KECE321 Communication Systems I

(Haykin Sec. 4.4)

Lecture #14, May 2, 2012

Prof. Young-Chai Ko

Summary

- Narrow-band frequency modulation
- Wideband frequency modulation

Introduction

- FM wave is a nonlinear function of the modulating wave.
  - Spectral analysis of the FM wave is much more difficult than other modulation schemes.
- Approaches of the spectral analysis of FM wave
  - Simple case by considering the single-tone modulation that produces a narrow-band FM wave
  - More general case with single-tone modulation but the FM wave is wide-band.

Narrow-Band FM

- Message signal (or modulating wave)

$$m(t) = A_m \cos(2\pi f_m t)$$

- Instantaneous frequency of FM wave

$$f_i(t) = f_c + k_f A_m \cos(2\pi f_m t) = f_c + \Delta f \cos(2\pi f_m t)$$

where $\Delta f = k_f A_m$

- Δf is called the *frequency deviation.*
  - Frequency deviation is proportional to the amplitude of the modulating signal and is independent of the modulating frequency.
- Angle

$$\theta_i(t) = 2\pi f_c t + \frac{\Delta f}{f_m} \cos(2\pi f_m t)$$

- Modulation index

$$\beta = \frac{\Delta f}{f_m} \quad \text{rad}$$

  - Hence, the angle can be rewritten as

$$\theta_i(t) = 2\pi f_c t + \beta \sin(2\pi f_m t)$$

- FM wave

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

$$= A_c \cos(2\pi f_c t) \cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta \sin(2\pi f_m t)]$$

  - Narrow-band if $\beta << 1$ rad

- For narrow-band FM, that is, $\beta << 1 \text{ rad}$

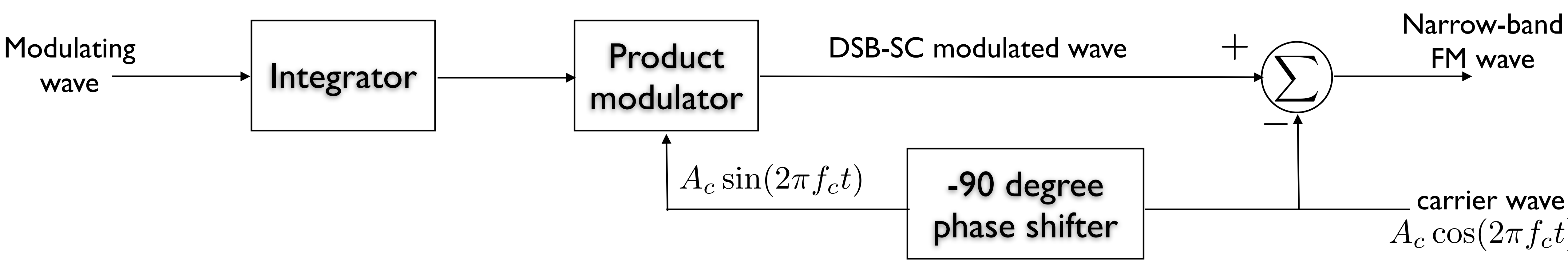
$$\cos[\beta \sin(2\pi f_m t)] \approx 1$$

