Mobile Communications (KECE425)

Lecture Note 25

6-9-2014

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Summary

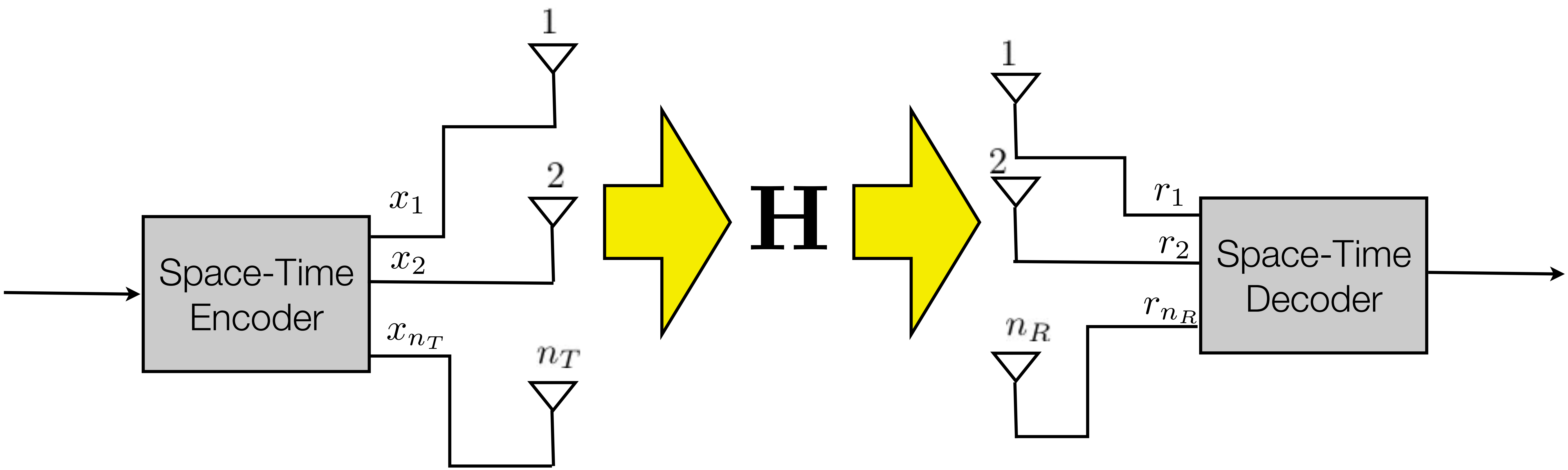
- Channel capacity of MIMO systems
- MIMO detection techniques
  - Maximum likelihood detection
  - Zero-forcing detection

- SVD for a matrix A is

$$A = UDV^H$$

  - Each column vector of V is the eigenvector of $A^H A$.
  - D is diagonal matrix with the square root of eigenvalues of $A^H A$ in its diagonal elements.
  - We can also show that each column vector of U is the eigenvector of AA^H.

MIMO for Spatial Multiplexing



- MIMO spatial multiplexing means the transmission of multiple data streams from the transmit antennas instead of signal data stream such as in diversity systems.

- Received signal over MIMO channels

$$
\begin{aligned}
r_1 &= h_{11}x_1 + h_{12}x_2 + \cdots + h_{1n_T}x_{n_T} + n_1 \\
r_2 &= h_{21}x_1 + h_{22}x_2 + \cdots + h_{2n_T}x_{n_T} + n_2 \\
&\vdots \\
r_{n_R} &= h_{n_R1}x_1 + h_{n_R2}x_2 + \cdots + h_{n_Rn_T}x_{n_T} + n_{n_R}
\end{aligned}
$$

or in vector form with the channel matrix H

$$\mathbf{r} = H\mathbf{x} + \mathbf{n}$$

where

$$
\begin{aligned}
\mathbf{r} &= [r_1\, r_2 \ldots r_{n_R}]^T, \\
\mathbf{x} &= [x_1\, x_2 \ldots x_{n_T}]^T, \\
\mathbf{n} &= [n_1\, n_2 \ldots n_{n_R}]^T
\end{aligned}
$$

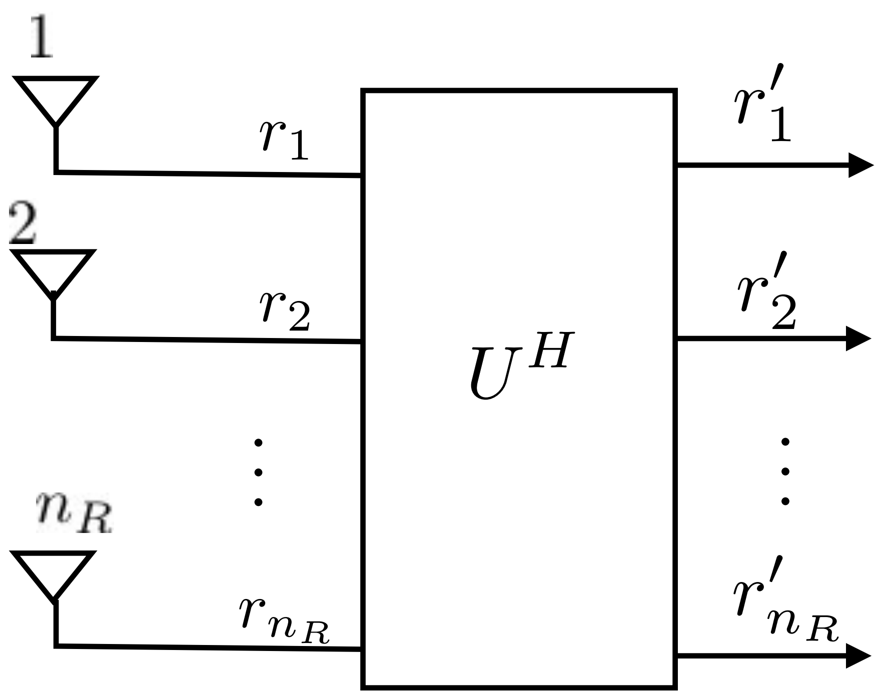
- MIMO channel matrix, H can be also factored by SVD such as:

$$H = UDV^H$$

- Then we can rewrite the received signal in vector form as

$$\mathbf{r} = UDV^H\mathbf{x} + \mathbf{n}$$

- At the receiver consider the following linear signal processing:



$$\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$$

where

$$\begin{aligned} \mathbf{r}' &= U^H\mathbf{r}, \\ \mathbf{x}' &= V^H\mathbf{x}, \\ \mathbf{n}' &= U^H\mathbf{n} \end{aligned}$$

where we assume that the receiver knows the channels, that is, H, by estimation, perfectly.

