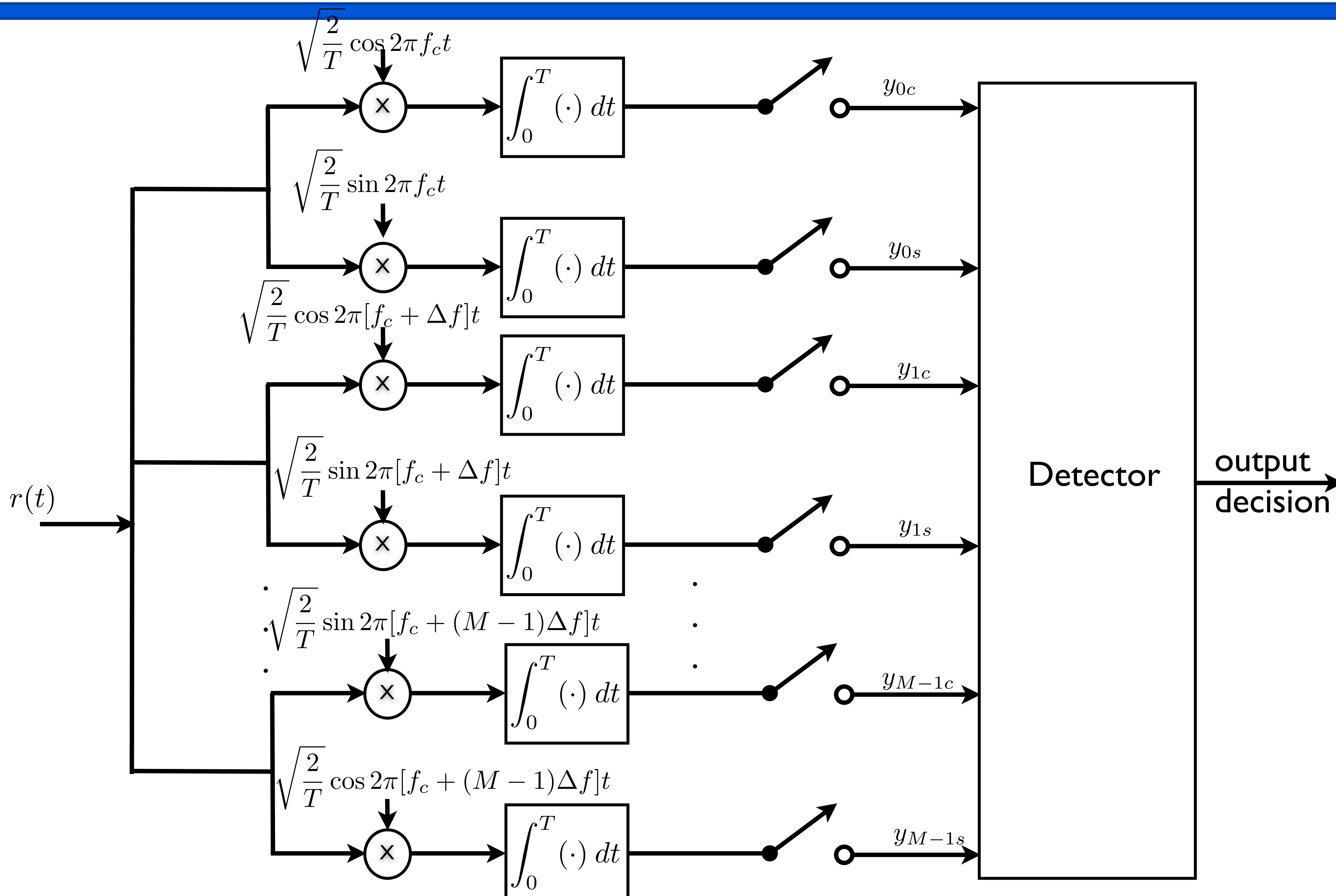
$$r(t) = \sqrt{\frac{2\mathcal{E}_s}{T}} \cos(2\pi f_c t + 2\pi m \Delta f t + \phi_m) + n(t) \qquad m = 0, 1, \ldots, M-1$$

phase error

- Coherent detection



Non-coherent Detection of M-FSK



- Non-coherent detection does not require the estimation of phase error.