$$\sin[\beta \sin(2\pi f_m t)] \approx \beta \sin(2\pi f_m t)$$

  - which gives the approximate form of the FM wave such as

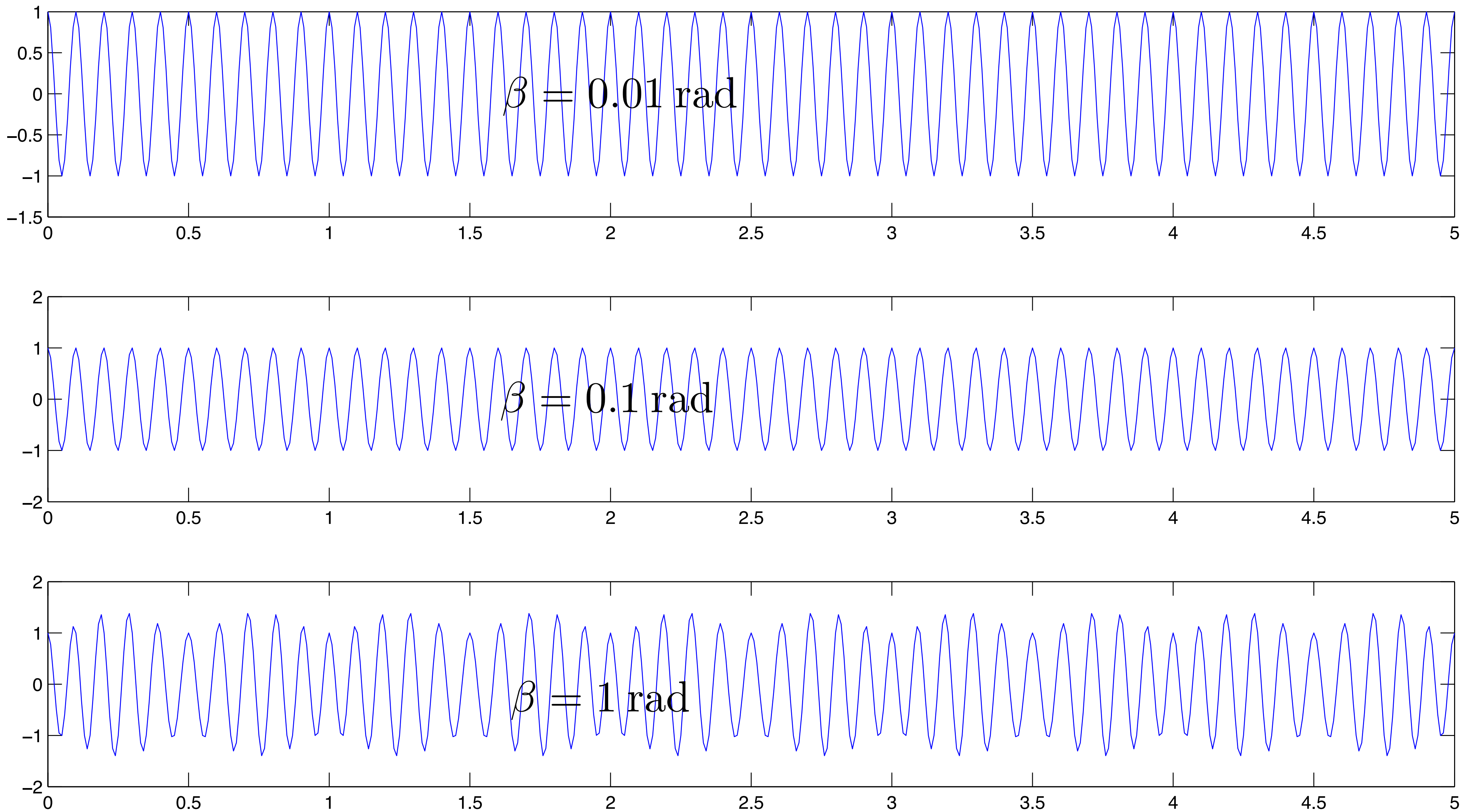
$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$

  - Block diagram



Example

$$f_c = 10\ \text{Hz} \qquad f_m = 1\ \text{Hz} \quad A_c = 1$$



Approximate narrow-band FM signal

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t)$$

$$= A_c\sqrt{1 + \beta^2 \sin^2(2\pi f_m t)}\cos[2\pi f_c t + \phi(t)]$$

envelope   phase

where $\phi(t) = \tan^{-1}[\beta \sin(2\pi f_m t)]$

- Using the approximation of $\tan^{-1}(x) \approx x - \frac{1}{3}x^3$ for small x

$$\phi(t) \approx \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

- Envelope: $A_c\sqrt{1 + \beta^2 \sin^2(2\pi f_m t)}$

- Phase: $\phi(t) \approx \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$

