Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #3

2012. 09. 03

School of Electrical Engineering

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Outline

- Review of probability and random variables (Secs. 5.1.1 - 5.1.4)
  - Example of binary symmetric channel
  - random variables
  - functions of random variable

Bayes' Theorem

- Bayes' theorem

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A)}$$

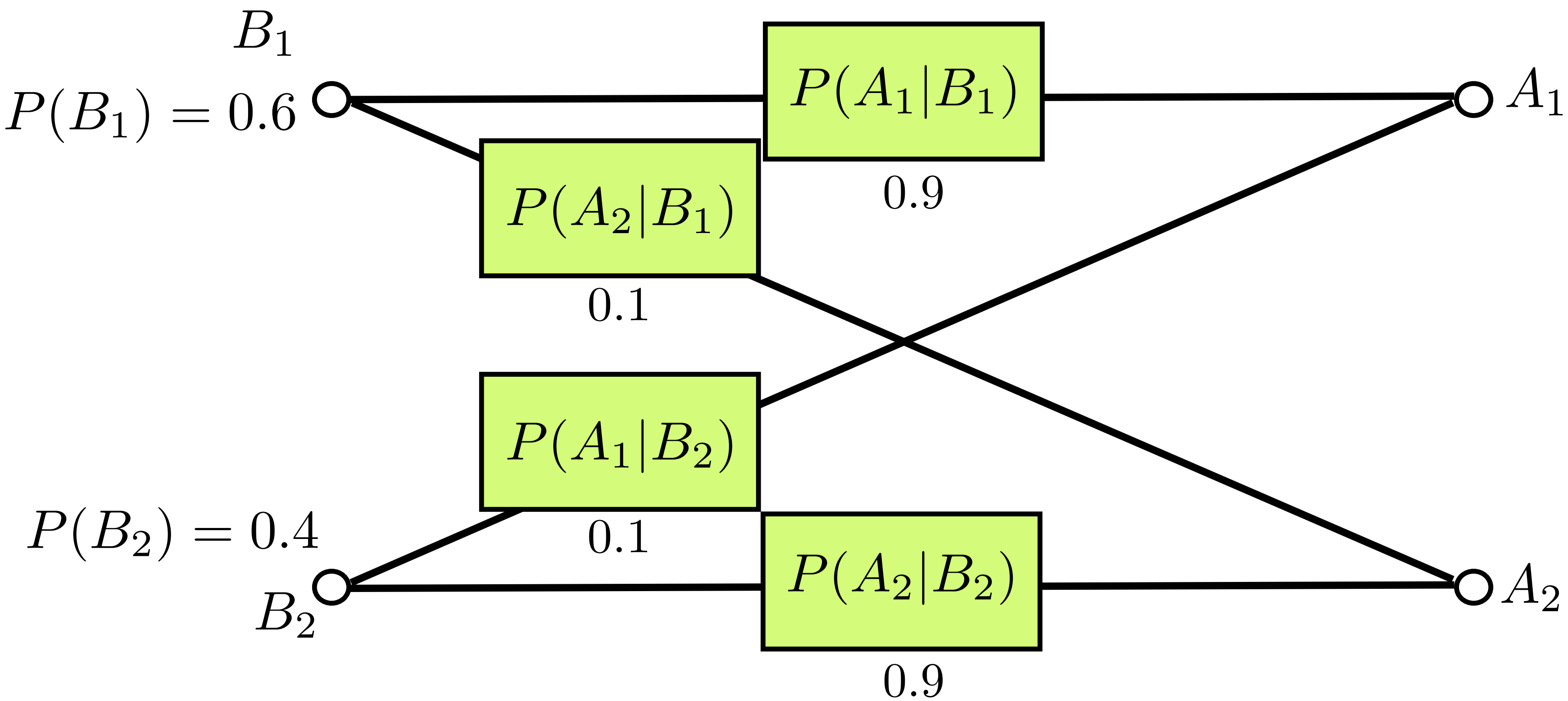
$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} = \frac{P(B_n|A)P(A)}{P(B_n)}$$

- We can also rewrite

$$\begin{aligned} P(B_n|A) &= \frac{P(A \cap B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A|B_1)P(B_1) + \cdots + P(A|B_N)P(B_N)} \\ &= \frac{P(A|B_n)P(B_n)}{\sum_{j=1}^{N} P(A|B_j)P(B_j)}. \end{aligned}$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Example: Binary Symmetric Channel (BSC)



$P(A_1)$ and $P(A_2)$?

