Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #22

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Outline

- Approximate symbol error rate of PSK (phase-shift keying) modulation
- Binary differential PSK (BDPSK)
- Quadrature Amplitude Modulation (QAM)

Transformation of y_1 and y_2

$$\begin{aligned} V &= \sqrt{y_1^2 + y_2^2}, \\ \Theta &= \tan^{-1}\frac{y_2}{y_1}. \end{aligned}$$

- Joint PDF of V and Θ

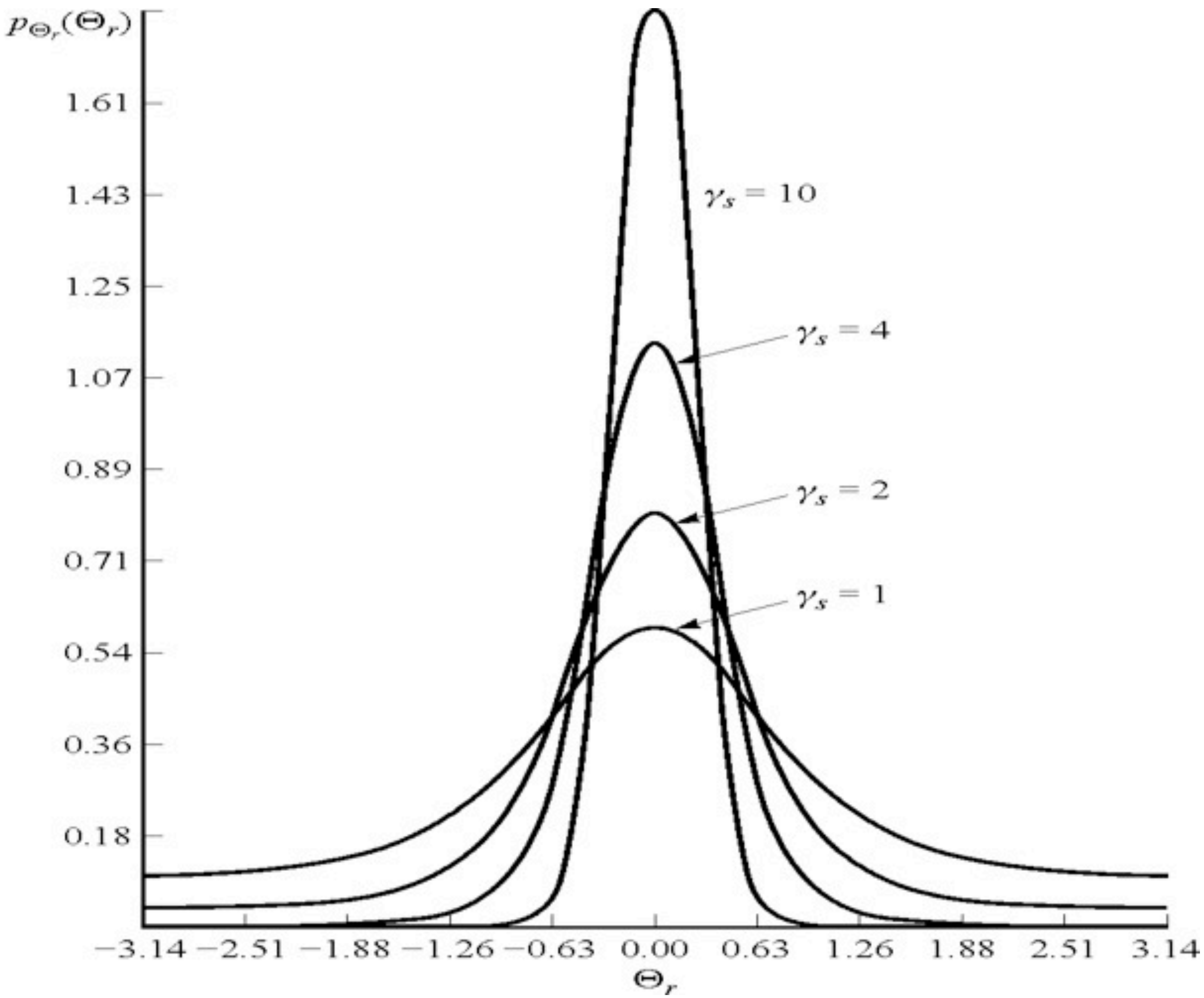
$$f_{V,\Theta}(v,\theta) = \frac{v}{2\pi\sigma_y^2} e^{-(v^2+\mathcal{E}_s-2\sqrt{\mathcal{E}_s}v\cos\theta)/2\sigma_y^2}.$$

- Marginal PDF of Θ

$$f_\Theta(\theta) = \int_0^\infty f_{V,\Theta}(v,\theta)\,dv = \frac{1}{2\pi}e^{-\rho_s\sin^2\theta}\int_0^\infty ve^{-(v-\sqrt{2}\rho_s\cos\theta)^2/2}\,dv$$

where $\rho_s = \dfrac{\mathcal{E}_s}{N_0}$ is the SNR per symbol.