Remarks

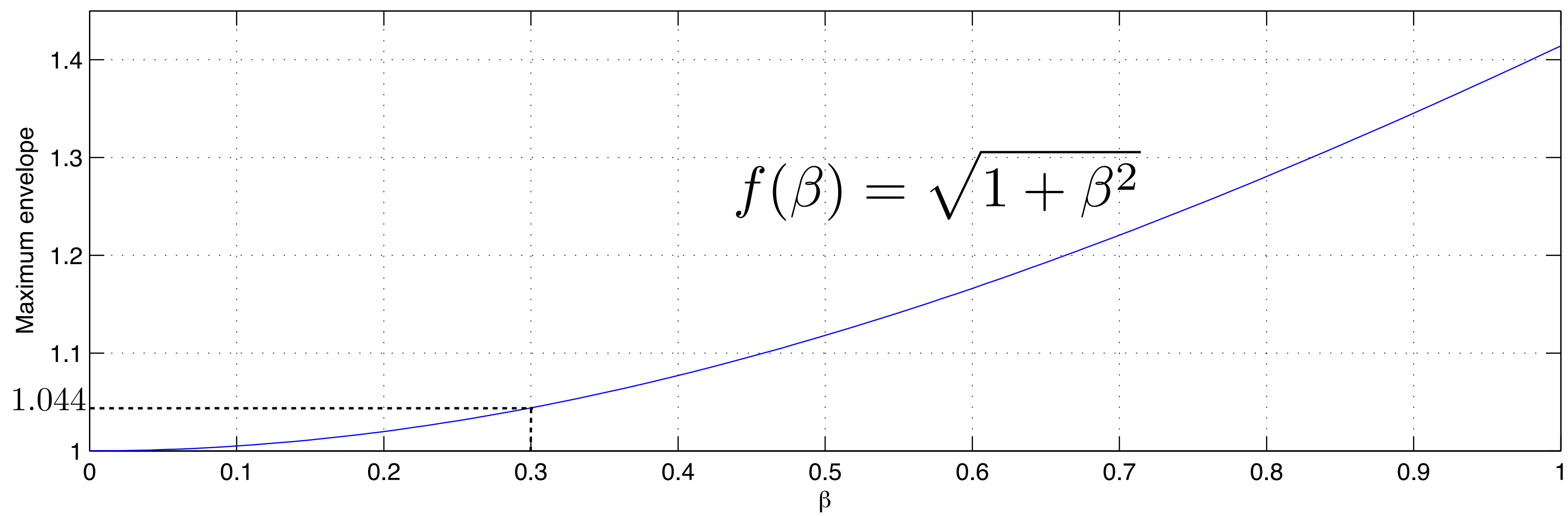
- Envelope is not constant due to the approximation of narrow-band FM and there exists residual amplitude modulation that varies with time.

- The angle contains harmonic distortion of third and higher order harmonics of the modulation frequency f_m.

- However, for $\beta < 0.3$, the effects of residual amplitude modulation and harmonic distortion are limited to negligible levels.

