The Uncertainty Principle

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November 19, 2013 (v5.4)

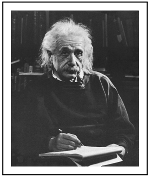
Marcel Duchamp, Nude Decending a Staircase, No. 2 (1912).
Philadelphia Museum of Art, Philadelphia. Image from Wikipedia.

Wave-Particle Duality

(revisited)

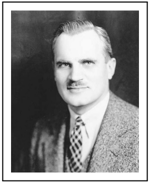
Wave-Particle Duality

(A. Einstein, 1902; A. H. Compton, 1923; L. de Broglie, 1924)



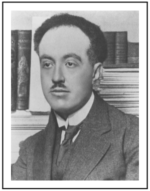
Let light (wave) have discrete energies!

$$(\text{energy}) = h \times (\text{frequency})$$



Let light (wave) have momentum!

$$(\text{momentum}) = \frac{h}{(\text{wavelength})}$$



Let particles behave like a wave with:

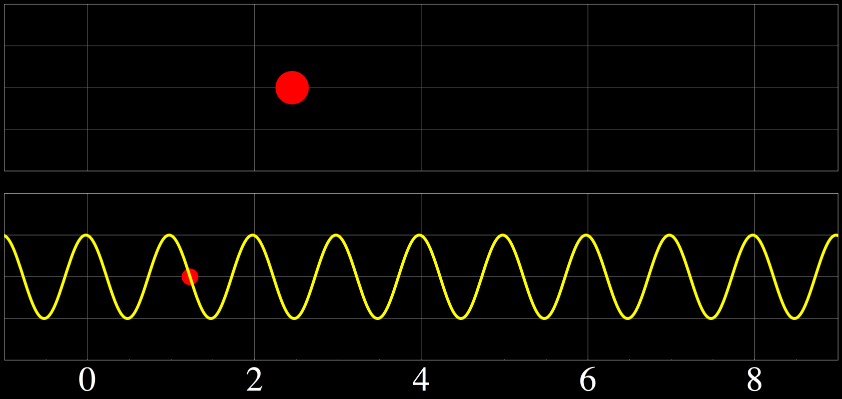
$$(\text{frequency}) = h \times (\text{energy}) \,,$$

$$(\text{wavelength}) = \frac{h}{(\text{momentum})}$$

Imeages courtesy of Wikipedia

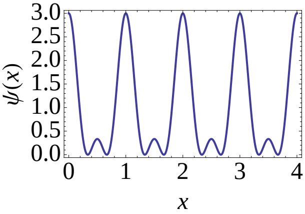
$$(\text{energy}) = h \times (\text{frequency})$$

$$(\text{momentum}) = \frac{h}{(\text{wavelength})}$$



Uncertainty in Momentum

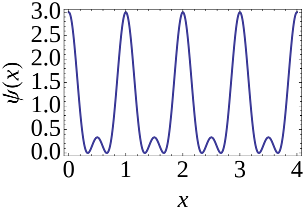
Momentum and Wave Number



$$\psi(x) = \frac{e^{i2\pi x} + e^{i4\pi x} + e^{i6\pi x}}{\sqrt{3}}$$

- The period is ☐.
- The primary wave length is ☐.
- The primary wave number is ☐.

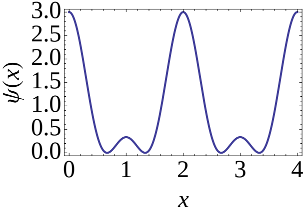
Momentum and Wave Number



$$\psi(x) = \frac{e^{i2\pi x} + e^{i4\pi x} + e^{i6\pi x}}{\sqrt{3}}$$

- The period is $\boxed{1\,\text{m}}$.
- The primary wave length is $\boxed{\lambda_0 = 1\,\text{m}}$.
- The primary wave number is $\boxed{k_0 = 2\pi\,\text{m}^{-1}}$.

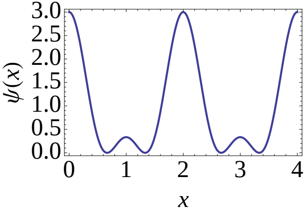
Momentum and Wave Number



$$\psi(x) = \frac{e^{i4\pi x} + e^{i8\pi x} + e^{i12\pi x}}{\sqrt{3}}$$

- The period is ☐.
- The primary wave length is ☐.
- The primary wave number is ☐.

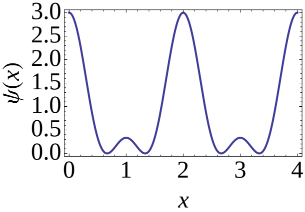
Momentum and Wave Number



$$\psi(x) = \frac{e^{i4\pi x} + e^{i8\pi x} + e^{i12\pi x}}{\sqrt{3}}$$

- The period is $\boxed{2\,\mathrm{m}}$.
- The primary wave length is $\boxed{\lambda_0 = 2\,\mathrm{m}}$.
- The primary wave number is $\boxed{k_0 = \pi\,\mathrm{m}^{-1}}$.