$P(B_1|A_1)$ and $P(B_2|A_2)$?

$P(B_1|A_2)$ and $P(B_2|A_1)$?

$$\begin{aligned} P(A_1) &= P(A_1|B_1)P(B_1) + P(A_1|B_2)P(B_2) = 0.9(0.6) + 0.1(0.4) = 0.58 \\ P(A_2) &= P(A_2|B_1)P(B_1) + P(A_2|B_2)P(B_2) = 0.1(0.6) + 0.9(0.4) = 0.42 \end{aligned}$$

$$P(B_1|A_1) = \frac{P(A_1|B_1)P(B_1)}{P(A_1)} = \frac{0.9(0.6)}{0.58} = \frac{0.54}{0.58} \approx 0.931$$

$$P(B_2|A_2) = \frac{P(A_2|B_2)P(B_2)}{P(A_2)} = \frac{0.9(0.4)}{0.42} = \frac{0.36}{0.42} \approx 0.857$$

$$P(B_1|A_2) = \frac{P(A_2|B_1)P(B_1)}{P(A_2)} = \frac{0.1(0.6)}{0.42} = \frac{0.06}{0.42} \approx 0.143$$

$$P(B_2|A_1) = \frac{P(A_1|B_2)P(B_2)}{P(A_1)} = \frac{0.1(0.4)}{0.58} = \frac{0.04}{0.58} \approx 0.069$$

Independent Events

- Statistically independent if

$$P(A|B) = P(A) \qquad P(B|A) = P(B)$$

- We also have for statistically events

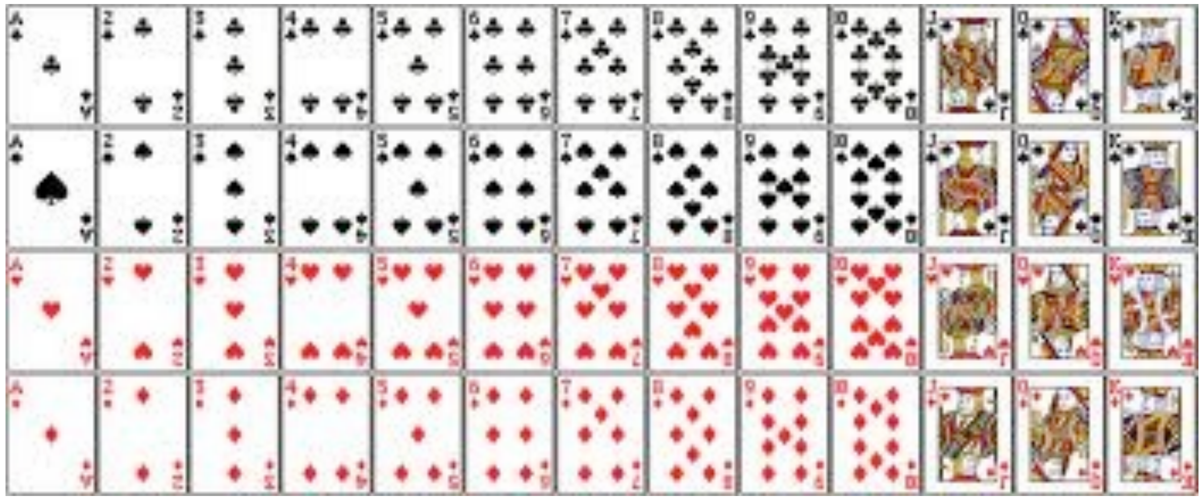
$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

- If A and B are statistically independent,

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B) \neq 0$$

- Note
  - If A and B are nonzero probabilities of occurrences and statistically independent,
  - which means $A \cap B \neq \phi$.
- In order for two events to be independent they must have an intersection $A \cap B \neq \phi$