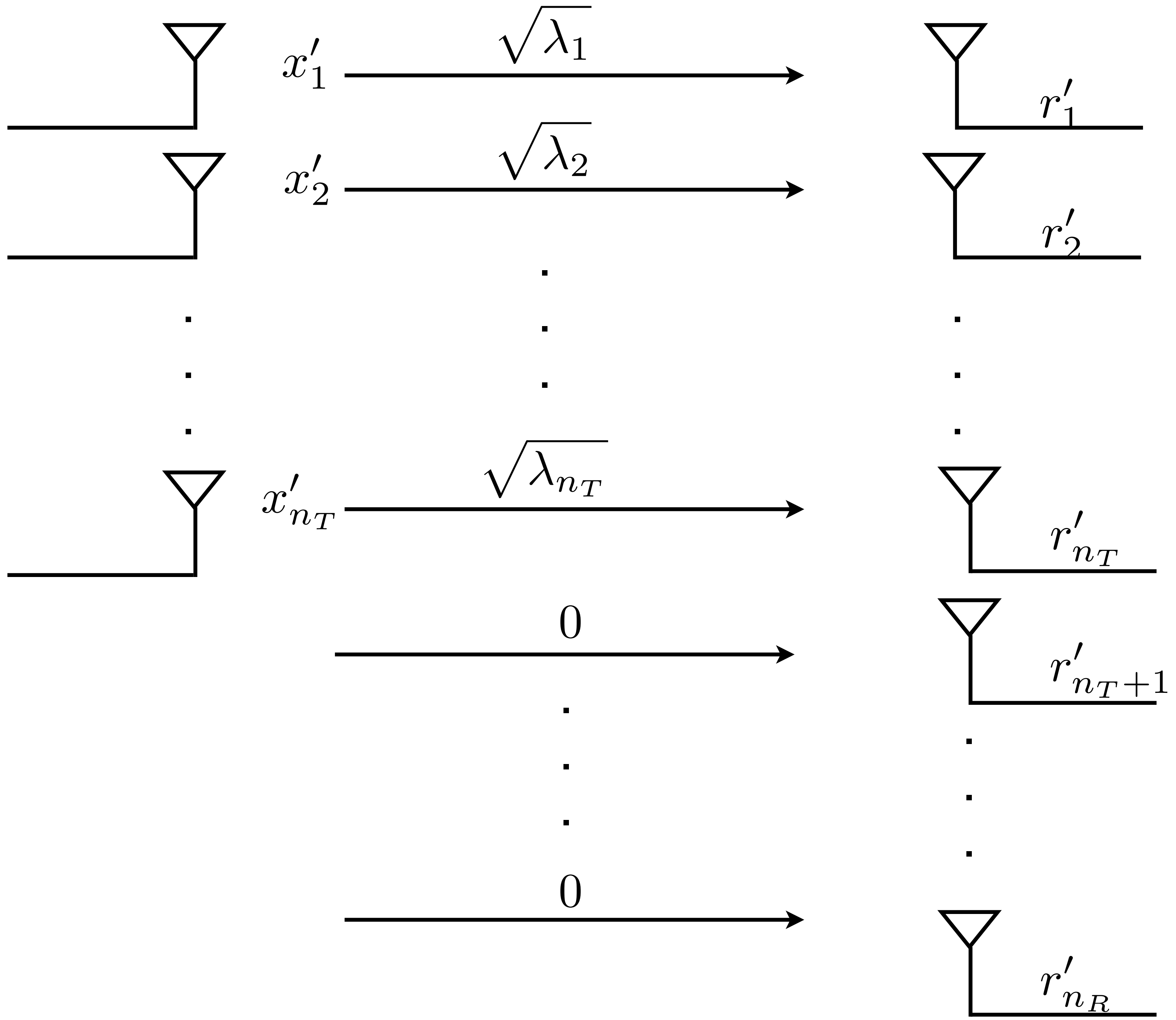
1) When $n_R > n_T$, $\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$ has the form of:

$$\begin{bmatrix} r'_1 \\ r'_2 \\ \vdots \\ r'_{n_T} \\ r'_{n_T+1} \\ \vdots \\ r_{n_R} \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_{n_T}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_{n_T} \end{bmatrix} + \begin{bmatrix} n'_1 \\ n'_2 \\ \vdots \\ n'_{n_T} \\ n'_{n_T+1} \\ \vdots \\ n'_{n_R} \end{bmatrix}$$

$*$ That is, we can write

$$\begin{aligned} r'_k &= \sqrt{\lambda_k} x'_k + n'_k, \quad \text{for } k = 1, 2, ..., n_T \\ r'_k &= n'_k, \quad \text{for } k = n_T + 1, ..., n_R \end{aligned}$$

- In this case, the MIMO channel can be modeled as n_T parallel channels with the channel coefficient $\sqrt{\lambda_k}$ for $k = 1, 2, ..., n_T$.