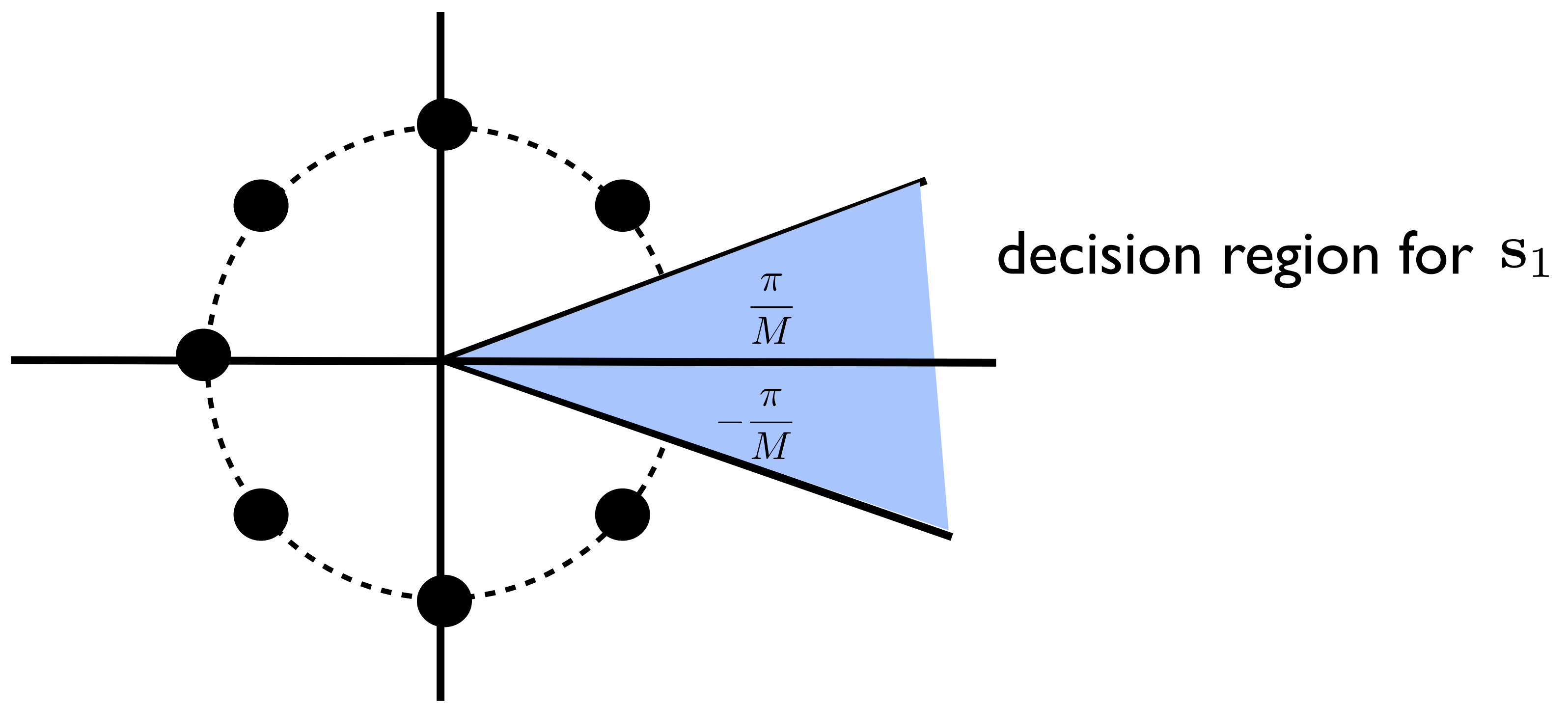
PDF of $p_\Theta(\theta)$



[Fig. 10. 15, Proakis textbook]

- Symbol error rate

$$P_M = 1 - \int_{-\pi/M}^{\pi/M} f_\Theta(\theta)\, d\theta.$$



$$P_M = 1 - \int_{-\pi/M}^{\pi/M} f_\Theta(\theta)\, d\theta = 1 - \int_{-\pi/M}^{\pi/M} \frac{e^{-\rho_s \sin^2\theta}}{2\pi} \int_0^\infty v e^{-(v-\sqrt{2}\rho_s \cos\theta)^2/2}\, dv\, d\theta$$

- BER of BPSK

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

- SER of QPSK

$$P_c = (1 - P_2)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2,$$

$$P_4 = 1 - P_c$$

$$P_4 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right] \approx 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right).$$