Momentum and Wave Number

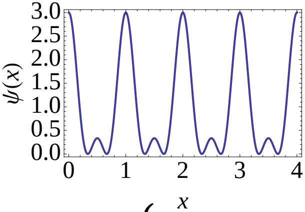


$$\psi(x) = \frac{e^{i4\pi x} + e^{i8\pi x} + e^{i12\pi x}}{\sqrt{3}}$$

- The period is $\boxed{2\,\mathrm{m}}$.
- The primary wave length is $\boxed{\lambda_0 = 2\,\mathrm{m}}$.
- The primary wave number is $\boxed{k_0 = \pi\,\mathrm{m}^{-1}}$.

$$\frac{2\pi}{\lambda_0}, \quad 2\times\frac{2\pi}{\lambda_0}, \quad 3\times\frac{2\pi}{\lambda_0}, \quad \cdots$$

Uncertainty in Momentum

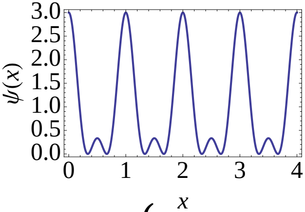


$$\psi(x) = \frac{e^{i2\pi x} + e^{i4\pi x} + e^{i6\pi x}}{\sqrt{3}}$$

$$\psi(x) \Longrightarrow \begin{cases} e^{ik_0 x}, & \text{Probability} = \boxed{},\ P = \boxed{} \\ e^{i2k_0 x}, & \text{Probability} = \boxed{},\ P = \boxed{} \\ e^{i3k_0 x}, & \text{Probability} = \boxed{},\ P = \boxed{} \end{cases}$$

- The average of P is $\boxed{}$.
- The uncertainty in P is $\boxed{}$.

Uncertainty in Momentum

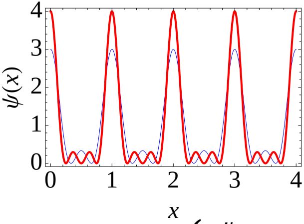


$$\psi(x) = \frac{e^{i2\pi x} + e^{i4\pi x} + e^{i6\pi x}}{\sqrt{3}}$$

$$\psi(x) \Longrightarrow \begin{cases} e^{ik_0 x}, & \text{Probability} = \boxed{1/3},\ P = \boxed{h\,\mathrm{m}^{-1}} \\ e^{i2k_0 x}, & \text{Probability} = \boxed{1/3},\ P = \boxed{2h\,\mathrm{m}^{-1}} \\ e^{i3k_0 x}, & \text{Probability} = \boxed{1/3},\ P = \boxed{3h\,\mathrm{m}^{-1}} \end{cases}$$

- The average of P is $\boxed{\langle P \rangle = 2h\,\mathrm{m}^{-1}}$.
- The uncertainty in P is $\boxed{\Delta P = \sqrt{2/3}\,h\,\mathrm{m}^{-1}}$.

Uncertainty in Momentum

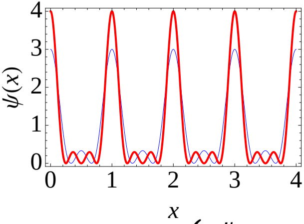


$$\psi(x) = \frac{e^{i2\pi x} + e^{i4\pi x} + e^{i6\pi x} + e^{i8\pi x}}{\sqrt{4}}$$

$$\psi(x) \Longrightarrow \begin{cases} e^{ik_0 x}, & \text{Probability} = \boxed{}, \quad P = \boxed{} \\ e^{i2k_0 x}, & \text{Probability} = \boxed{}, \quad P = \boxed{} \\ e^{i3k_0 x}, & \text{Probability} = \boxed{}, \quad P = \boxed{} \\ e^{i4k_0 x}, & \text{Probability} = \boxed{}, \quad P = \boxed{} \end{cases}$$

- The average of P is $\boxed{}$.
- The uncertainty in P is $\boxed{}$.

Uncertainty in Momentum

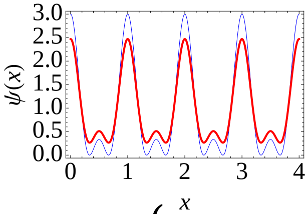


$$\psi(x) = \frac{e^{i2\pi x} + e^{i4\pi x} + e^{i6\pi x} + e^{i8\pi x}}{\sqrt{4}}$$

$$\psi(x) \Longrightarrow \begin{cases} e^{ik_0 x}, & \text{Probability} = \boxed{1/4},\ P = \boxed{h\,\mathrm{m}^{-1}} \\ e^{i2k_0 x}, & \text{Probability} = \boxed{1/4},\ P = \boxed{2h\,\mathrm{m}^{-1}} \\ e^{i3k_0 x}, & \text{Probability} = \boxed{1/4},\ P = \boxed{3h\,\mathrm{m}^{-1}} \\ e^{i4k_0 x}, & \text{Probability} = \boxed{1/4},\ P = \boxed{4h\,\mathrm{m}^{-1}} \end{cases}$$

- The average of P is $\boxed{\langle P \rangle = 2.5\,h\,\mathrm{m}^{-1}}$.
- The uncertainty in P is $\boxed{\Delta P = \sqrt{5/4}\,h\,\mathrm{m}^{-1}}$.