2) When $n_T > n_R$, $\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$ has the form of:

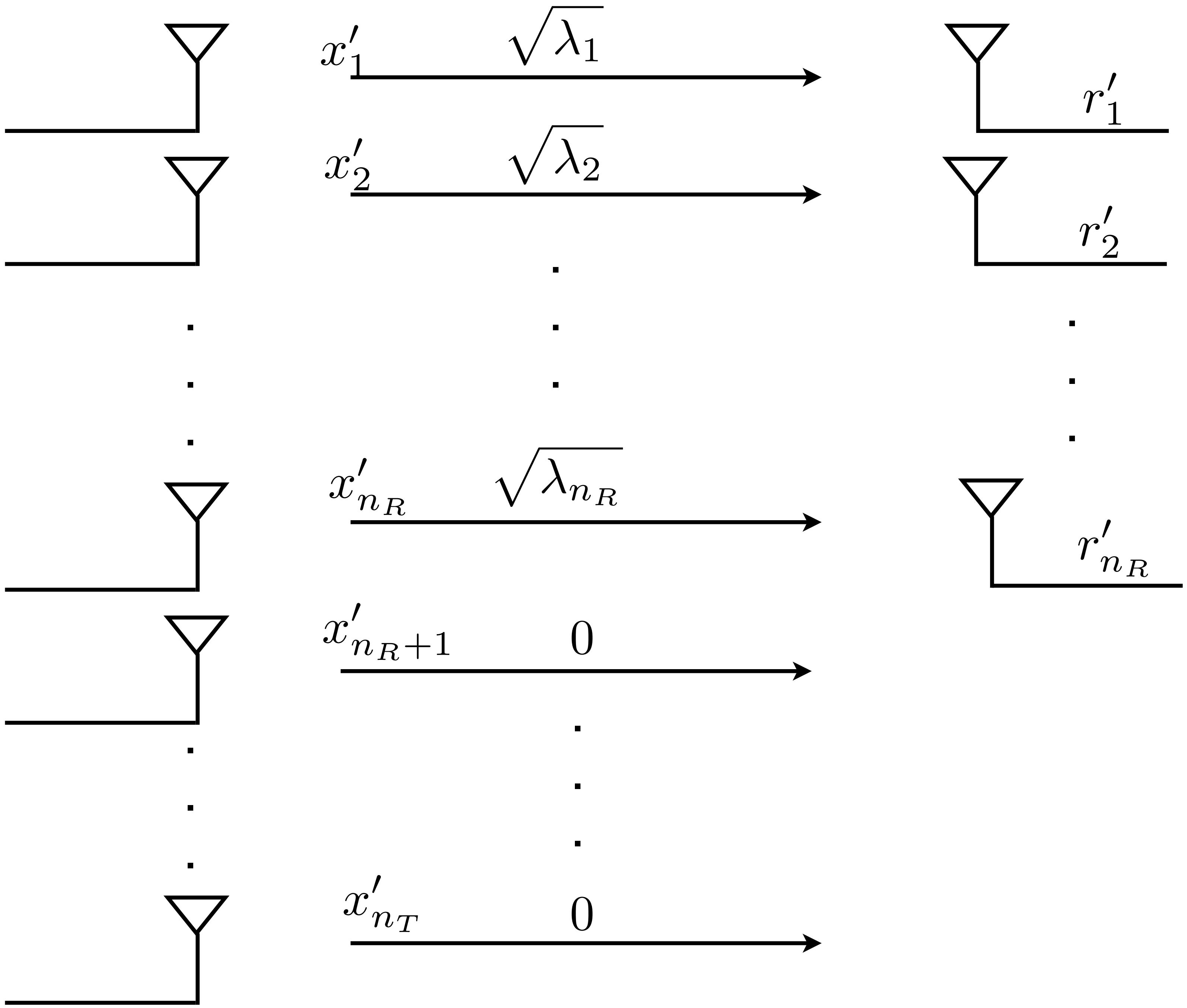
$$\begin{bmatrix} r'_1 \\ r'_2 \\ \vdots \\ r'_{n_R} \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_{n_R}} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_{n_R} \\ x'_{n_R+1} \\ \vdots \\ x'_{n_T} \end{bmatrix} + \begin{bmatrix} n'_1 \\ n'_2 \\ \vdots \\ n'_{n_R} \end{bmatrix}$$

* Then we have

$$r'_k = \sqrt{\lambda_k} x'_k + n'_k, \quad \text{for } k = 1, 2, ..., n_R$$

☐ Note that x'_k for $k \geq n_R + 1$ is not received, or we can say, located in the null space of the channel.

- In this case, the MIMO channel can be modeled as n_R parallel channels with the channel coefficient $\sqrt{\lambda_k}$ for $k = 1, 2, ..., n_R$.