Approximation of SER for large $\mathcal{E}_s/N_0$

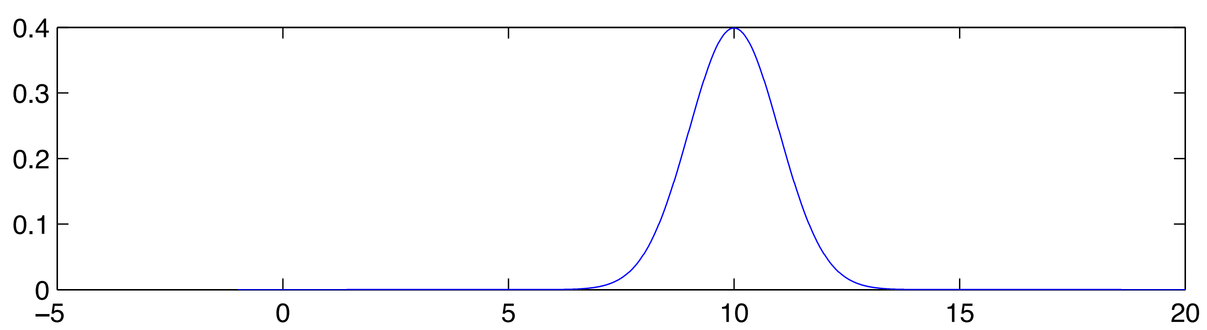
$$f_\Theta(\theta) = \frac{1}{2\pi} e^{-\rho_s \sin^2\theta} \int_0^\infty v e^{-(v-\sqrt{2}\rho_s\cos\theta)^2/2}\, dv$$

$$\approx \sqrt{\frac{\rho_s}{\pi}} \cos\theta e^{-\rho_s \sin^2\theta}$$

Note

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(v-m)^2}{2}}\, dv = 1 \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty v e^{-\frac{(v-m)^2}{2}}\, dv = m$$

For high SNR per symbol, $\sqrt{2}\rho_s \cos\theta >> 1$



$$\int_{-\infty}^\infty v e^{-(v-m)^2/2}\, dv \approx \int_0^\infty v e^{-(v-m)^2/2}\, dv = \sqrt{2\pi} m$$

$$
\begin{aligned}
P_M &\approx 1 - \int_{-\pi/M}^{\pi/M} \sqrt{\frac{\rho_s}{\pi}} \cos\theta e^{-\rho_s \sin^2\theta}\, d\theta \\
&\approx \frac{2}{\sqrt{2\pi}} \int_{\sqrt{2\rho_s}\sin\pi/M}^{\infty} e^{-u^2/2}\, du \\
&= 2Q\left(\sqrt{2\rho_s}\sin\frac{\pi}{M}\right) \\
&= 2Q\left(\sqrt{2k\sin^2\left(\frac{\pi}{M}\right)\frac{\mathcal{E}_b}{N_0}}\right) \\
&\approx 2Q\left(\sqrt{\frac{2\pi^2\log_2 M}{M^2}\frac{\mathcal{E}_b}{N_0}}\right)
\end{aligned}
$$

change in variable:

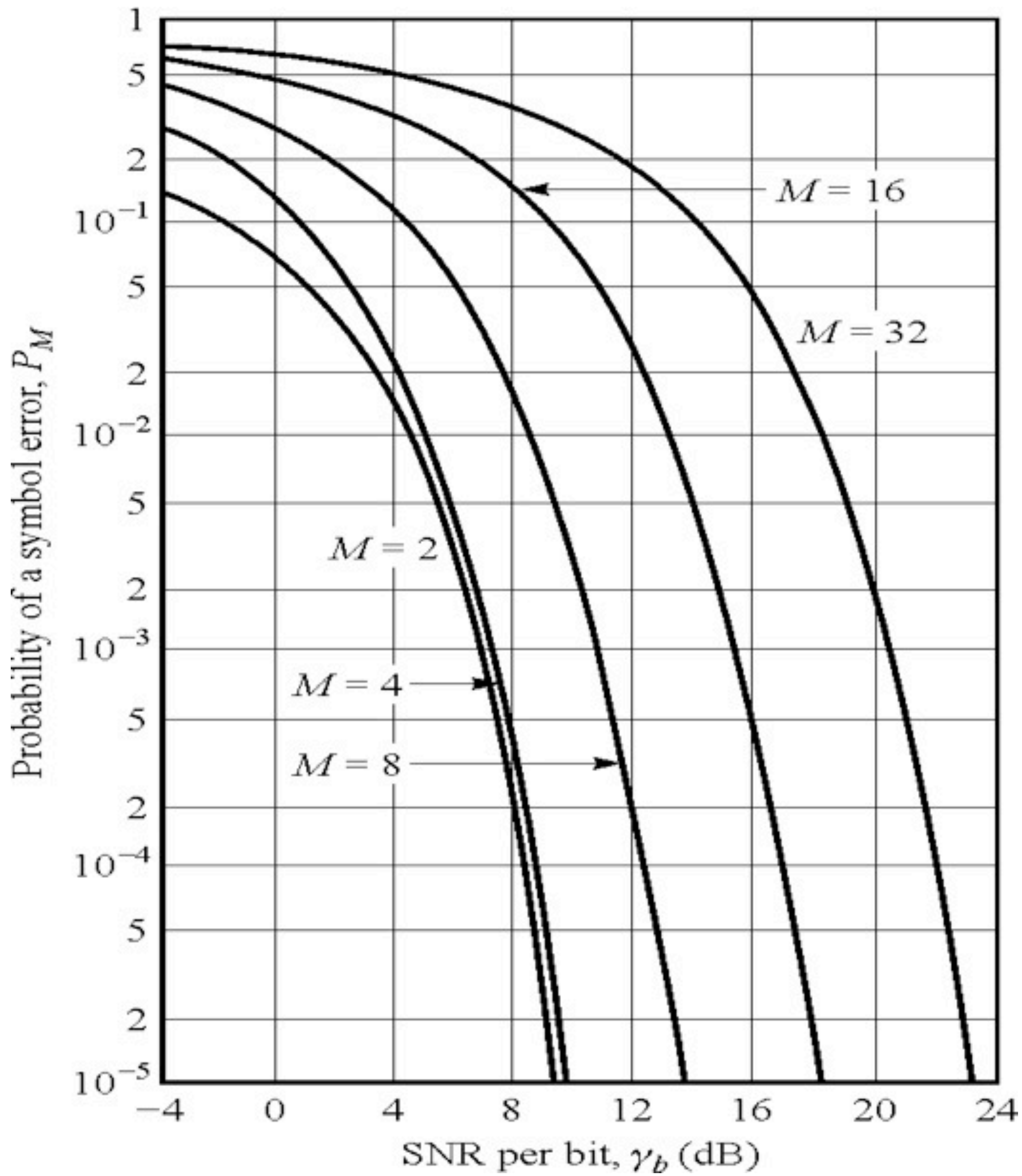
$$u = 2\sqrt{\rho_s}\sin^2\theta$$

where $k = \log_2 M$

$\rho_s = k\rho_b$

$\sin\frac{\pi}{M} \approx \frac{\pi}{M}$ for large M

Approximation of SER for M-PSK

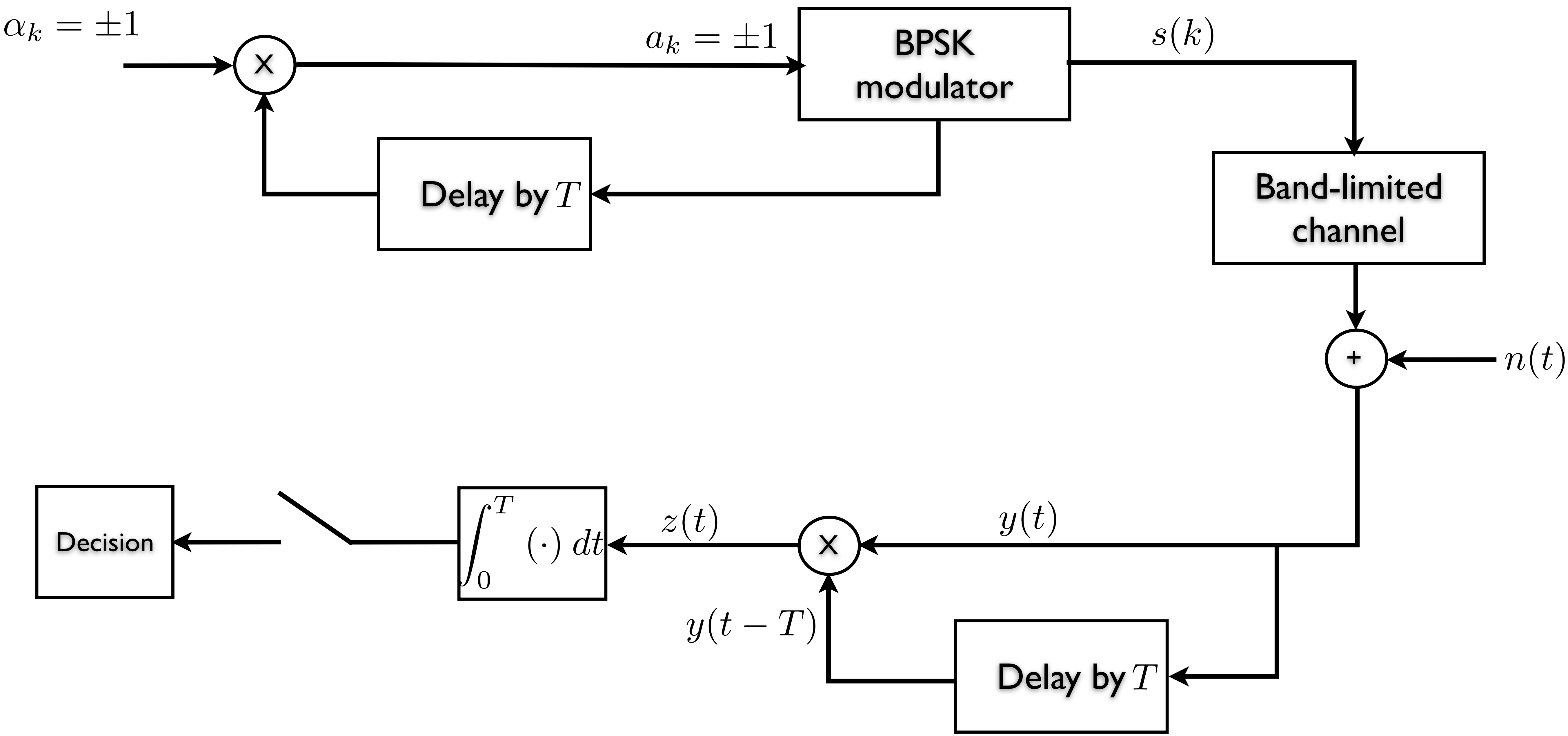


[Fig. 10. 6, Proakis textbook]