Uncertainty in Momentum

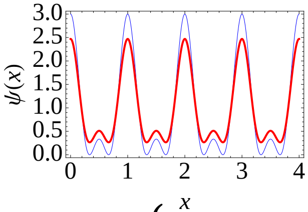


$$\psi(x) = \frac{6e^{i2\pi x} + 3e^{i4\pi x} + 2e^{i6\pi x}}{7}$$

$$\psi(x) \Longrightarrow \begin{cases} e^{ik_0 x}, & \text{Probability} = \boxed{}, \ P = \boxed{} \\ e^{i2k_0 x}, & \text{Probability} = \boxed{}, \ P = \boxed{} \\ e^{i3k_0 x}, & \text{Probability} = \boxed{}, \ P = \boxed{} \end{cases}$$

- The average of P is $\boxed{}$.
- The uncertainty in P is $\boxed{}$.

Uncertainty in Momentum



$$\psi(x) = \frac{6e^{i2\pi x} + 3e^{i4\pi x} + 2e^{i6\pi x}}{7}$$

$$\psi(x) \Longrightarrow \begin{cases} e^{ik_0 x}, & \text{Probability} = \boxed{36/49},\; P = \boxed{h\,\text{m}^{-1}} \\ e^{i2k_0 x}, & \text{Probability} = \boxed{09/49},\; P = \boxed{2h\,\text{m}^{-1}} \\ e^{i3k_0 x}, & \text{Probability} = \boxed{04/49},\; P = \boxed{3h\,\text{m}^{-1}} \end{cases}$$

- The average of P is $\boxed{\langle P \rangle = 1.35\,h\,\text{m}^{-1}}$.
- The uncertainty in P is $\boxed{\Delta P = 0.62\,h\,\text{m}^{-1}}$.

Break and you will see!

(Reduction vs Emergence)

Break and you will see!

(Reduction vs Emergence)

Ask and it will be given to you; seek and you will find; knock and the door will be opened to you. — Matthew 7:7

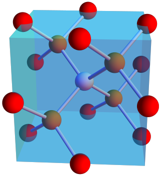
Parts and The Whole

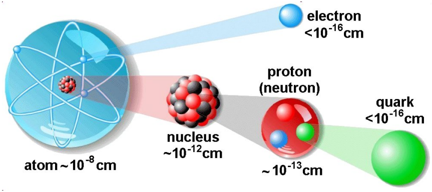
(Reduction vs Emergence)

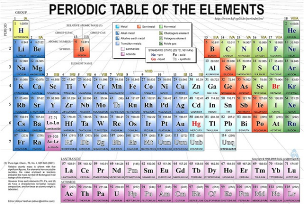
Images courtesy of http://t2yes.blogspot.com/ and http://www.blue-inc-solutions.co.uk/, respectively.

Parts and The Whole

(The Structure of Matter)

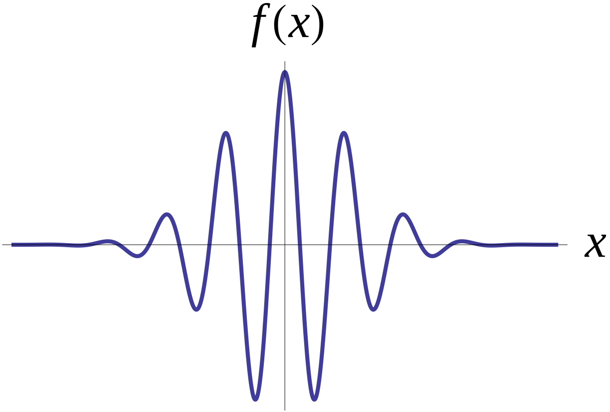






Top left image courtesy of `http://www.garysuttonshow.com/`; top right courtesy of `http://www.ipp.phys.ethz.ch/`; bottom courtesy of `http://www.ktf-split.hr/`

Decomposition of a Function?



Fourier Decomposition

(decomposition of functions into plain waves)

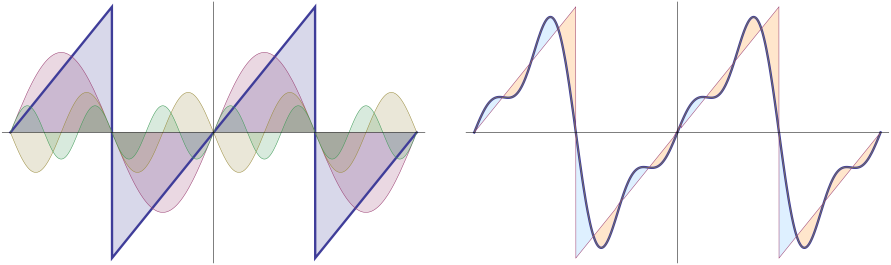
(Joseph Fourier, 1768–1830)
Courtesy of Wikipedia