  - The received signal is correlated with the basis function (quadrature carriers)

$$\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t) \quad \text{and} \quad \sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t) \qquad \text{for} \quad m = 0, 1, \ldots, M-1$$

  - Assume m-th signal is transmitted. Then 2M samples at the detector may be written as

$$y_{kc} = \sqrt{\mathcal{E}_s}\left[\frac{\sin 2\pi(k-m)\Delta f t}{2\pi(k-m)\Delta f T}\cos\phi_m - \frac{\cos 2\pi(k-m)\Delta f t - 1}{2\pi(k-m)\Delta f T}\sin\phi_m\right] + n_{k1}$$

$$y_{ks} = \sqrt{\mathcal{E}_s}\left[\frac{\cos 2\pi(k-m)\Delta f t - 1}{2\pi(k-m)\Delta f T}\cos\phi_m - \frac{\sin 2\pi(k-m)\Delta f t}{2\pi(k-m)\Delta f T}\sin\phi_m\right] + n_{k2}$$

    - which can be rewritten as

$$\begin{aligned} y_{mc} &= \sqrt{\mathcal{E}_s}\cos\phi_m + n_1 \\ y_{ms} &= \sqrt{\mathcal{E}_s}\sin\phi_m + n_2 \end{aligned} \quad \text{for } k = m \qquad \text{and} \qquad \begin{aligned} y_{kc} &= n_{k1} \\ y_{ks} &= n_{k2} \end{aligned} \quad \text{for } k \neq m$$

- Assume $\Delta f = 1/T$, so that the signals are orthogonal.

- It can be easily shown that the 2M noise samples are zero-mean, mutually uncorrelated Gaussian random variables with an equal variance $\sigma^2 = N_0/2$.

  - Joint PDF of y_{mc} and y_{ms}

$$f_{\mathbf{Y}_m}(y_{mc}, y_{ms}|\phi_m) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(y_{mc} - \sqrt{\mathcal{E}_s}\cos\phi_m)^2 + (y_{ms} - \sqrt{\mathcal{E}_s}\sin\phi_m)^2}{2\sigma^2}\right] \qquad \text{for } k = m$$

$$f_{\mathbf{Y}_k}(y_{kc}, y_{ks}|\phi_m) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{y_{kc}^2 + y_{ks}^2}{2\sigma^2}\right] \qquad \text{for } k \neq m$$

Detection

- Choose maximum a posteriori probability (MAP)

$$P[\mathbf{s}_m \text{ was transmitted}|\mathbf{y}] \equiv P(\mathbf{s}_m|\mathbf{y}), \quad m = 0, 1, \ldots, M-1$$

Optimum Detection for Binary PSK

- Two posteriori probabilities

$$P(\mathbf{s}_0|\mathbf{y}) = \frac{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_0)P(\mathbf{s}_0)}{f_{\mathbf{Y}}(\mathbf{y})},$$
$$P(\mathbf{s}_1|\mathbf{y}) = \frac{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_1)P(\mathbf{s}_1)}{f_{\mathbf{Y}}(\mathbf{y})}$$

- Optimum detection rule

$$P(\mathbf{s}_0|\mathbf{y}) \mathop{\gtrless}_{\mathbf{s}_1}^{\mathbf{s}_0} P(\mathbf{s}_1|\mathbf{y})$$

or equivalently,

$$\frac{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_0)P(\mathbf{s}_0)}{f_{\mathbf{Y}}(\mathbf{y})} \mathop{\gtrless}_{\mathbf{s}_1}^{\mathbf{s}_0} \frac{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_1)P(\mathbf{s}_1)}{f_{\mathbf{Y}}(\mathbf{y})}$$

- Note that
  - The received vector $\mathbf{y}$ is four-dimensional vector:

$$\mathbf{y} = (y_{0c},\ y_{0s},\ y_{1c},\ y_{1s})$$

  - Optimum detection can be simplified to

$$\frac{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_0)}{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_1)} \quad \overset{\mathbf{s}_0}{\underset{\mathbf{s}_1}{\gtrless}} \quad \frac{P(\mathbf{s}_1)}{P(\mathbf{s}_0)}.$$

  - Likelihood ratio function

$$\Lambda(\mathbf{y}) = \frac{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_0)}{f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_1)}$$

