Quantum Mechanics 1

Assignment 1

Due: March 19 (Tuesday), 2013

1. Some students may have wondered if the choice of arbitrary boudary conditions did not really matter in computing the number of states for blackbody radiation. Consider a cubic blackbody with sides L, and apply the fixed-end boundary conditions. That is, the electric fields are zero at the surface. By following the methods used in class, show that the number of states between ν and $\nu + d\nu$ is the same as the case with the periodic boudary conditions.

   Hint: The electromagnetic wave can be written in the form $A \sin k_1 x \sin k_2 y \sin k_3 z$. And since $\sin(-k_1 x) = -\sin k_1 x$ is not an independent function, only the positive numbers of k_1 should be included.

2. In class, we derived the energy density $u(\nu, T)$ of a blackbody à la Planck. Let us consider a two-dimensional blackbody.

   (a) Obtain the energy density $u_2(\nu, T)$ of the two-dimensional blackbody. Assume that there are two possible polarizations for the electromagnetic waves even in two dimensions.

   (b) The total energy $U_2(T)$ can be written as AT^b. Obtain A and b.

3. *Taste of statistical mechanics*

   In the derivation of the Planck's formula, we have computed the average energy $\langle E_n \rangle = \langle n \rangle h\nu$ for a given frequency ν at temperature T.

   (a) Compute the energy fluctuation defined by

$$\Delta E_n = \sqrt{\langle E_n^2 \rangle - \langle E_n \rangle^2}. \tag{1}$$

   (b) Write ΔE_n in the low-frequency limit, and in the high-frequency limit.

4. We briefly reviewed the Bohr's model of a hydrogen atom. From spectroscopy, the wavelength λ of the absorbed line spectrum from the state n_1 to the state n_2 is given by

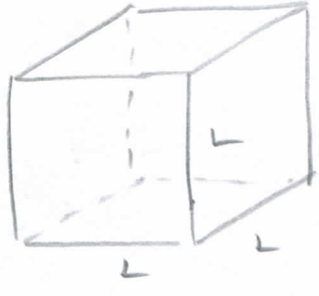
$$\frac{1}{\lambda} = \mathrm{Ry.}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \tag{2}$$

where Ry. is called the Rydberg constant. By deriving it carefully again, obtain the Rydberg constant and express it in SI units.

1. Some students may have wondered if the choice of arbitrary boundary conditions did not really matter in computing the number of states for blackbody radiation. Consider a cubic blackbody with sides L, and apply the fixed-end boundary conditions. That is, the electric fields are zero at the surface. By following the methods used in class, show that the numbers of states between ν and $\nu+d\nu$ is the same as the case with the periodic boundary conditions.

Hint: The electromagnetic wave can be written in the form $A \sin k_1 x \sin k_2 y \sin k_3 z$. And since $\sin(-k_1 x) = -\sin k_1 x$ is not an independent function, only the positive numbers of k_1 should be included.

⇒ 1)



$x=0$, $y=0$, $z=0$ 에서..
field가 0이 되는 것은.. 자동 만족.

$x=L$, $y=L$, $z=L$ 에서..
fixed boundary condition을 만족하려면..

$$k_1 L = n_1 \pi, \quad k_2 L = n_2 \pi, \quad k_3 L = n_3 \pi$$

이고 n_1, n_2, n_3는 자연수여야 함.

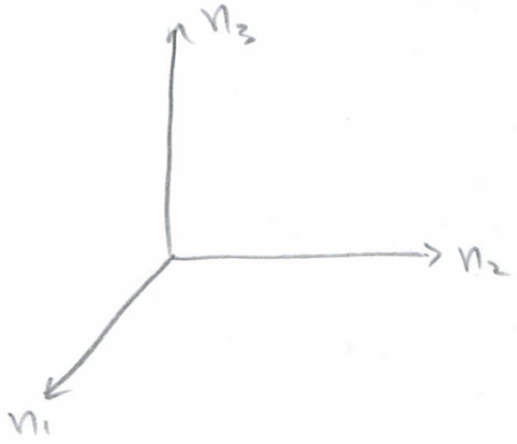
$\therefore\ k_1 = \frac{n_1 \pi}{L},\quad k_2 = \frac{n_2 \pi}{L},\quad k_3 = \frac{n_3 \pi}{L}$

where.. $n_1, n_2, n_3 \in \mathbb{N}$.

ii) 이제.. 특정 (n_1, n_2, n_3) 는.. electromagnetic field의 진행 방향을 나타내게 된다.