$$f(x) = \sum_{n=1}^{\infty} \left[S_n \sin(k_n x) + C_n \cos(k_n x) \right]$$

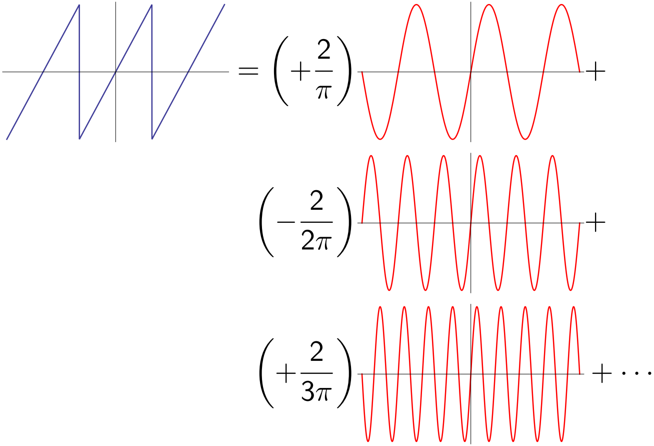
$$k_n = \frac{2\pi}{\lambda}, \frac{4\pi}{\lambda}, \frac{6\pi}{\lambda}, \cdots$$

Fourier Decomposition

(decomposition into plain waves)



$$f(x) = \sum_{n=1}^{\infty} \boxed{(-1)^{n+1}\frac{2}{n\pi}} \sin(n\pi x)$$

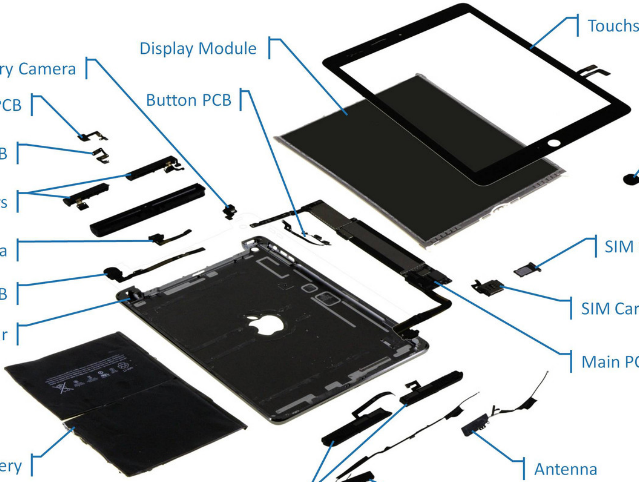


$= \left(+\frac{2}{\pi}\right)$ +
$\left(-\frac{2}{2\pi}\right)$ +
$\left(+\frac{2}{3\pi}\right)$ $+ \cdots$

Apple iPad Air (MF496LL/A)

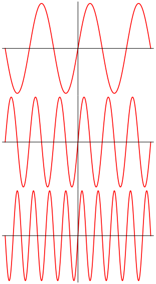
Exploded View

Teardown Analysis



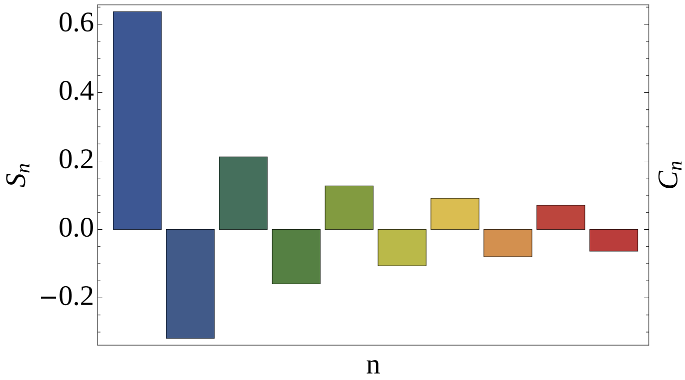
Bookkeeping of the Coefficients, How?

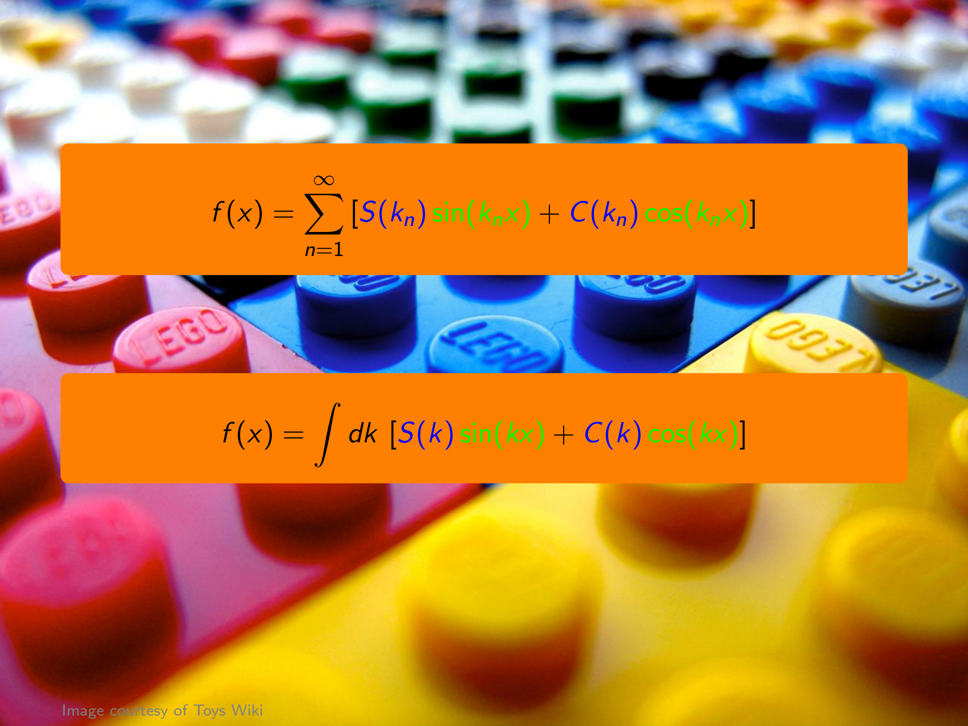
Bookkeeping of the Coefficients, How?