- Recall the PDFs

$$f_{\mathbf{Y}_m}(y_{mc}, y_{ms}|\phi_m) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(y_{mc} - \sqrt{\mathcal{E}_s}\cos\phi_m)^2 + (y_{ms} - \sqrt{\mathcal{E}_s}\sin\phi_m)^2}{2\sigma^2}\right] \qquad \text{for } k = m$$

$$f_{\mathbf{Y}_k}(y_{kc}, y_{ks}|\phi_m) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{y_{kc}^2 + y_{ks}^2}{2\sigma^2}\right] \qquad \text{for } k \neq m$$

- The PDF's $f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_0)$ and $f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_1)$ in the likelihood ratio may be expressed as

$$f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_0) = f_{\mathbf{Y}_1}(y_{1c}, y_{1s}) \int_0^{2\pi} f_{\mathbf{Y}_0}(y_{0c}, y_{0s}|\phi_0) f_\Phi(\phi_0)\, d\phi_0$$

$$f_{\mathbf{Y}}(\mathbf{y}|\mathbf{s}_1) = f_{\mathbf{Y}_0}(y_{0c}, y_{0s}) \int_0^{2\pi} f_{\mathbf{Y}_1}(y_{1c}, y_{1s}|\phi_1) f_\Phi(\phi_1)\, d\phi_1$$

  - Assume

$$f_\Phi(\phi_1) = \frac{1}{2\pi}, \qquad 0 \leq \phi_m \leq 2\pi.$$

- Then the integral becomes

$$\frac{1}{2\pi}\int_0^{2\pi} f_{\mathbf{Y}_m}(y_{mc}, y_{ms}|\phi_m)\, d\phi_m = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{y_{mc}^2+y_{ms}^2+\mathcal{E}_s}{2\sigma^2}\right]$$

$$\times \frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{\sqrt{\mathcal{E}_s}(y_{mc}\cos\phi_m + y_{ms}\sin\phi_m)}{\sigma^2}\right] d\phi_m;$$

- But

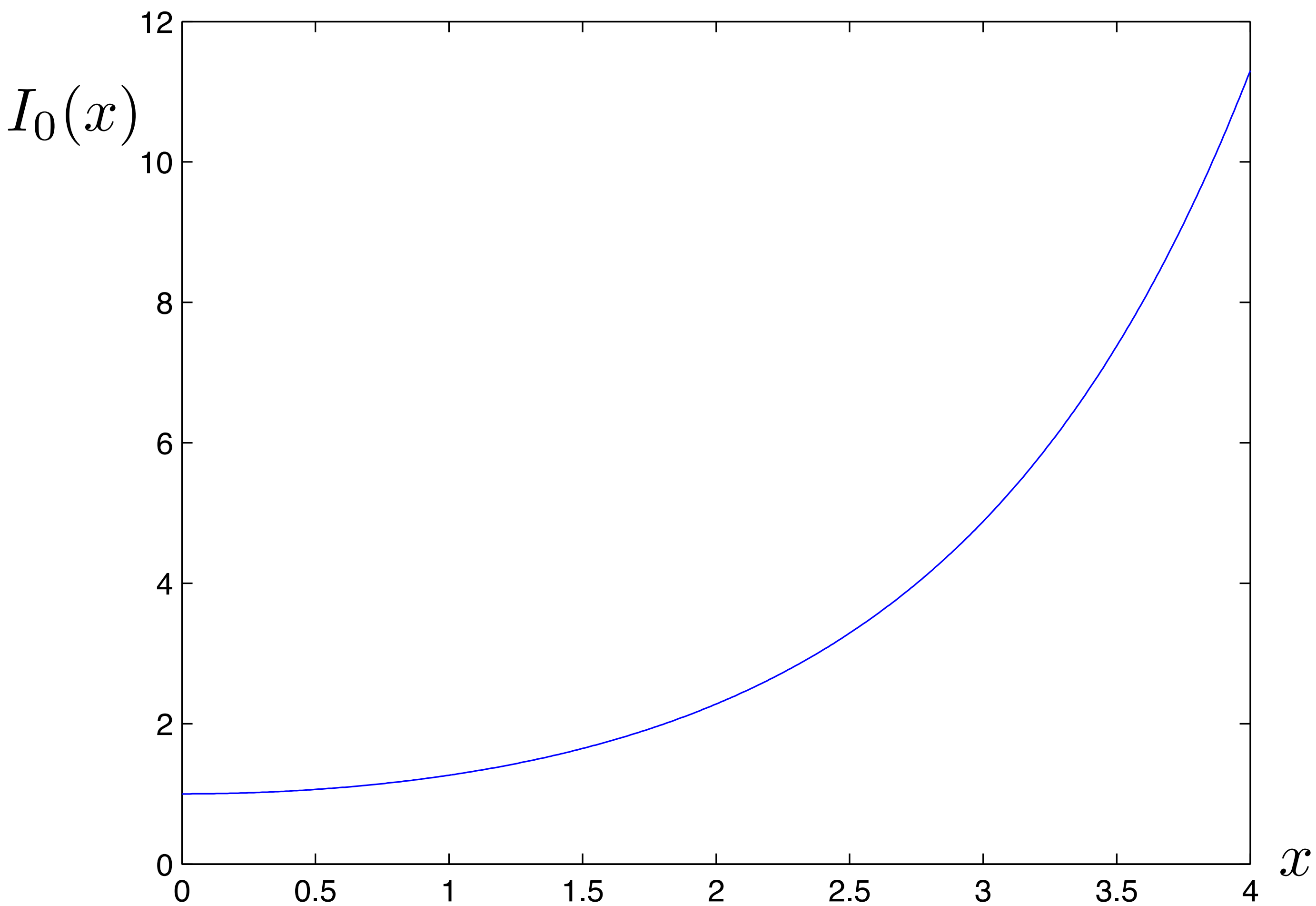
$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{\sqrt{\mathcal{E}_s}(y_{mc}\cos\phi_m + y_{ms}\sin\phi_m)}{\sigma^2}\right] d\phi_m = I_0\left(\frac{\sqrt{\mathcal{E}_s(y_{mc}^2+y_{ms}^2)}}{\sigma^2}\right)$$