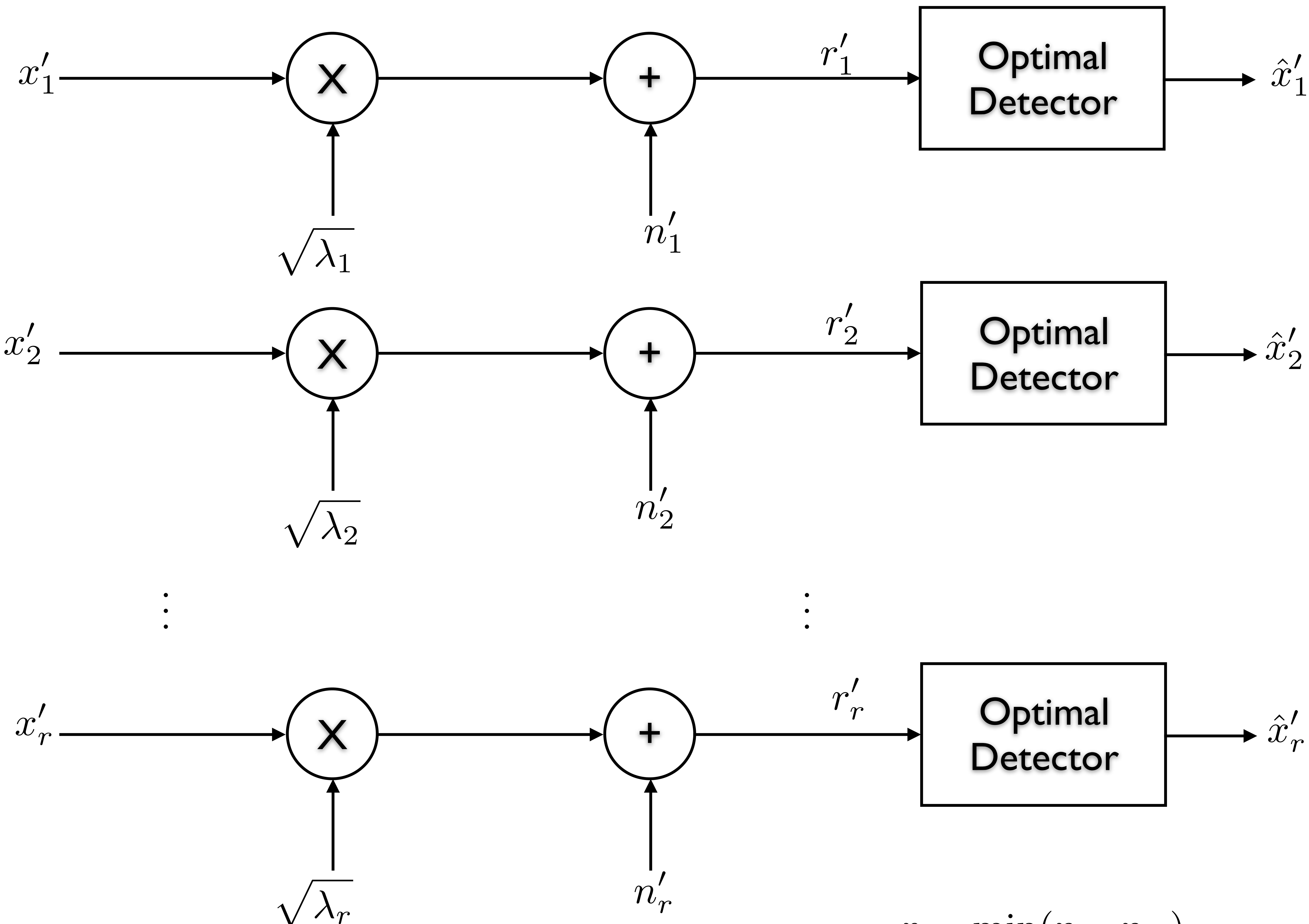
- Rank of a matrix is equal to the number of non-zero eigenvalues.
  - For a matrix of A with the size of $m \times n$, the rank r is given as

$$r = \min(m, n)$$

- Hence, the MIMO channel with n_T and n_R antennas at the transmitter and the receiver, respectively, the rank r is given as

$$r = \min(n_T, n_R)$$

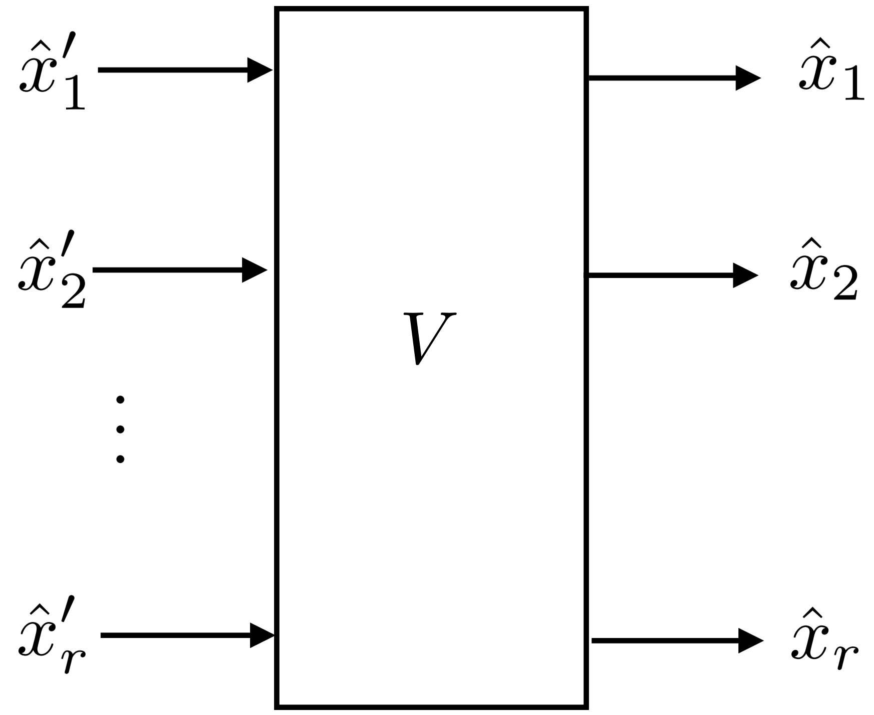
  - So we have r parallel channels.



x'_1
X
+
r'_1
Optimal Detector
$\hat{x}'_1$
$\sqrt{\lambda_1}$
n'_1
x'_2
X
+
r'_2
Optimal Detector
$\hat{x}'_2$
$\sqrt{\lambda_2}$
n'_2
x'_r
X
+
r'_r
Optimal Detector
$\hat{x}'_r$
$\sqrt{\lambda_r}$
n'_r

$$r = \min(n_T, n_R)$$

- Recall $\mathbf{x}' = V^H \mathbf{x}$.