(그 wave vector는.. (k_1, k_2, k_3))

그렇다면.. 특정 wave 의 진행방향은 다음 n-space 상의 한 점으로 표현가능하다.



okay.. $n_1, n_2, n_3 > 0$ 이고 n_1, n_2, n_3가 자연수인 n-space 상 한 점은 field의 진행방향을 나타내게된다.

n_1, n_2, n_3는 모두 자연수이므로.. n-space 부피 1당 하나의 진행 방향 vector가 대응된다.

iii) $n \equiv \sqrt{n_1^2 + n_2^2 + n_3^2}$ 이라하면..

n 과 n+dn 사이 state 개수는.. 대략적으로

(n-space 부피 1당.. 진행 방향 vector 1개)

$\sim \frac{1}{8} \cdot 4\pi n^2 \cdot dn \times 2 \times 2$

($n_1, n_2, n_3 > 0$)

진행 방향 vector의 개수 (n ~ n+dn 사이)

각 진행 방향에.. electric field, magnetic field 존재

각 electric, magnetic field 는 진행 방향에 수직한 두 방향으로 진동.

iii) $k_1 = \frac{n_1\pi}{L}$, $k_2 = \frac{n_2\pi}{L}$ $k_3 = \frac{n_3\pi}{L}$.

이므로.. $k \equiv \sqrt{k_1^2+k_2^2+k_3^2}$ 이라 하면..

$k = \left(\frac{\pi}{L}\right)\cdot n$ 이라고 할 수 있다.

따라서.. $k \sim k+dk$ 에 들어 있는 state 갯수는..

$$\frac{1}{8}\cdot 4\pi\left(\frac{L}{\pi}\right)^2 k^2\cdot dk\cdot\left(\frac{L}{\pi}\right) \times 2 \times 2$$

iv) electromagnetic field 의 dispersion relation 은..

$\omega = ck$ 이고.. $\omega = 2\pi\nu$ 이다

$\therefore 2\pi\nu = ck$ $k = \frac{2\pi}{c}\nu$.

v) 따라서. $\nu \sim \nu+d\nu$ 에 들어있는 state 갯수는..

$$\frac{1}{8}\cdot 4\pi\left(\frac{L}{\pi}\right)^2\cdot\left(\frac{2\pi}{c}\right)^2\nu^2\cdot d\nu\cdot\left(\frac{2\pi}{c}\right)\cdot\left(\frac{L}{\pi}\right)\cdot 2\times 2$$

$$= 16\pi\cdot\left(\frac{L}{\pi}\right)^3\cdot\left(\frac{\pi}{c}\right)^3\cdot\nu^2 d\nu$$

$$= \frac{16\pi\nu^2 L^3}{c^3}\, d\nu.$$

↳ Periodic B.C. 과 완전히 동일!

2. In class, we derived the energy density $u(\nu, T)$ of a blackbody à la Planck. Let us consider a two-dimensional blackbody.

(a) Obtain the energy density $u_2(\nu, T)$ of the two-dimensional blackbody. Assume that there are two possible polarizations for the electromagnetic waves even in two dimensions.

→ i) The energy density per frequency

$$= \frac{1}{L^2} \times \left(\begin{array}{c} \text{\# of states} \\ \text{per frequency} \end{array} \right) \times \underbrace{\left(\begin{array}{c} \text{Thermal equilibrium 에서} \\ \text{frequency } \nu \text{ 인 빛이} \\ \text{가지는 평균 에너지} \end{array} \right)}_{\langle E \rangle}$$

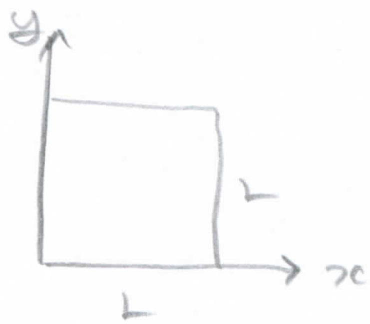
unit

전체 공간에 대해 density 고려하므로

ii) # of states per frequency

unit

• periodic boundary condition 사용



$\sim e^{ik_1 x} e^{ik_2 y}$

boundary condition

$e^{ik_1 L} = e^0 = 1$

$e^{ik_2 L} = e^0 = 1.$

$\therefore k_1 L = 2\pi n_1$

$k_2 L = 2\pi n_2$

(n_1, n_2는 정수)