Index	Wave length	"Momentum"	Coefficient	Shape
1	$\frac{2}{1}$	$\frac{1h}{2}$	$+\frac{2}{\pi}$	
2	$\frac{2}{2}$	$\frac{2h}{2}$	$-\frac{2}{2\pi}$	
3	$\frac{2}{3}$	$\frac{3h}{2}$	$+\frac{2}{3\pi}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Bookkeeping of the Coefficients, How?

Momentum-Space Wave Function

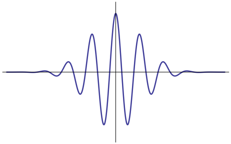
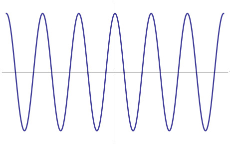


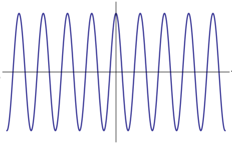


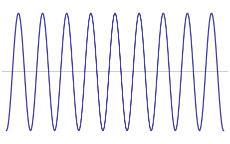
$$f(x) = \sum_{n=1}^{\infty} \left[S(k_n) \sin(k_n x) + C(k_n) \cos(k_n x) \right]$$

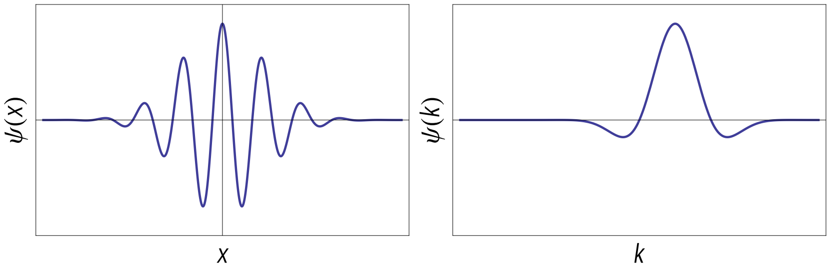
$$f(x) = \int dk \, \left[S(k) \sin(kx) + C(k) \cos(kx) \right]$$

Image courtesy of Toys Wiki

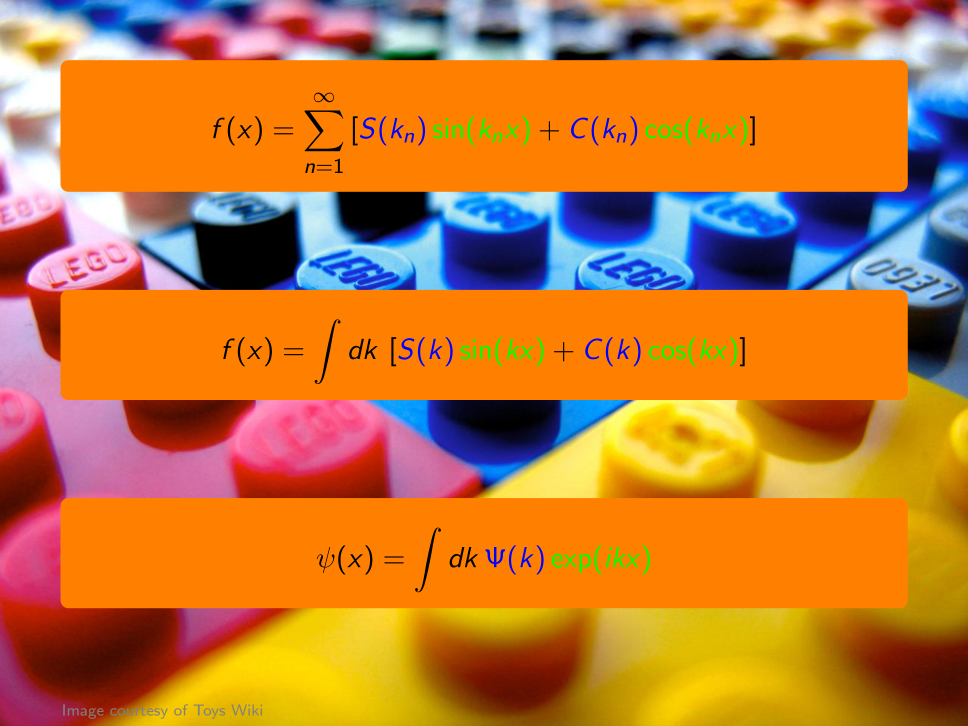
 $= \left(-e^{-\pi^2/4}\right)$  $+$

1  $+$

$\left(-e^{-\pi^2/4}\right)$  $+ \cdots$



ψ(x)
x
ψ(k)
k



$$f(x) = \sum_{n=1}^{\infty} \left[S(k_n)\sin(k_n x) + C(k_n)\cos(k_n x) \right]$$