  - Also recall $VV^H = I$.

$$\hat{\mathbf{x}} = V\hat{x}' = VV^H\mathbf{x} + V\mathbf{n}' = \mathbf{x} + \mathbf{n}'$$

Channel Capacity

- Maximum data rate with error-free communication

$$C \;=\; W\sum_{n=1}^{r}\log_2\left(1+\frac{\lambda_n P}{n_{n_T}N_0}\right) \qquad \text{[bits/sec]}$$

  - Note that $\{\lambda_n\}_{n=1}^{r}$ are the eigenvalues of $H^H H$.

  - Also note that λ_ns are random variable since the elements of H are random variable.

  - For the mean capacity, which we call it ergodic capacity, can be found as

$$\bar{C} = E[C] \;=\; W\sum_{n=1}^{r}\int_0^{\infty}\log_2\left(1+\frac{\lambda_n P}{n_{n_T}N_0}\right)p_{\lambda_n}(\lambda_n)\,d\lambda_n$$

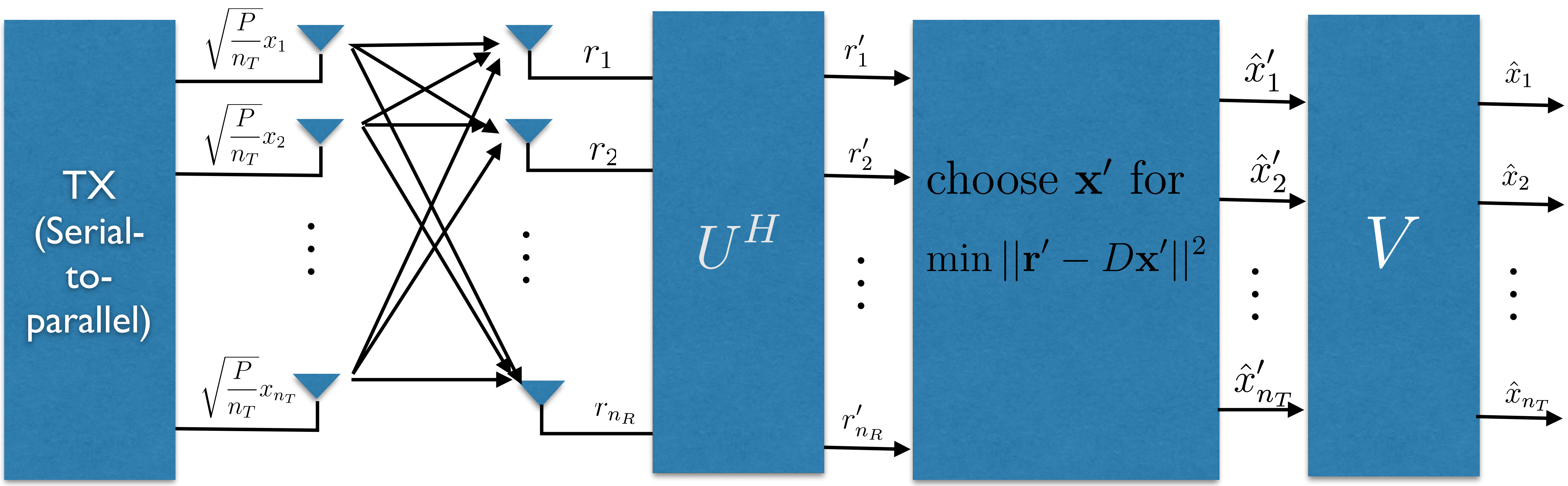
MIMO Detection

- Maximum likelihood detection (MLD)
- Zero-forcing detection (ZFD)
- Minimum mean square error detection (MMSED)
- Sphere decoding (SD)

- In our class, we only cover MLD and ZFD by examples.

Maximum-Likelihood Detection (MLD) for MIMO

- We only consider $n_R \geq n_T$.



$$\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_{n_T}]^T$$

$$||\mathbf{x}||^2 = 1$$

Total transmit power $= P$

- Example for $n_T = 3$ and $n_R = 3$ with the channel matrix H given as

$$H = \begin{bmatrix} 0.5761 + j0.6823 & 0.6459 + j0.6768 & 0.1969 + j0.1003 \\ 0.6405 + j0.1114 & 0.4471 + j0.3432 & 0.3867 + j0.2982 \\ 0.0898 + j0.6863 & 0.0690 + j0.5659 & 0.6771 + j0.6475 \end{bmatrix}$$

  - We can write $H = UDV^H$ which can be calculated as

$$U = \begin{bmatrix} -0.4399 - j0.4524 & -0.3912 - j0.4568 & 0.4897 - j0.0168 \\ -0.4175 - j0.2261 & -0.2101 + j0.3441 & -0.4080 + j0.6675 \\ -0.1631 - j0.5914 & 0.4312 + j0.5383 & 0.1415 - j0.3576 \end{bmatrix}$$

$$D = \begin{bmatrix} 1.9745 & 0 & 0 \\ 0 & 0.6601 & 0 \\ 0 & 0 & 0.2182 \end{bmatrix}$$

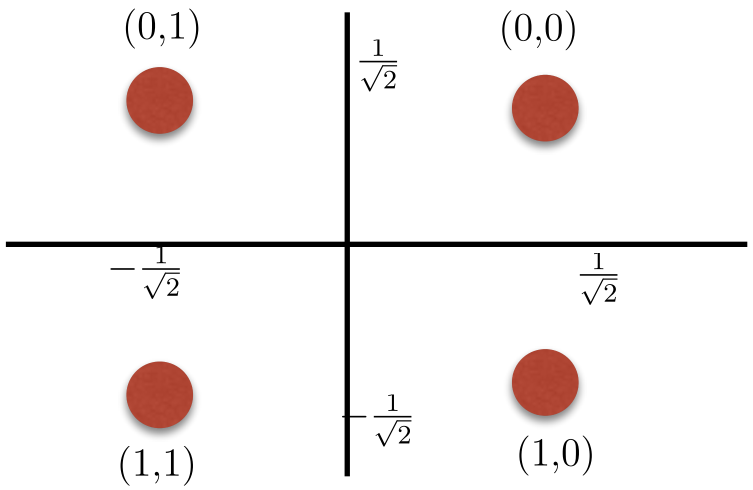
$$V = \begin{bmatrix} -0.6458 & -0.3411 & -0.6830 \\ -0.6080 + j0.0502 & -0.3081 - j0.0169 & 0.7288 - j0.0390 \\ -0.4326 - j0.1533 & 0.8165 + j0.3489 & 0.0013 - j0.0293 \end{bmatrix}$$