Binary Differential Phase-Shift-Keying (BDPSK)

- Differential coherent modulations are used when tracking the phase is difficult or costly.
- These techniques allow the extraction/recovery of the information bits without knowledge of phase.
- Let us treat only binary differential phase-shift-keying (BDPSK).

Block diagram



- Received signal for $t \in [kT, \ (k+1)T]$

$$y(t) = a_k \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \theta) + n(t)$$

- After delay by

$$y(t-T) = a_{k-1} \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c (t-T) + \theta) + n(t)$$

$$= a_{k-1} \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \theta) + n(t)$$

⟵ Assume $f_c T = \text{integer}$

- Key assumption

  - Assume that $\theta(t)$ is unknown to the receiver but slowly varying such as it is roughly constant over 2 successive bits.

$$\frac{1+\cos(4\pi f_c t + 2\theta)}{2}$$

$$\begin{aligned} z(t) &= y(t)y(t-T) \\ &= a_k a_{k-1}\frac{2\mathcal{E}_b}{T}\cos(2\pi f_c t+\theta)\cos(2\pi f_c t+\theta) + a_k\sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_c t+\theta)n(t) \\ &\quad + a_{k-1}\sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_c t+\theta)n(t) + n^2(t) \end{aligned}$$

- After integration

$$z = a_k a_{k-1}\frac{2\mathcal{E}_b}{T}\int_{kT}^{(k+1)T}\frac{1+\cos(4\pi f_c t+2\theta)}{2}\,dt + \text{ noise}$$

$$= a_k a_{k-1}\frac{2\mathcal{E}_b}{T}\frac{T}{2} + \text{ noise}$$

$$= a_k a_{k-1}\mathcal{E}_b + \text{noise}$$

$$\text{noise} = \begin{cases} 2n_1 + n_2 \\ n_2 \end{cases}$$

$$n_1 = \sqrt{\frac{2\mathcal{E}_b}{N_0}}\int_0^T \cos(2\pi f_c t+\theta)n(t)\,dt$$

$$n_2 = \int_0^T n^2(t)\,dt$$

Differential encoding and decoding

- If the symbol $a_k \in \{-1,\ 1\}$ are obtained from the information symbols $\alpha_k \in \{-1,\ 1\}$.

  From the differential encoder we have:

$$a_k = a_{k-1} \cdot \alpha_k$$

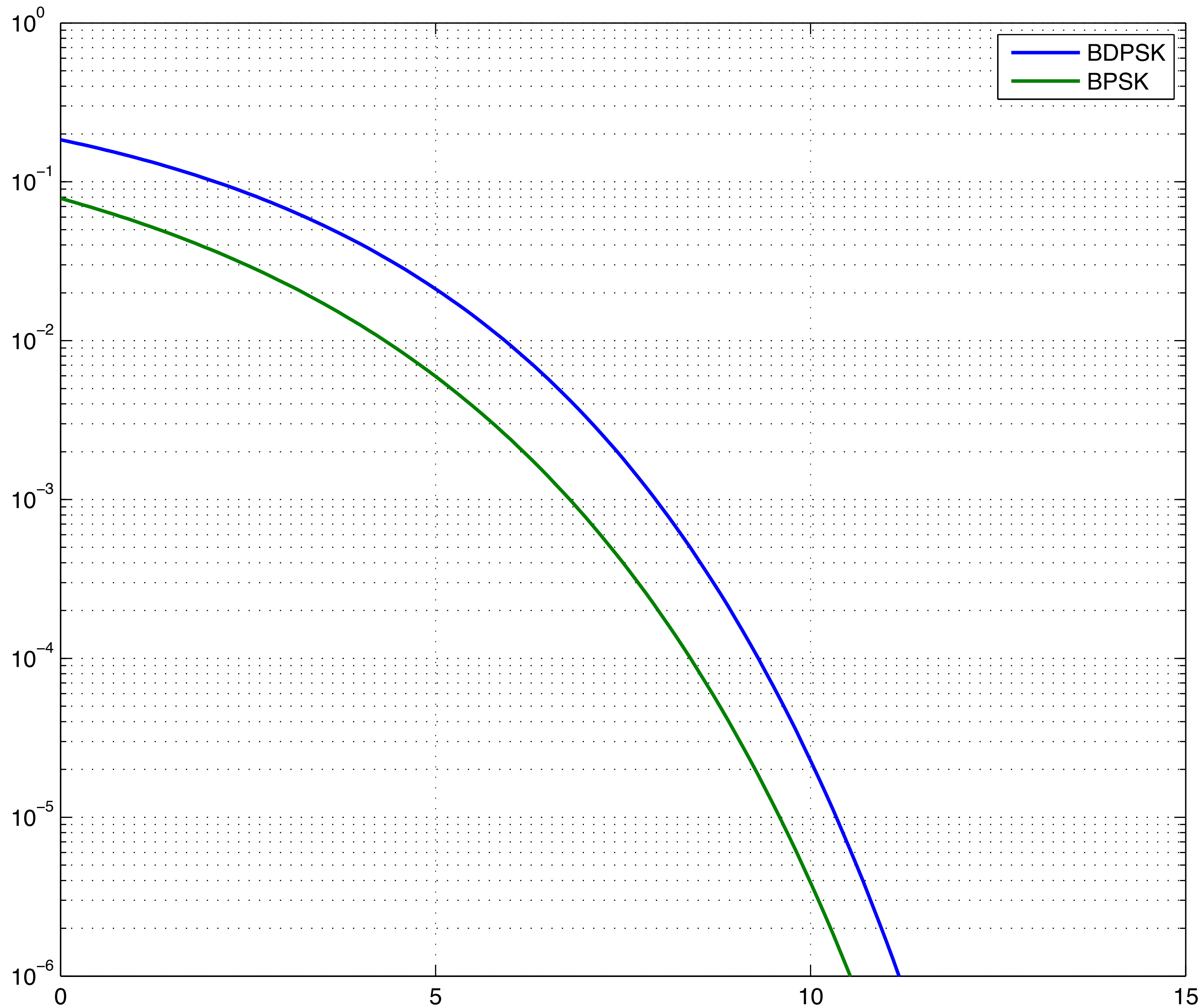
a_k	a_{k-1}	α_k
1	1	1
1	-1	-1
-1	1	-1
-1	-1	1