- where $I_0(x)$ is the modified Bessel function of order zero. This function is a monotonically increasing function of its argument and can be expressed in power series form as

$$I_0(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}(k!)^2}$$

- Then, likelihood function can be written as

$$\Lambda(\mathbf{y}) = \frac{I_0\left(\frac{\sqrt{\mathcal{E}_s(y_{0c}^2+y_{0s}^2)}}{\sigma^2}\right)}{I_0\left(\frac{\sqrt{\mathcal{E}_s(y_{1c}^2+y_{1s}^2)}}{\sigma^2}\right)} \quad \mathop{\gtrless}_{\mathbf{s}_1}^{\mathbf{s}_0} \quad \frac{P(\mathbf{s}_1)}{P(\mathbf{s}_0)}.$$



- Thus, the optimum detector computes the two envelopes

$$y_0 = \sqrt{y_{0c}^2 + y_{0s}^2}$$

and

$$y_1 = \sqrt{y_{1c}^2 + y_{1s}^2}$$

and the corresponding values of the Bessel function

$$I_0\left(\sqrt{\mathcal{E}_s y_0^2}\sigma^2\right)$$

and

$$I_0\left(\sqrt{\mathcal{E}_s y_1^2}\sigma^2\right)$$

- We observe that this computation requires knowledge of the noise variance σ^2 and the signal energy $\mathcal{E}_s$.
- The likelihood ratio is then compared with the threshold $\dfrac{P(\mathbf{s}_1)}{P(\mathbf{s}_0)}$.