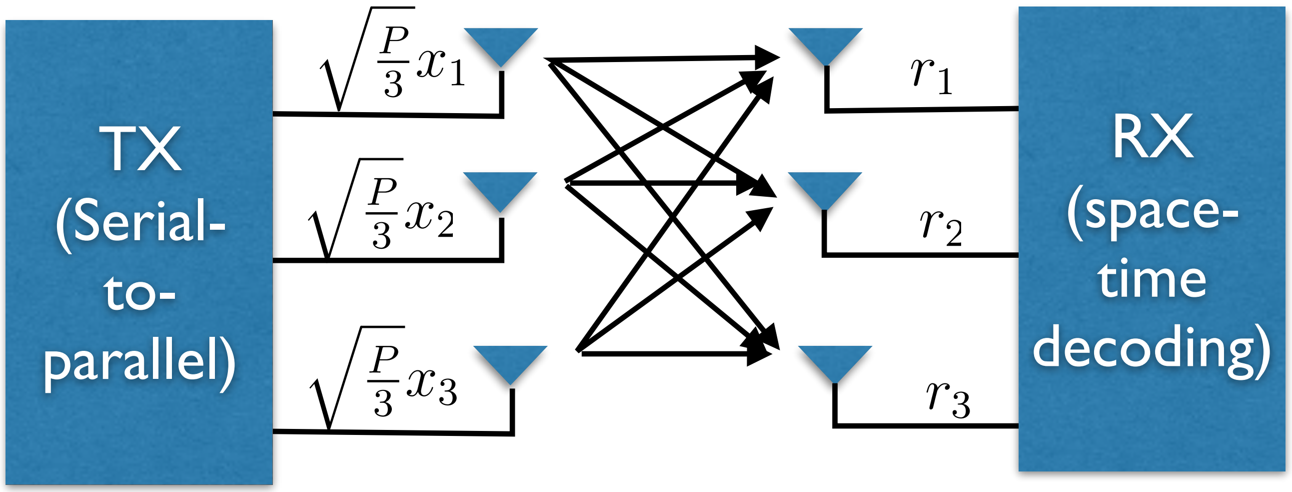
- Note that there are 64 possible cases for the symbol transmission with three antennas for QPSK.

$$\sqrt{2}\mathbf{x} = \begin{bmatrix} 1+j \\ 1+j \\ 1+j \end{bmatrix}, \begin{bmatrix} 1+j \\ 1+j \\ -1+j \end{bmatrix}, \begin{bmatrix} 1+j \\ 1+j \\ 1-j \end{bmatrix}, \begin{bmatrix} 1+j \\ 1+j \\ -1-j \end{bmatrix}$$

$$\vdots$$

$$= \begin{bmatrix} 1+j \\ -1-j \\ -1-j \end{bmatrix}, \begin{bmatrix} -1+j \\ -1-j \\ -1-j \end{bmatrix}, \begin{bmatrix} 1-j \\ -1-j \\ -1-j \end{bmatrix}, \begin{bmatrix} -1-j \\ -1-j \\ -1-j \end{bmatrix}$$





- Assume the transmit symbols in QPSK modulation are given as

$$\mathbf{x} = \sqrt{\frac{P}{3}} \begin{bmatrix} \frac{1+j}{\sqrt{2}} \\ \frac{-1+j}{\sqrt{2}} \\ \frac{1+j}{\sqrt{2}} \end{bmatrix}$$

  * where P is the total transmit signal power and the power to each antenna is allocated equally.

- Assume the noise vector at the receiver is given as

$$\mathbf{n} = \begin{bmatrix} 0.4456 - j0.3818 \\ 0.4482 + j0.2268 \\ 0.2123 + j0.5155 \end{bmatrix}$$

- Then the received vector $\mathbf{r}$ at the receiver with 3 antennas is

$$
\begin{aligned}
\mathbf{r} &= H\mathbf{x} + \mathbf{n} \\
&= \sqrt{\frac{P}{3}} \begin{bmatrix} 0.5761 + j0.6823 & 0.6459 + j0.6768 & 0.1969 + j0.1003 \\ 0.6405 + j0.1114 & 0.4471 + j0.3432 & 0.3867 + j0.2982 \\ 0.0898 + j0.6863 & 0.0690 + j0.5659 & 0.6771 + j0.6475 \end{bmatrix} \begin{bmatrix} (1+j)/\sqrt{2} \\ (-1+j)/\sqrt{2} \\ (1+j)/\sqrt{2} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0.4456 - j0.3818 \\ 0.4482 + j0.2268 \\ 0.2123 + j0.5155 \end{bmatrix} \\
&= \sqrt{\frac{P}{3}} \begin{bmatrix} -0.9421 + j1.0781 \\ -0.1221 + j1.0894 \\ -0.8498 + j1.1341 \end{bmatrix} + \begin{bmatrix} 0.4456 - j0.3818 \\ 0.4482 + j0.2268 \\ 0.2123 + j0.5155 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}' &= UH\mathbf{x} + U\mathbf{n} \\
&= \sqrt{\frac{P}{3}} \begin{bmatrix} -0.8008 - j2.0705 \\ 0.5207 - j0.0926 \\ -0.2281 + j0.0057 \end{bmatrix} + \begin{bmatrix} -0.6012 + j0.4177 \\ 0.3530 + j0.2590 \\ 0.0388 - j0.4223 \end{bmatrix}
\end{aligned}
$$

- Just for simplicity, let us assume that $P = 3$. Then,

$$\mathbf{r}' = \begin{bmatrix} -0.8008 - j2.0705 \\ 0.5207 - j0.0926 \\ -0.2281 + j0.0057 \end{bmatrix} + \begin{bmatrix} -0.6012 + j0.4177 \\ 0.3530 + j0.2590 \\ 0.0388 - j0.4223 \end{bmatrix}$$

$$= \begin{bmatrix} -1.4020 - j1.6528 \\ 0.8737 + j0.1664 \\ -0.1893 - j0.4166 \end{bmatrix}$$

$$= D\mathbf{x}' + \mathbf{n}'$$

where $\mathbf{x}' = V^H\mathbf{x}$.