$$z = a_k a_{k-1} \mathcal{E}_b + \text{ noise } = (a_{k-1})^2 \alpha_k \mathcal{E}_b + \text{ noise}$$

$$= \alpha_k \mathcal{E}_b + \text{ noise}$$

Performance of BDPSK

$$P_b = \frac{1}{2} e^{-\mathcal{E}_b/N_0}$$



Quadrature Amplitude Modulation (QAM)

- Transmit signal waveforms

$$u_m(t) = A_{mc} g_T(t) \cos 2\pi f_c t + A_{ms} g_T(t) \sin 2\pi f_c t, \qquad m = 1, 2, \ldots, M$$

$$= \sqrt{A_{mc}^2 + A_{ms}^2}\, g_T(t) \cos(2\pi f_c t + \theta_m)$$

$$= \Re\{\sqrt{A_{mc}^2 + A_{ms}^2}\, g_T(t) e^{j(2\pi f_c t + \theta_m)}\}$$

$$= \Re\{s_{lm}(t) e^{2j\pi f_c t}\}$$

where $s_{lm}(t) = \sqrt{A_{mc}^2 + A_{ms}^2}\, g_T(t) e^{j\theta_m}$

$$= s_{m1}\phi_1(t) + s_{m2}\phi_2(t)$$

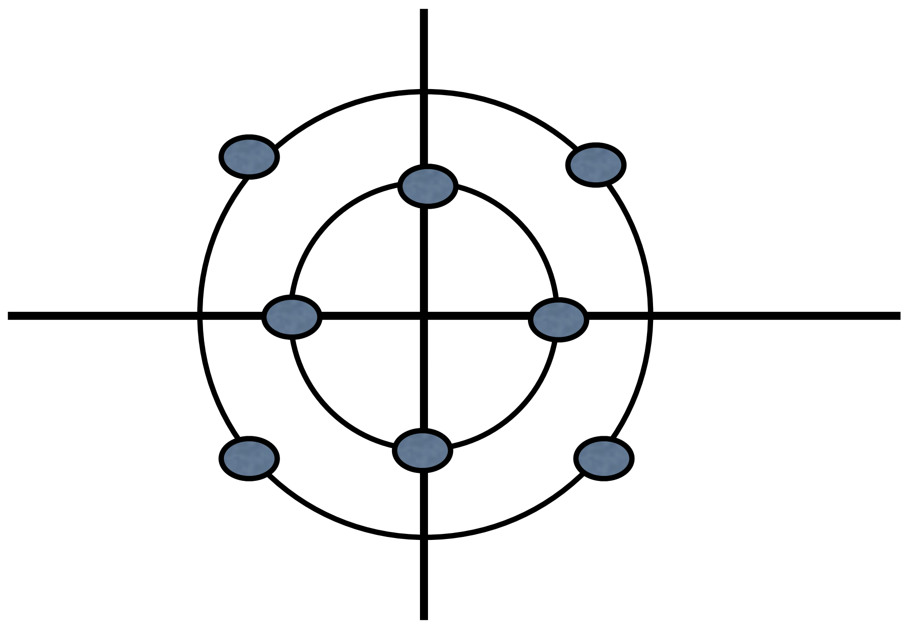
$$\theta_m = \tan^{-1} \frac{A_{ms}}{A_{mc}}$$