- n-space 넓이 1당 진행방향 vector 1개.

$$\downarrow \quad k = \frac{2\pi}{L} n$$

k-space 넓이 1당 진행방향 vector $\left(\frac{2\pi}{L}\right)^2$개.

$$\downarrow \quad \nu = \frac{c}{2\pi} k$$

$$\left(\begin{array}{c} \nu \sim \nu + d\nu \text{ 사이} \\ \text{진행 방향 vector} \\ \text{갯수} \end{array} \right) = \frac{2\pi k \cdot dk}{\left(\frac{2\pi}{L}\right)^2} = \frac{2\pi \left(\frac{2\pi}{c}\right) \nu \left(\frac{2\pi}{c}\right) \cdot d\nu}{\left(\frac{2\pi}{L}\right)^2}$$

$$= \frac{2\pi L^2 \cdot \nu}{c^2} d\nu$$

- 이번에도 역시 polarization 에 의해 특정 진행방향에 대해 2개의 state 존재.

하지만.. Planck 의 가정에는.. 전기장과 자기장이 각각 energy를 가져가지 않고 전자기장 자체가 $h\nu$ 에너지를 가져감.

$$\therefore \left(\begin{array}{c} \text{\# of states} \\ \text{per unit frequency} \end{array} \right) = \frac{4\pi L^2}{c^2} \nu$$

iii) Planck의 가정에 의하면.. 주파수 ν인 빛(전자기장)은

오로지 $h\nu$의 정수배인 에너지를 가질수 밖에 없다.

$$\therefore \langle E \rangle = \sum_{n=0}^{\infty} nh\nu \cdot \left[\frac{\exp\left[-\frac{nh\nu}{k_B T}\right]}{\underbrace{\sum_{l=0}^{\infty} \exp\left[-\frac{lh\nu}{k_B T}\right]}} \right]$$

주파수가 ν인 빛이 가지는 평균에너지 in thermal equilibrium

thermal equilibrium 상태에 있다면.. 에너지 분포는 Boltzmann 분포를 따른다

$$\sum_{l=0}^{\infty} \exp\left(-\frac{lh\nu}{k_B T}\right) = \frac{1}{1-\exp\left(-\frac{h\nu}{k_B T}\right)}$$

$$\therefore \langle E \rangle = \sum_{n=0}^{\infty} nh\nu \cdot \exp\left(-\frac{nh\nu}{k_B T}\right) \times \left[1-\exp\left(-\frac{h\nu}{k_B T}\right)\right]$$

$$= \left[1-\exp\left(-\frac{h\nu}{k_B T}\right)\right] \cdot \sum_{n=0}^{\infty} nh\nu \exp\left[-\frac{nh\nu}{k_B T}\right].$$

※ $\frac{1}{k_B T} = \beta$ 라고 하면..

$$\sum_{n=0}^{\infty} nh\nu \exp\left[-\frac{nh\nu}{k_B T}\right] = \sum_{n=0}^{\infty} nh\nu \exp[-nh\nu \cdot \beta]$$

$$= \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial \beta}\right) \exp[-nh\nu\beta]$$

$$= \left(-\frac{\partial}{\partial \beta}\right) \sum_{n=0}^{\infty} \exp[-nh\nu\beta]$$

$$= -\frac{\partial}{\partial \beta} \cdot \frac{1}{1-\exp(-h\nu\beta)}$$

$$= + \frac{-\exp(-h\nu\beta)\cdot(-h\nu)}{\left(1-\exp(-h\nu\beta)\right)^2} = \frac{\exp(-h\nu\beta)\cdot h\nu}{[1-\exp(-h\nu\beta)]^2}$$

$$\therefore \langle E \rangle = \left[1-\exp\left(-\frac{h\nu}{k_B T}\right)\right] \cdot \frac{\exp\left(-\frac{h\nu}{k_B T}\right) h\nu}{\left[1-\exp\left(-\frac{h\nu}{k_B T}\right)\right]^2}$$

$$= \frac{\exp\left(-\frac{h\nu}{k_B T}\right) h\nu}{1-\exp\left(-\frac{h\nu}{k_B T}\right)} = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}.$$

iv)

$$\therefore u_2(\nu, T) = \frac{1}{L^2} \cdot \frac{4\pi L^2 \nu}{c^2} \cdot \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

$$= \frac{4\pi h}{c^2} \frac{\nu^2}{\exp(h\nu/k_B T) - 1}$$

(b) The total energy $U_2(T)$ can be written as AT^b. Obtain A and b.