- At the receiver, we have

$$\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$$

$$= \begin{bmatrix} \sqrt{\lambda_1}x_1' \\ \sqrt{\lambda_2}x_2' \\ \sqrt{\lambda_3}x_3' \end{bmatrix} + \begin{bmatrix} n_1' \\ n_2' \\ n_3' \end{bmatrix} = \mathbf{x}'' + \mathbf{n}'$$

where $\mathbf{x}'' = D\mathbf{x}' = DV^H\mathbf{x}$.

* There are also 64 possible cases for $\mathbf{x}''$. For example, $\mathbf{x} = \left[\frac{1+j}{\sqrt{2}}, \frac{1-j}{\sqrt{2}}, \frac{1+j}{\sqrt{2}}\right]^T$, we have

$$\mathbf{x}'' = DV^H \begin{bmatrix} \frac{1+j}{\sqrt{2}} \\ \frac{1-j}{\sqrt{2}} \\ \frac{1+j}{\sqrt{2}} \end{bmatrix}$$

- ML decision rule

$$\text{Choose } k \text{ to give} \quad \min_{k=1,2,\ldots,64} ||\mathbf{r}' - \mathbf{x}''_k||^2$$

* From my calculation by my Matlab program, I have found $\mathbf{x}'_k$ to give the smallest value of metric:

$$\hat{\mathbf{x}}'' = \begin{bmatrix} -0.8008 - j2.0705 \\ 0.5207 - j0.0926 \\ -0.2281 + j0.0057 \end{bmatrix}$$

* Finally, we can obtain $\hat{\mathbf{x}}$ by

$$\hat{\mathbf{x}} = V^H D^{-1} \hat{\mathbf{x}}'' = \begin{bmatrix} 0.7071 + j0.7071 \\ -0.7071 + j0.7071 \\ 0.7071 + j0.7071 \end{bmatrix} \Longrightarrow \text{No error occurs.}$$

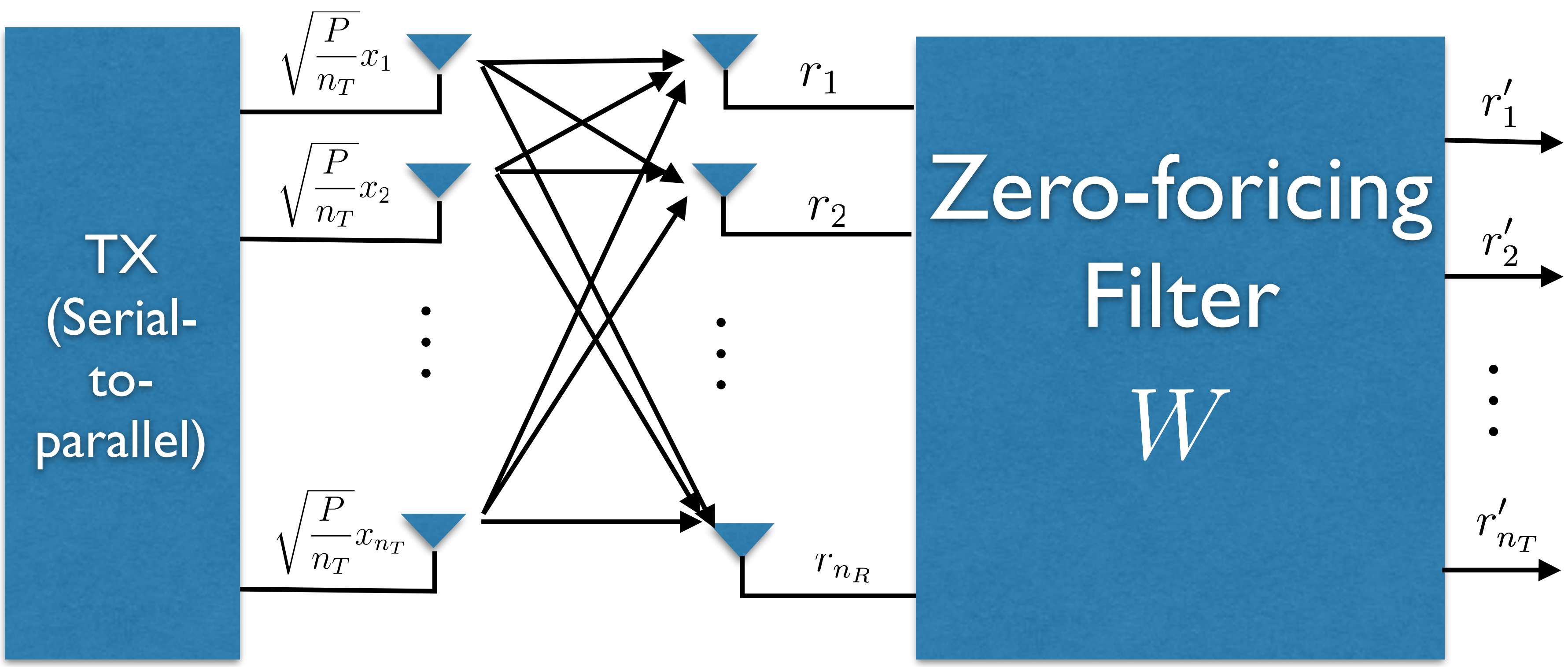
Remarks on MLD

- MLD is known for optimum decision rule for MIMO detection.
- However, it requires too much computations.
  - For M-ary modulation with $r = \min(n_T, n_R)$ case, we need the metric calculation as many as M^r.
    - For 16-QAM with 8 antennas, we need $16^8 = 4295 \times 10^6$.
- Hence, many other sub optimum algorithms have been developed such as sphere detection (SD), minimum mean-square error detection (MMSED), and zero-forcing detection (ZFD).
  - Among these, we only study ZFD in this semester.