$$\phi_1(t) = \sqrt{\frac{1}{\mathcal{E}_s}}\, g_T(t) \cos(2\pi f_c t)$$

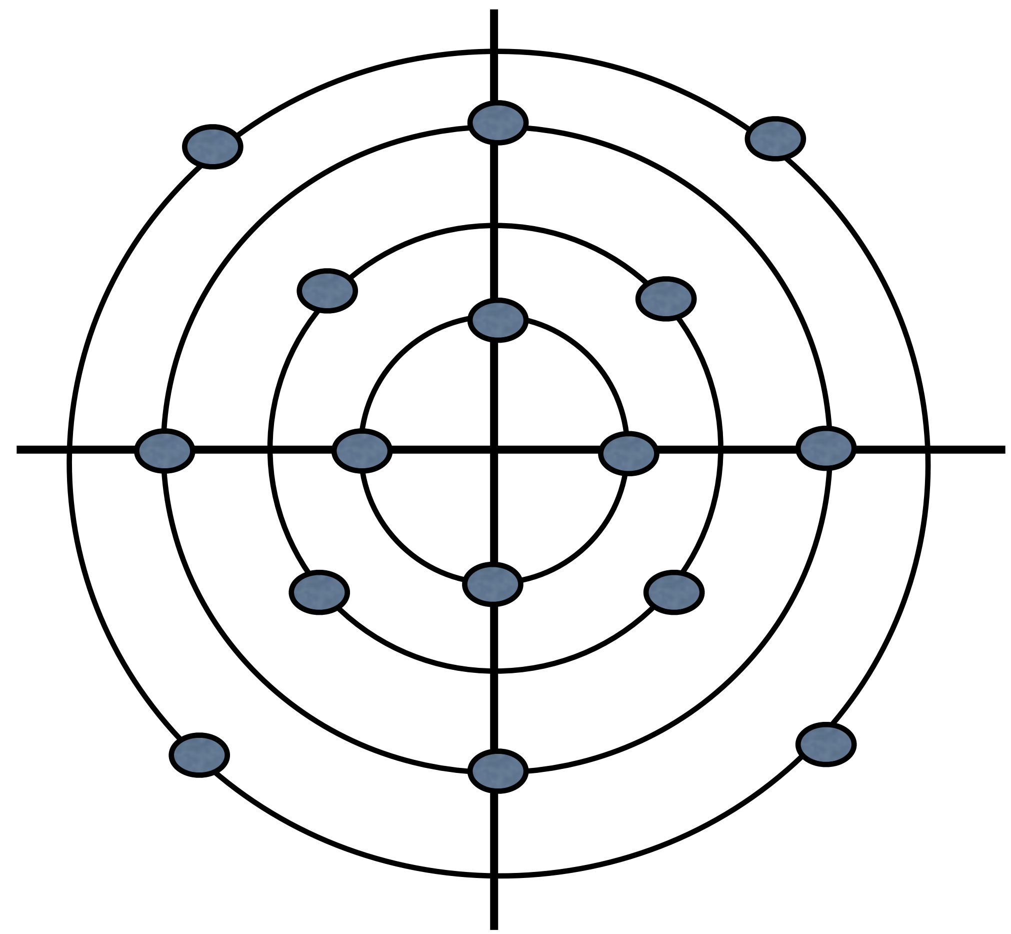
$$\mathbf{s}_m = \left(\sqrt{\mathcal{E}_s} A_{mc}, \quad \sqrt{\mathcal{E}_s} A_{ms}\right)$$

$$\phi_2(t) = \sqrt{\frac{1}{\mathcal{E}_s}}\, g_T(t) \sin(2\pi f_c t)$$

- Example



$M = 8$



$M = 16$

Spectral efficiency

$$u_m(t) = \sqrt{A_{mc}^2 + A_{ms}^2} g_T(t) \cos(2\pi f_c t + \theta_m) = A_m g_T \cos(2\pi f_c t + \theta_n),$$