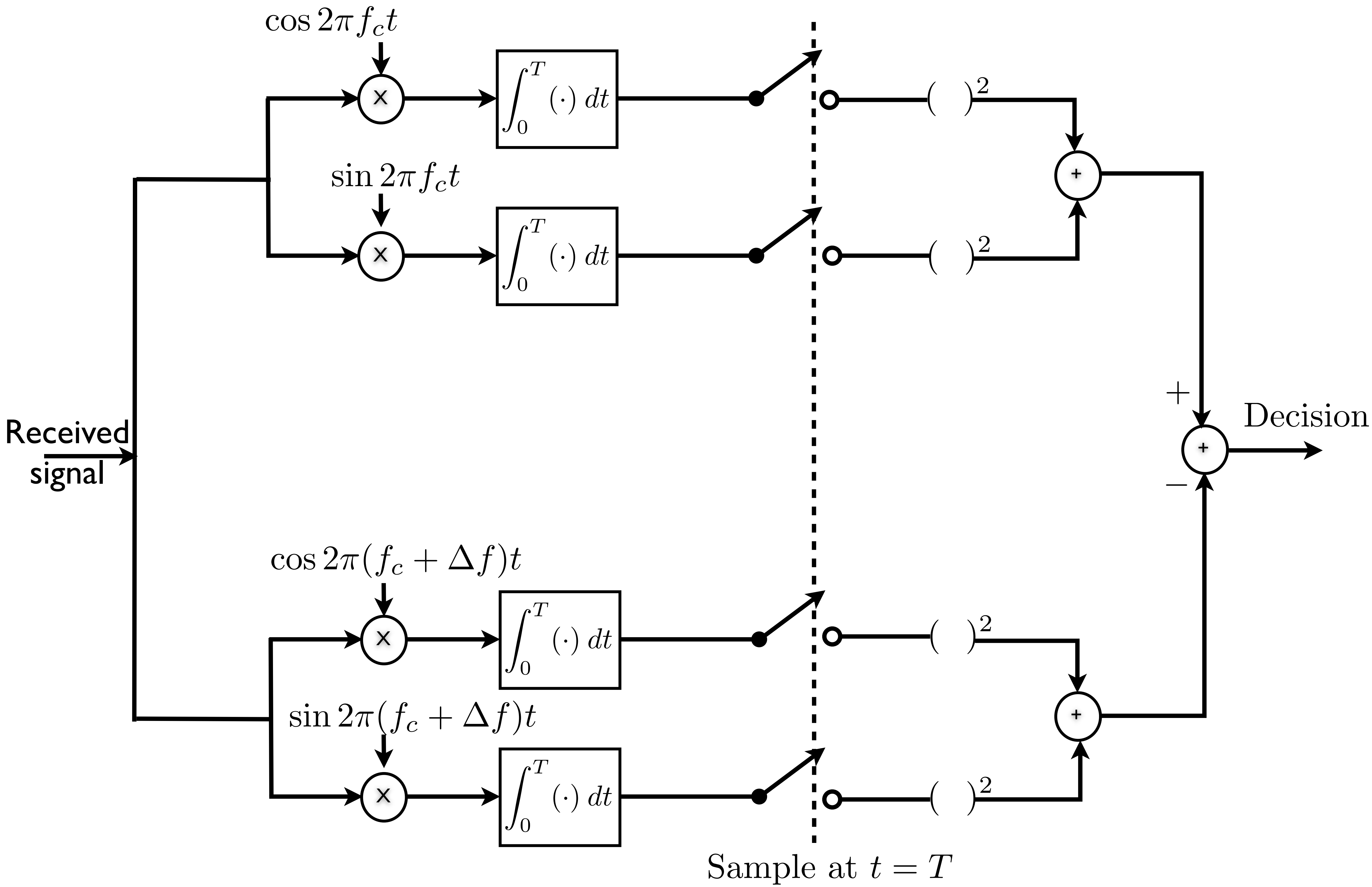
- When the two signals are equally probable, the optimum detector can be much simplified. Also due to the monotonicity of the Bessel function, the optimum detector rule simplifies to

$$\sqrt{y_{0c}^2 + y_{0s}^2} \quad \mathop{\gtrless}_{\mathbf{s}_1}^{\mathbf{s}_0} \sqrt{y_{1c}^2 + y_{1s}^2}$$

- Thus, the optimum detector bases its decision on the two envelopes

$$\begin{aligned} y_0 &= \sqrt{y_{0c}^2 + y_{0s}^2} \\ y_1 &= \sqrt{y_{1c}^2 + y_{1s}^2}; \end{aligned}$$

Square-law detector



- Generalization of the optimum detector to M-ary orthogonal FSK signals is straightforward.

  - Compute the M envelopes as

$$y_m = \sqrt{y_{mc}^2 + y_{ms}^2}, \quad m = 0, 1, \ldots, M-1.$$

  - For equally probable case, the optimum detector selects the signal corresponding to the largest envelope (or squared envelope).