$\Rightarrow$ i) $$U_2(T) = \int_0^\infty d\nu \, u_2(\nu, T)$$

$$= \frac{4\pi h}{c^2} \cdot \int_0^\infty d\nu \cdot \frac{\nu^2}{\exp(h\nu/k_B T) - 1}$$

$x = \frac{h\nu}{k_B T}$

$$= \frac{4\pi h}{c^2} \cdot \left(\frac{k_B T}{h}\right)^3 \cdot \underbrace{\int_0^\infty dx \frac{x^2}{e^x - 1}}_{A}$$

ii) Ⓐ를 구해보자.

$$Ⓐ = \int_0^\infty dx \; e^{-x} \cdot \frac{x^2}{1-e^{-x}}$$

$x>0$ 이므로. $0<e^{-x}<1$

$$= \int_0^\infty dx \; e^{-x} \cdot x^2 \cdot \sum_{n=0}^{\infty} e^{-nx}$$

$$= \sum_{n=0}^{\infty} \underbrace{\int_0^\infty dx \; e^{-(n+1)x} \, x^2}_{\text{integration by parts.}}$$

$$= \sum_{n=0}^{\infty} \frac{2}{(n+1)^3} = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{(n+1)^3}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^3} = 2\,\zeta(3)$$

↳ Zeta function.

iii) $\therefore U_2(T) = \frac{4\pi h}{c^2} \cdot \left(\frac{k_B T}{h}\right)^3 \cdot 2\zeta(3)$

$$= \frac{8\pi k_B^3 \zeta(3)}{c^2 h^2} \cdot T^3.$$

$$\therefore A = \frac{8\pi k_B^3 \zeta(3)}{c^2 h^2}, \quad b = 3$$

3. Taste of statistical mechanics.

In the derivation of the Planck's formula, we have computed the average energy $\langle E_n \rangle = \langle n \rangle h\nu$ for a given frequency ν at temperature T.

(a) Compute the energy fluctuation defined by..

$$\Delta E_n = \sqrt{\langle E_n^2 \rangle - \langle E_n \rangle}$$

i) $$\langle E \rangle = \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}.$$

ii) $$\langle E^2 \rangle = \sum_{n=0}^{\infty} (nh\nu)^2 \cdot \underbrace{\exp\left[-\frac{nh\nu}{k_B T}\right]\left[1 - \exp\left(-\frac{h\nu}{k_B T}\right)\right]}_{\text{Boltzmann 확률}}$$

$$= \left[1 - \exp\left(-\frac{h\nu}{k_B T}\right)\right] \cdot \sum_{n=0}^{\infty} (nh\nu)^2 \cdot \exp(-nh\nu\beta)$$

$$\left(\text{where } \beta = \frac{1}{k_B T} \right)$$

$$= \left[1 - \exp\left(-\frac{h\nu}{k_B T}\right)\right] \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial \beta}\right)^2 \exp(-nh\nu\beta)$$

$$= \left[1 - \exp\left(-\frac{h\nu}{k_B T}\right)\right] \cdot \left(-\frac{\partial}{\partial \beta}\right)^2 \underbrace{\sum_{n=0}^{\infty} \exp(-nh\nu\beta)}_{B}$$

iii) Ⓑ $= \left(-\frac{\partial}{\partial\beta}\right)^2 \cdot \frac{1}{1-\exp(-h\nu\beta)}$

$$= \left(-\frac{\partial}{\partial\beta}\right) \cdot \left\{ \frac{\exp(-h\nu\beta)\, h\nu}{[1-\exp(-h\nu\beta)]^2} \right\}$$