$$\begin{aligned} m &= 1, 2, \ldots, M_1 \\ n &= 1, 2, ..., M_2 \end{aligned}$$

- Number of bits per symbol

  Let $M_1 = 2^{k_1}$ and $M_2 = 2^{k_2}$

  $$k_1 + k_2 = \log_2 M_1 + \log_2 M_2 \quad \text{bits/symbol}$$

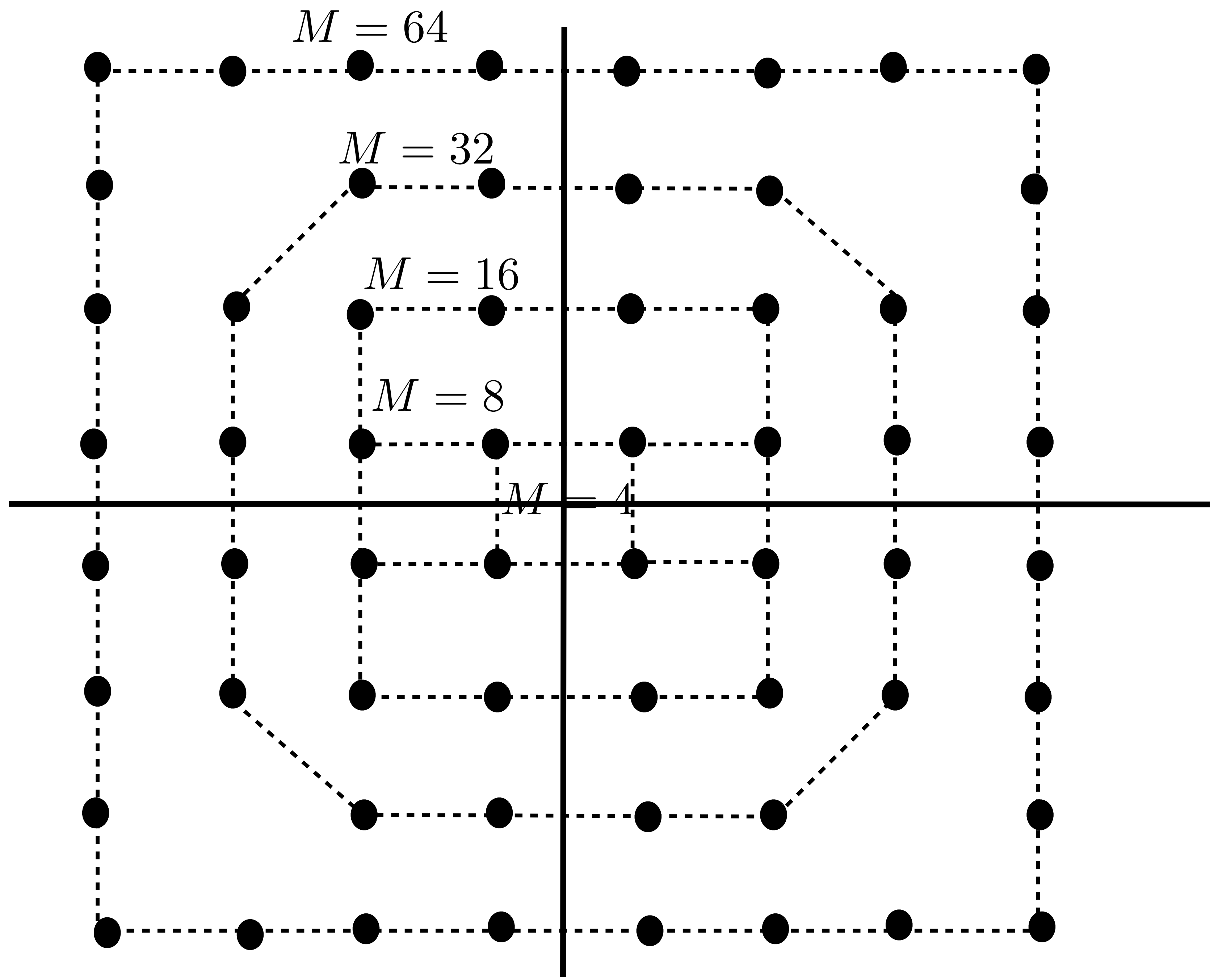
- Symbol rate

  $$R_s = \frac{R_b}{k_1 + k_2}$$

Rectangular QAM

- Signal amplitudes take the set of values $\{(2m-1-M)d,\ \ m=1,2,\ldots,M\}$

- Signal space diagram



Rectangular QAM

- Average energy per symbol

$$\mathcal{E}_{av} = \frac{1}{M}\sum_{i=1}^{M} ||\mathbf{s}_i||^2.$$

- Distance between two symbols

$$d_{mn} = \sqrt{||\mathbf{s}_m - \mathbf{s}_n||^2}.$$

Probability of Error for QAM

- For rectangular signal constellations in which $M = 2^k$ where k is even, the QAM signal constellation is equivalent to two PAM signals on quadrature carriers, each having $\sqrt{M} = 2^{k/2}$ signal points.
  - Since the signals in the phase-quadrature components can be perfectly separated at the demodulator, the probability of error for QAM is easily determined from the probability of error for PAM.
  - Specifically, the probability of a correct decision for the M-ary QAM system is

$$P_c = (1 - P_{\sqrt{M}})^2$$

    - where $P_{\sqrt{M}}$ is the probability of error of an $\sqrt{M}$ -ary PAM with one-half the average power in each quadrature signal of the equivalent QAM system.
  - By appropriately modifying the probability of error for M-ary QAM, we obtain

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_{av}}{N_0}}\right)$$

~ where E_{av}/N_0 is the SNR per symbol.

- Therefore, the probability of a symbol error for the M-ary QAM is

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

  - Note that this result is exact for $M = 2^k$ when k is even.