Zero-Forcing MIMO Detection

- Block diagram of ZFD

$$\mathbf{r}' = W\mathbf{r}$$



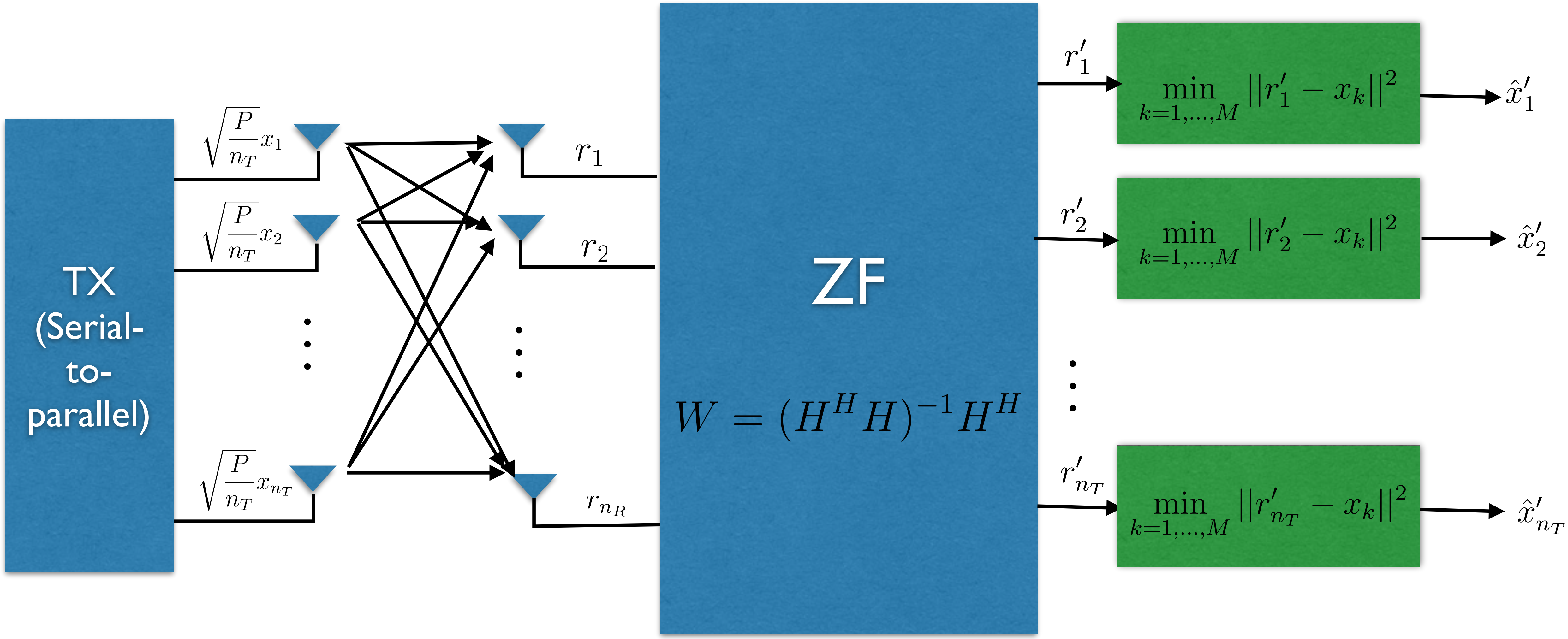
$$\mathbf{r} = H\mathbf{x} + n$$

Let us design W such that $W\mathbf{r} = \mathbf{x} + \mathbf{n}'$.

We can simply let $W = (H^H H)^{-1} H^H$.

Then, we have $W\mathbf{r}$ as

$$\begin{aligned} W\mathbf{r} &= \underbrace{(H^H H)^{-1} H^H H}_{= I}\mathbf{x} + W\mathbf{n} \\ &= \mathbf{x} + \mathbf{n}' \end{aligned}$$



$$r'_k = x_k + n'_k, \quad k = 1, 2, \dots, n_T$$

Total computation of the metric $\min_{k=1,\dots M} ||r_j - x_k||^2 = M \times n_T$.

- In our previous example, zero-forcing filter W can be calculated as

$$W = (H^H H)^{-1} H^H = \begin{bmatrix} -1.1869 - j0.4365 & 1.5224 + j2.1935 & -0.6125 - j1.0348 \\ 1.9568 - j0.3853 & -1.2703 - j2.0709 & 0.3569 + j1.2230 \\ -0.5887 + j0.2277 & -0.0610 - j0.5031 & 0.9483 - j0.5717 \end{bmatrix}$$

- Then the signal output is

$$\mathbf{r}' = W\mathbf{r} = \begin{bmatrix} 0.5998 + j1.7588 \\ -0.6366 - j0.9844 \\ 1.1145 + j1.1615 \end{bmatrix}$$

$$\hat{x}_1 = (1+j)/\sqrt{2}$$

$$\hat{x}_2 = (-1-j)/\sqrt{2} \longrightarrow \text{error}$$

$$\hat{x}_3 = (1+j)/\sqrt{2}$$