$$= \frac{(h\nu)^2 \exp(-h\nu\beta)}{[1-\exp(-h\nu\beta)]^2}$$

$$+ \exp(-h\nu\beta) \cdot (h\nu)^2 \frac{2\exp(-h\nu\beta)}{[1-\exp(-h\nu\beta)]^3}$$

$$= \frac{(h\nu)^2}{[1-\exp(-h\nu\beta)]^3} \Big[\exp(-h\nu\beta) \cdot [1-\exp(-h\nu\beta)] + 2\exp(-2h\nu\beta) \Big]$$

$$= \frac{(h\nu)^2 \exp(-h\nu\beta)}{[1-\exp(-h\nu\beta)]^3} \left(1 + \exp(-h\nu\beta) \right)$$

$$\therefore \langle E^2 \rangle = \frac{(h\nu)^2 \exp(-h\nu\beta)\, [1+\exp(-h\nu\beta)]}{[1-\exp(-h\nu\beta)]^2}$$

$$= \frac{(h\nu)^2 [\exp(h\nu\beta)+1]}{[\exp(h\nu\beta)-1]^2}$$

iv)

$$\therefore \langle E^2\rangle - \langle E\rangle^2 = \frac{(h\nu)^2\left[\exp(h\nu\beta)+1\right]}{\left[\exp(h\nu\beta)-1\right]^2}$$

$$-(h\nu)^2 \cdot \frac{1}{\left[\exp(h\nu\beta)-1\right]^2}$$

$$= (h\nu)^2 \cdot \frac{\exp(h\nu\beta)}{\left[\exp(h\nu\beta)-1\right]^2}$$

v)

$$\therefore \Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$$

$$= h\nu \cdot \frac{\exp\left(\frac{h\nu}{2k_BT}\right)}{\exp\left(\frac{h\nu}{k_BT}\right)-1}$$

(b) Write ΔE_n in the low-frequency limit, and in the high-frequency limit.

i) $$\Delta E = h\nu \cdot \frac{\exp\left(\frac{h\nu}{2k_BT}\right)}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}$$

ii) $\nu \ll 1$ 일때.. $\exp(x) = 1 + x + O(x^2)$

$$\therefore \Delta E \simeq h\nu \cdot \frac{1 + \frac{h\nu}{2k_BT}}{1 + \frac{h\nu}{k_BT} - 1} = k_BT\left(1 + \frac{h\nu}{2k_BT}\right)$$

$$= k_BT + \frac{1}{2}h\nu + O(\nu^2) \xrightarrow{\nu \ll 1} k_BT$$

iii) $\nu \gg 1$ 일때. $\exp\left(\frac{h\nu}{k_BT}\right) - 1 \simeq \exp\left(\frac{h\nu}{k_BT}\right)$

$$\therefore \Delta E \simeq h\nu \cdot \frac{\exp\left(\frac{h\nu}{2k_BT}\right)}{\exp\left(\frac{h\nu}{k_BT}\right)} = h\nu \cdot \exp\left(-\frac{h\nu}{2k_BT}\right)$$

$$\xrightarrow{\nu \gg 1} 0.$$

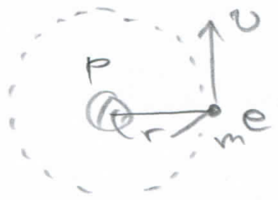
4. We briefly reviewed the Bohr's model of a hydrogen atom. From spectroscopy, the wavelength λ of the absorbed line spectrum from the state n_1 to the state n_2 is given by

$$\frac{1}{\lambda} = R_y \cdot \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right),$$

where R_y is called the Rydberg constant. By deriving it carefully again, obtain the Rydberg constant and express it in SI units.

⇒ i) Bohr model 에서는.. 핵을 중심으로 전자가 한 평면 상에서 등속 원운동 한다고 가정.

ii) 그리고 핵 주위로 도는 전자의 각운동량은 quantize 되어 있다.



$$mvr = n\hbar$$ (n은 자연수)

iii) 전자가 n_1 orbital 에서.. n_2 orbital로 이동할 때.. 광자를 흡수하거나 방출하는데..

그 광자의 에너지 $h\nu$ 는..