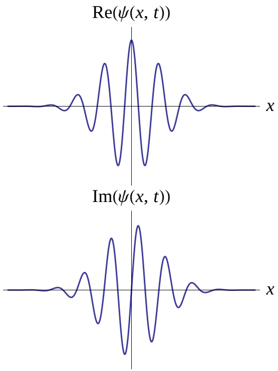
$$f(x) = \int dk \, \left[S(k)\sin(kx) + C(k)\cos(kx) \right]$$

$$\psi(x) = \int dk \, \Psi(k)\exp(ikx)$$

Image courtesy of Toys Wiki

The Uncertainty Principle

Any info about position and momentum?



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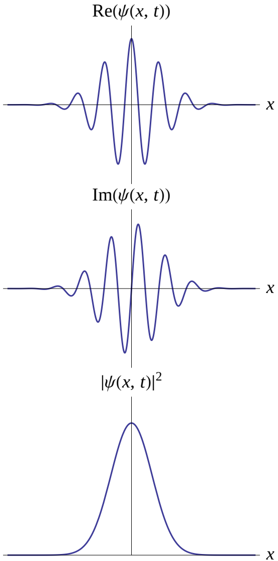

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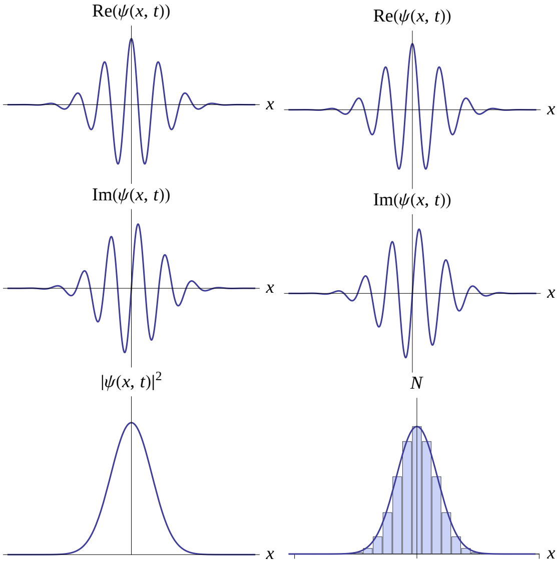
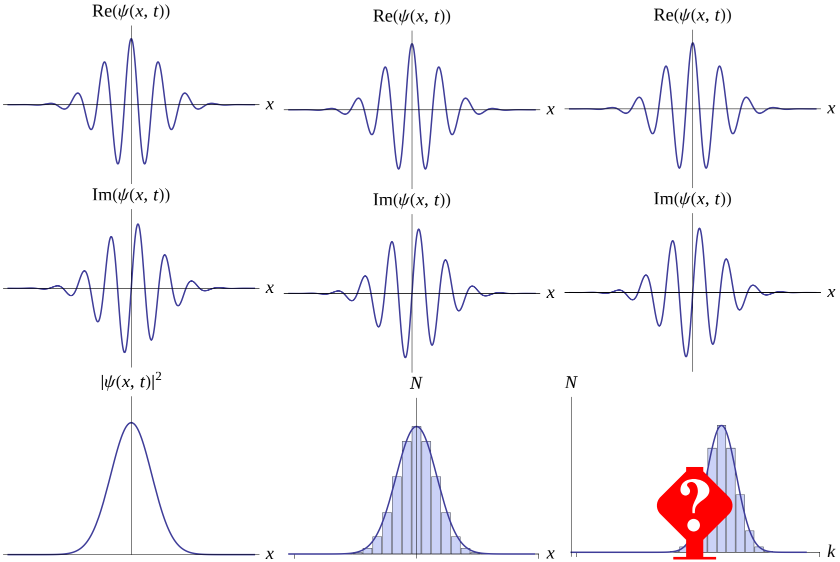
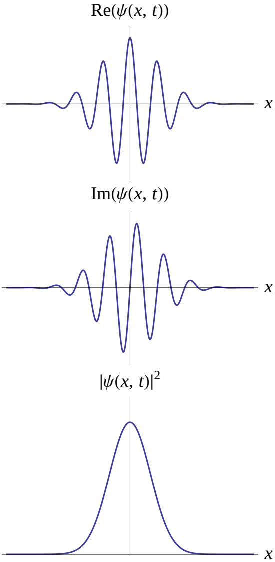
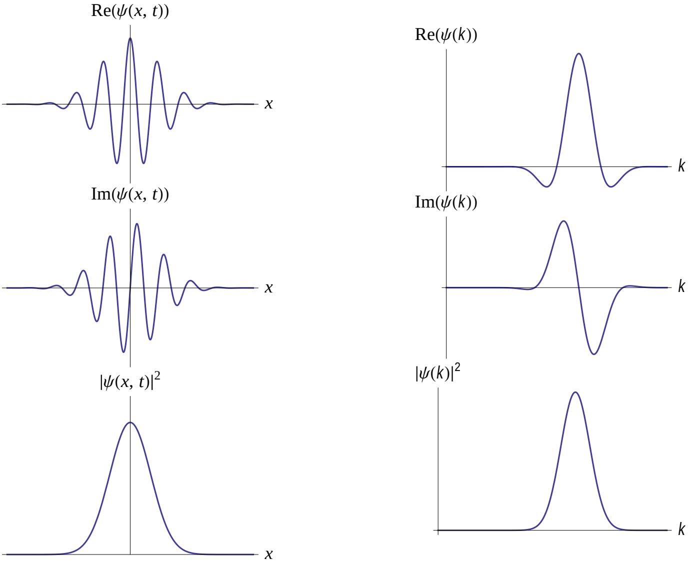


Any info about position and momentum?