  - In the case of non-equally probable transmitted signals, the optimum detector must compute the M posteriori probabilities and then select the signal corresponding to the largest posteriori probability.

Probability of Error for Non-coherent Detection of FSK

- Assume that the M signal are equally probable a priori.
- Assume that $u_0(t)$ was transmitted in the interval $0 \le t \le T$.
- The M-decision metrics at the detector are the M envelopes

$$y_m^2 = \sqrt{y_{mc}^2 + y_{ms}^2}, \quad m = 0, 1, \ldots, M-1,$$

where

$$\begin{aligned} y_{0c}^2 &= \sqrt{\mathcal{E}_s} \cos \phi_0 + n_{0c} \\ y_{0s}^2 &= \sqrt{\mathcal{E}_s} \sin \phi_0 + n_{0s} \end{aligned}$$

and

$$\begin{aligned} y_{mc} &= n_{mc}, \quad m = 1, 2, \ldots, M-1, \\ y_{ms} &= n_{ms}, \quad m = 1, 2, \ldots, M-1, \end{aligned}$$

and

$$n_{mc}, n_{ms} \sim \mathcal{N}(0, N_0/2)$$

- PDFs

$$f_{Y_0}(y_{0c}, y_{0s}) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{y_{0c}^2 + y_{0s}^2 + \mathcal{E}_s}{2\sigma^2}\right] I_0\left(\sqrt{\frac{\mathcal{E}_s(y_{0c}^2 + y_{0s}^2)}{\sigma^2}}\right)$$

$$f_{Y_m}(y_{mc}, y_{ms}) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{y_{mc}^2 + y_{ms}^2}{2\sigma^2}\right], \quad m = 1, 2, \ldots, M-1.$$

- Change of variables

$$R_m = \frac{\sqrt{Y_{mc}^2 + Y_{ms}^2}}{\sigma} \quad \text{and} \quad \Theta_m = \tan^{-1}\frac{Y_{ms}}{Y_{mc}}.$$

$$Y_{mc} = \sigma R_m \cos\Theta_m \quad \text{and} \quad Y_{ms} = \sigma R_m \sin\Theta_m$$

  - Jacobian

$$|\mathbf{J}| = \begin{vmatrix} \sigma\cos\Theta_m & \sigma\sin\Theta_m \\ -\sigma R_m \sin\Theta_m & \sigma R_m \cos\Theta_m \end{vmatrix} = \sigma^2 R_m.$$

- Consequently,

$$f_{R_0,\Theta_0}(r_0,\theta_0) = \frac{r_0}{2\pi} e^{-(r_0^2+2\mathcal{E}_s/N_0)/2} I_0\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}} r_0\right)$$

$$f_{R_m,\Theta_m}(r_m,\theta_m) = \frac{r_m}{2\pi} e^{-r_m^2/2}, \quad m = 1, 2, \ldots, M-1.$$

- Probability of correct decision

$$P_c = P(R_1 < R_0,\ R_2 < R_0,\ \ldots,\ R_{M-1} < R_0)$$

$$= \int_0^\infty P(R_1 < R_0,\ R_2 < R_0,\ \ldots,\ R_{M-1} < R_0 | R_0 = x) f_{R_0}(x)\, dx.$$

$$= \int_0^\infty \left[P(R_1 < R_0 | R_0 = x)\right]^{M-1}\, dx,$$

where $P(R_1 < R_0 | R_0 = x) = \int_0^x f_{R_1}(r_1)\, dr_1 = 1 - e^{-x^2/2}.$

- Note that

$$\left[1 - e^{-x^2/2}\right]^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-nx^2/2}.$$

- Probability of correct decision can be written as

$$P_c = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \frac{1}{n+1} e^{-n\rho_s/(n+1)}.$$

where $\rho_s = \mathcal{E}_s/N_0$ is the SNR per symbol.

- Probability of a symbol error

$$P_M = \sum_{n=1}^{M-1} (-1)^{n+1} \binom{M-1}{n} \frac{1}{n+1} e^{-nk\rho_b/(n+1)},$$

where $\rho_b = \mathcal{E}_b/N_0$ is the SNR per bit.

- For binary FSK (M=2),

$$P_2 = \frac{1}{2} e^{-\rho_b/2}.$$