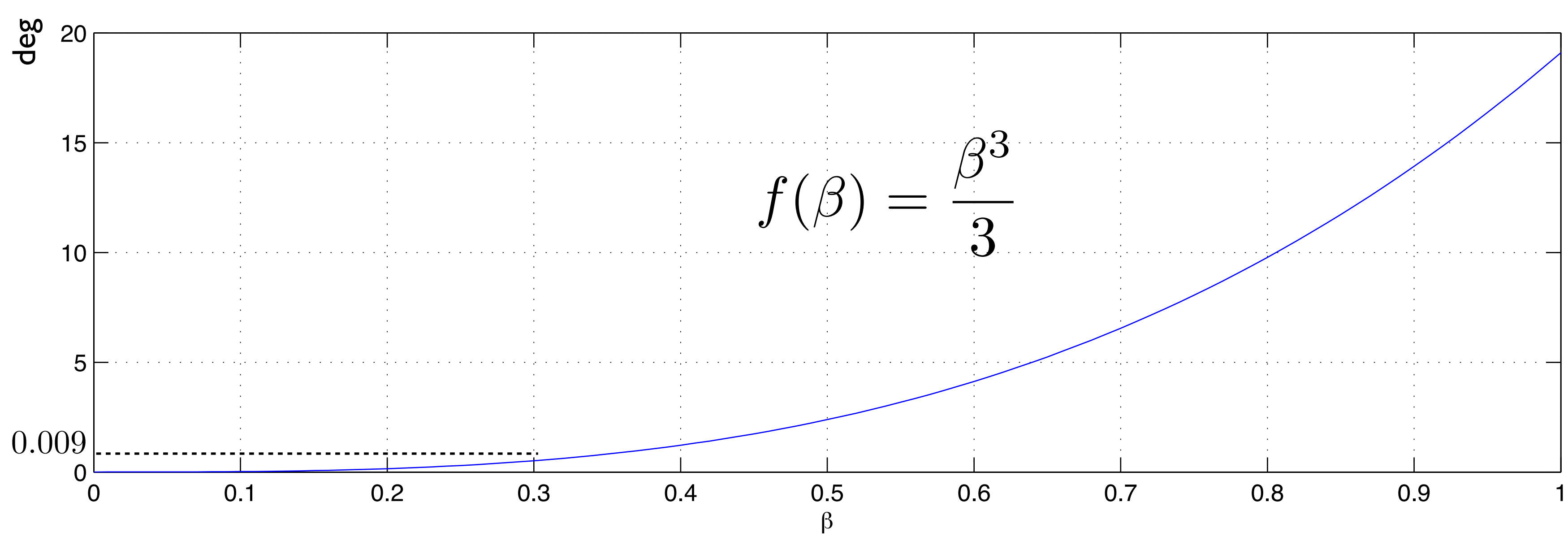
- Envelope of narrow-band FM is upper bounded by

$$\sqrt{1+\beta^2\sin^2(2\pi f_m t)} \le \sqrt{1+\beta^2}$$



- Phase $\phi(t) \approx \beta\sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$

third-order harmonics



- Recall the approximated narrow-band FM signal given as

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$

  - which can be rewritten as

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \{\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\}$$

  - On the other hand, the AM signal is

$$s_{\text{AM}}(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c \{\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\}$$

Wide-Band FM

- FM wave

$$s(t) = \mathbf{Re}[A_c \exp(j2\pi f_c t + j\beta \sin(2\pi f_m t))] = \mathbf{Re}[\tilde{s}(t) \exp(j2\pi f_c t)]$$

- where

$$\tilde{s}(t) = A_c \exp[j\beta \sin(2\pi f_m t)]$$

is called the complex envelope of the FM wave $s(t)$.

- Noting that $\tilde{s}(t)$ is periodic signal with fundamental frequency, f_m which can be proven as

$$\begin{aligned} \tilde{s}(t) &= A_c \exp[j\beta \sin(2\pi f_m (t + k/f_m))] \\ &= A_c \exp[j\beta \sin(2\pi f_m t + 2k\pi)] \\ &= A_c \exp[j\beta \sin(2\pi f_m t)] \end{aligned}$$

Fourier series form of $\tilde{s}(t)$

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t)$$

where

$$\begin{aligned} c_n &= f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t) \exp(-j2\pi n f_m t)\, dt \\ &= f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} \exp[j\beta \sin(2\pi f_m t) - j2\pi n f_m t]\, dt \end{aligned}$$

Changing the variable such as $x = 2\pi f_m t$ gives

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx \quad = A_c J_n(\beta)$$

where $J_n(\beta)$ is the nth order Bessel function of the first kind and argument β defined as

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx$$

- Complex envelope of the FM wave is in Fourier series form as

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi n f_m t)$$

- The FM wave can be now written as

$$\begin{aligned} s(t) &= \mathbf{Re}\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + nf_m)t]\right] \\ &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + nf_m)t] \end{aligned}$$

- and its Fourier transform is

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]$$