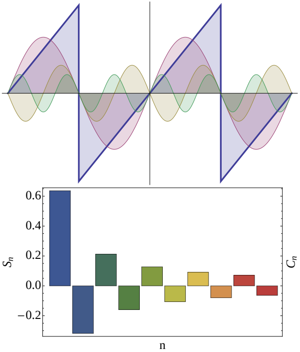


Re(ψ(x, t))
x
Im(ψ(x, t))
x
|ψ(x, t)|²
x


$\mathrm{Re}(\psi(x, t))$
x
$\mathrm{Im}(\psi(x, t))$
x
$|\psi(x, t)|^2$
x
$\mathrm{Re}(\psi(k))$
k
$\mathrm{Im}(\psi(k))$
k
$|\psi(k)|^2$
k

Uncertainty in Momentum

(Periodic Wave Functions)



A hypothetical example:

$$\psi(x) = \sum_{n=-\infty}^{\infty} S_n \exp(ik_n x)$$

$$S_n = (-1)^{n+1} \frac{2}{n\pi}$$

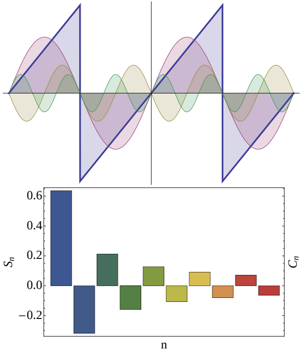
$$k_n = n\pi$$

$$\langle P \rangle = \boxed{?}$$

$$(\Delta P)^2 = \boxed{?}$$

Uncertainty in Momentum

(Periodic Wave Functions)



A hypothetical example:

$$\psi(x) = \sum_{n=-\infty}^{\infty} S_n \exp(ik_n x)$$

$$S_n = (-1)^{n+1} \frac{2}{n\pi}$$

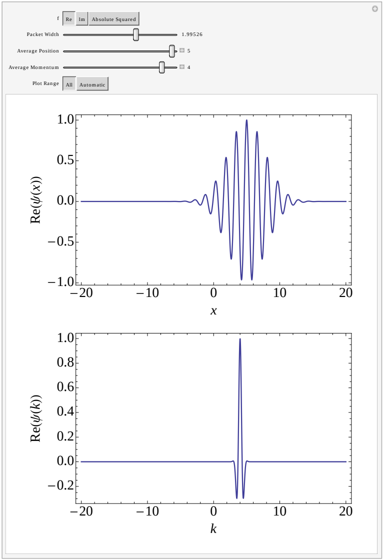
$$k_n = n\pi$$

$$\langle P \rangle = \boxed{\sum_{n=1}^{\infty} |S_n|^2 \hbar k_n}$$

$$(\Delta P)^2 = \boxed{?}$$

Uncertainty in Momentum

(Wave Packets)



$$\psi(x) = \int dk\, \tilde{\psi}(k) \exp(ikx)$$

$\langle x \rangle =$ ☐

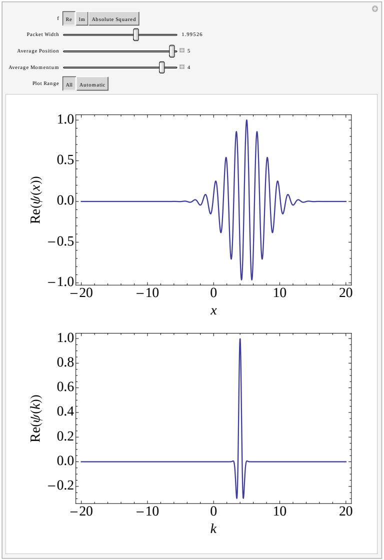
$\Delta x =$ ☐

$\langle p \rangle =$ ☐

$\Delta p =$ ☐

Uncertainty in Momentum

(Wave Packets)



$$\psi(x) = \int dk\, \tilde{\psi}(k) \exp(ikx)$$

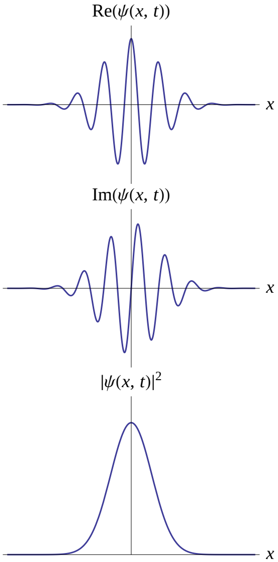
$$\langle x \rangle = \boxed{5\text{ m}}$$

$$\Delta x = \boxed{2\text{ m}}$$

$$\langle p \rangle = \boxed{4\hbar}$$

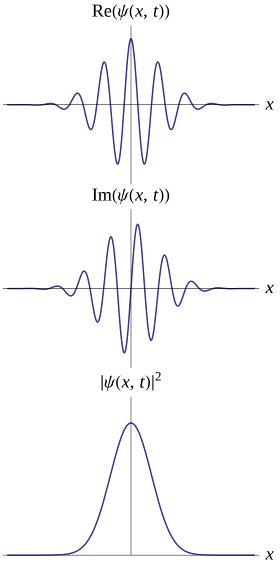
$$\Delta p = \boxed{0.5\hbar}$$

The Uncertainty Relation!