두 orbital 에서 전자가 가진 total energy의 차이 이다. $\therefore h\nu = E_{n_2} - E_{n_1}$

iv) i) ~ iii) 의 가정으로 R_y. 를 구해보자.

$$E_{n_1} = \frac{1}{2} m v_1^2 - \frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r_1}$$

$$E_{n_2} = \frac{1}{2} m v_2^2 - \frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r_2}$$

$n_2 > n_1$

이때. 전자는 전자기력을 구심력으로.. 등속원운동을 하므로..

$$\frac{m v_1^2}{r_1} = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_1^2} \rightarrow m v_1^2 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_1}$$

$$\frac{m v_2^2}{r_2} = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_2^2} \rightarrow m v_2^2 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_2}$$

$\therefore$

$$\left[\begin{array}{l} E_{n_1} = -\dfrac{e^2}{8\pi\varepsilon_0} \cdot \dfrac{1}{r_1} \\[2ex] E_{n_2} = -\dfrac{e^2}{8\pi\varepsilon_0} \cdot \dfrac{1}{r_2} \end{array}\right.$$

ii) 이제 각운동량 quantization 가정을 쓰자.

$$m v_1 r_1 = n_1 h \rightarrow v_1 = \frac{n_1 h}{m r_1}$$

$$m v_2 r_2 = n_2 h \rightarrow v_2 = \frac{n_2 h}{m r_2}$$

$$\therefore \; m v_1^2 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_1}$$

대입.

$$m v_2^2 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_2}$$

$$m\left(\frac{n_1 h}{m r_1}\right)^2 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_1}$$

$$m\left(\frac{n_2 h}{m r_2}\right)^2 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r_2}$$

$$\frac{1}{r_1} = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{m}{h^2} \cdot \frac{1}{n_1^2}$$

$$\frac{1}{r_2} = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{m}{h^2} \cdot \frac{1}{n_2^2}$$

vii)

$$\therefore E_{n_1} = -\frac{e^2}{8\pi\varepsilon_0}\cdot\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{m}{h^2}\cdot\frac{1}{n_1^2}$$

$$E_{n_2} = -\frac{e^2}{8\pi\varepsilon_0}\cdot\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{m}{h^2}\cdot\frac{1}{n_2^2}$$

$$E_{n_1} = -\frac{1}{2}m\left(\frac{e^2}{4\pi\varepsilon_0 h}\right)^2\frac{1}{n_1^2}$$

$$E_{n_2} = -\frac{1}{2}m\left(\frac{e^2}{4\pi\varepsilon_0 h}\right)^2\cdot\frac{1}{n_2^2}$$

viii) 이제 마지막 과정..

$$h\nu = E_{n_2} - E_{n_1}$$

이때.. $\nu = \frac{c}{\lambda}$ 이므로.

$$\frac{hc}{\lambda} = E_{n_2} - E_{n_1}$$

$$\frac{1}{\lambda} = \underbrace{\frac{1}{hc}\cdot\frac{1}{2}m\left(\frac{e^2}{4\pi\varepsilon_0 h}\right)^2}_{R_y.}\cdot\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$(n_2 > n_1)$.

$$\therefore R_y = \frac{m}{2hc}\left(\frac{e^2}{4\pi\varepsilon_0 h}\right)^2.$$

$$= \frac{mc}{2h}\left(\frac{e^2}{4\pi\varepsilon_0 hc}\right)^2$$

$\hbar = \frac{h}{2\pi}$

$$= \frac{mc}{2h}\cdot\frac{1}{(2\pi)^2}\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2$$

$$= \frac{mc}{4\pi\hbar}\alpha^2 \qquad \left(\alpha \fallingdotseq \frac{1}{137}\right)$$

(fine structure constant)

$$\fallingdotseq \frac{(9.1\times10^{-31}\,kg)\cdot(3.0\times10^{8}\,m/s)}{4\pi\cdot(1.1\times10^{-34}\,m^2\,kg/s)}\left(\frac{1}{137}\right)^2$$

$$= \frac{9.1\times3.0}{4\pi\cdot1.1}\cdot\left(\frac{1}{137}\right)^2\cdot10^{-31+8+34}\ m^{-1}.$$

$$= 2.0\times\left(\frac{1}{137}\right)^2\cdot10^{11}\ m^{-1}$$

$$= 2.0\times(5.3\times10^{-5})\cdot10^{11}\ m^{-1}$$

$$= 1.1\times10^{7}\ m^{-1}.$$