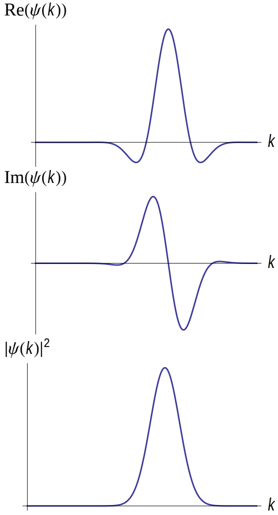


$$\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$$

The Uncertainty Relation!



$$\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$$



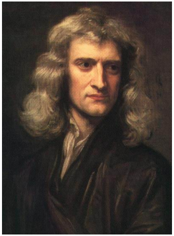
Conjugate Variables

$x \leftrightarrow p$	:	$\Delta x \Delta p \geq \frac{1}{2}\hbar$
$L_x \leftrightarrow L_y$	:	$\Delta L_x \Delta L_y \geq \frac{1}{2}\hbar \lvert\langle L_z \rangle\rvert$
$L_y \leftrightarrow L_z$	:	$\Delta L_y \Delta L_z \geq \frac{1}{2}\hbar \lvert\langle L_x \rangle\rvert$
$L_z \leftrightarrow L_x$	:	$\Delta L_z \Delta L_x \geq \frac{1}{2}\hbar \lvert\langle L_y \rangle\rvert$
$S_x \leftrightarrow S_y$	:	$\Delta S_x \Delta S_y \geq \frac{1}{2}\hbar \lvert\langle S_z \rangle\rvert$
$S_y \leftrightarrow S_z$	:	$\Delta S_y \Delta S_z \geq \frac{1}{2}\hbar \lvert\langle S_x \rangle\rvert$
$S_z \leftrightarrow S_x$	:	$\Delta S_z \Delta S_x \geq \frac{1}{2}\hbar \lvert\langle S_y \rangle\rvert$

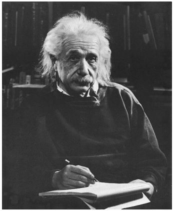
Time-Energy Uncertainty Relation

The Concept of Time

(“simultaneity”)



Time is considered to be “absolute” and to flow “equably” for all observers. Events seen by two different observers in motion relative to each other produces a mathematical concept of time.



Invoking a method of synchronizing clocks using the constant, finite speed of light as the maximum signal velocity. This led directly to the result that observers in motion relative to one another will measure different elapsed times for the same event.

Source: Wikipedia
See also: Jammer (2006)

Position-Momentum vs Time-Energy Relation

(space vs time)

Time in classical/quantum mechanics is merely a "parameter".

	Nonrelativistic	Relativistic
Classical	Classical mechanics	Special theory of relativity General theory of relativity Maxwell equations
Quantum	Quantum mechanics	Dirac equation Quantum field theory QED, QCD, etc. String theory (?)

The Question

Let the initial ($t = 0$) wave function be $\psi(x, t = 0)$.
How long would it take for $\psi(x, t)$ be "different significantly" from $\psi(x, 0)$?

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How long would it take for $\psi(x, t)$ be "different significantly" from $\psi(x, 0)$?

$$\int dx\, \psi^*(x, t)\psi(x, 0) \approx 0 \quad \text{for } t \gg \Delta t$$

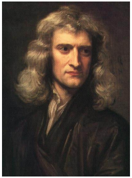
The Time-Energy Uncertainty Relation

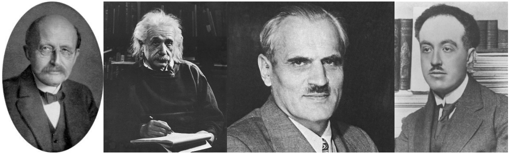
$$\boxed{\Delta t \, \Delta E \geq \frac{1}{2}\hbar}$$

Summary

The Uncertainty Family

(Newton, Planck, Einstein, Compton, de Broglie, Heisenberg)





$$E = \hbar\omega\,, \quad P = \hbar k$$

The wave-particle duality $\Longrightarrow$ The uncertainty principle

Images on the upper row from Wikipedia; lower courtesy of `http://www.pjcj.net/` & `http://www.canstockphoto.com/`, respectively.

Summary

1. Wave-particle duality
2. Fourier transformation
3. Position-momentum uncertainty relation
4. Time-energy uncertainty relation

References

M. Jammer, *Concepts of simultaneity: from antiquity to einstein and beyond*, (Johns Hopkins U. Press, Baltimore, 2006).