Example



- Define events as follows:
  - Event A : select a king
  - Event B: select a jack or queen
  - Event C: select a heart

$$P(A)=\frac{4}{52},\ P(B)=\frac{8}{52},\ \text{and}\ P(C)=\frac{13}{52}$$

- Joint probabilities

$$P(A\cap B)=0,\ P(A\cap C)=\frac{1}{52},\ P(B\cap C)=\frac{2}{52}$$

- Independent?

$$P(A\cap B)=0 \neq P(A)P(B)=\frac{32}{52^2}$$

$$P(A\cap C) = \frac{1}{52}=P(A)P(C)=\frac{1}{52}$$

$$P(B\cap C) = \frac{2}{52}=P(B)P(C)=\frac{2}{52}$$

Multiple Independent Events

- Three independents

$$\begin{aligned}
P(A_1 \cap A_2) &= P(A_1)P(A_2) \\
P(A_1 \cap A_3) &= P(A_1)P(A_3) \\
P(A_2 \cap A_3) &= P(A_2)P(A_3) \\
P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3)
\end{aligned}$$

Permutation and Combination

- Permutation

$$\begin{aligned}\text{ordering of } r \text{ elements taken from } n &= n(n-1)(n-2)\cdots(n-r+1)\\ &= \frac{n!}{(n-r)!} = P_r^n \quad r-1,2,\ldots,n\end{aligned}$$

- Combination

binomial coefficient

$$r \text{ elements taken from } n = \binom{n}{r} = \frac{n!}{(n-r)!r!} =_n C_r$$

- Binomial coefficient

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

- Symmetry of binomial coefficient

$$\binom{n}{r} = \binom{n}{n-r}$$

The Random Variable (RV)

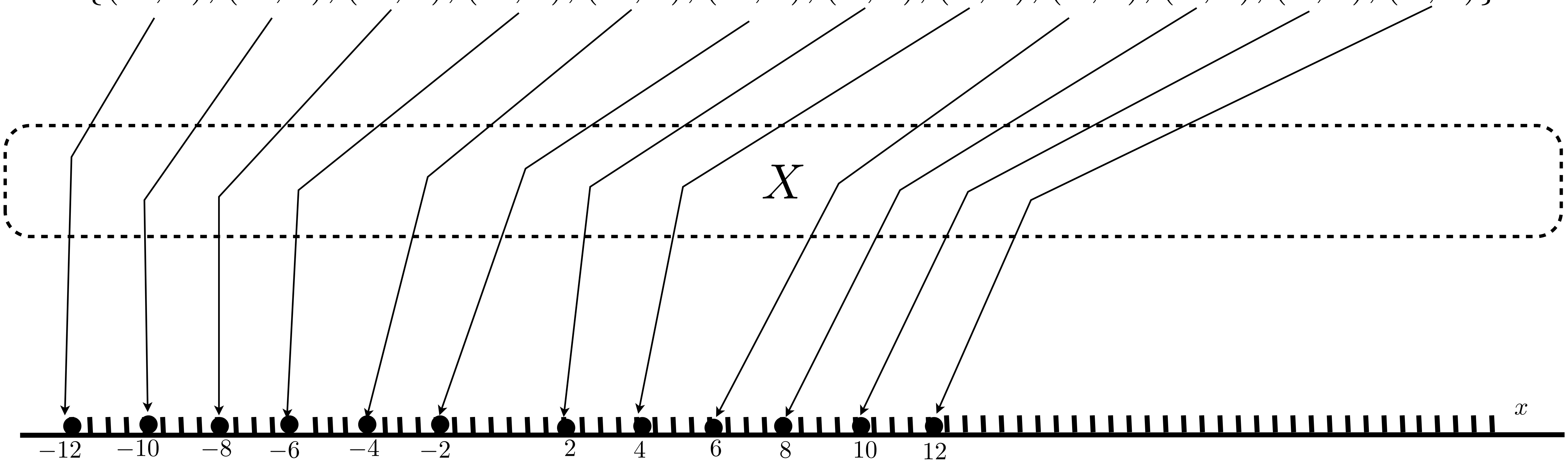
- A real random variable is defined as
  - a real function of the elements of a sample space Ω

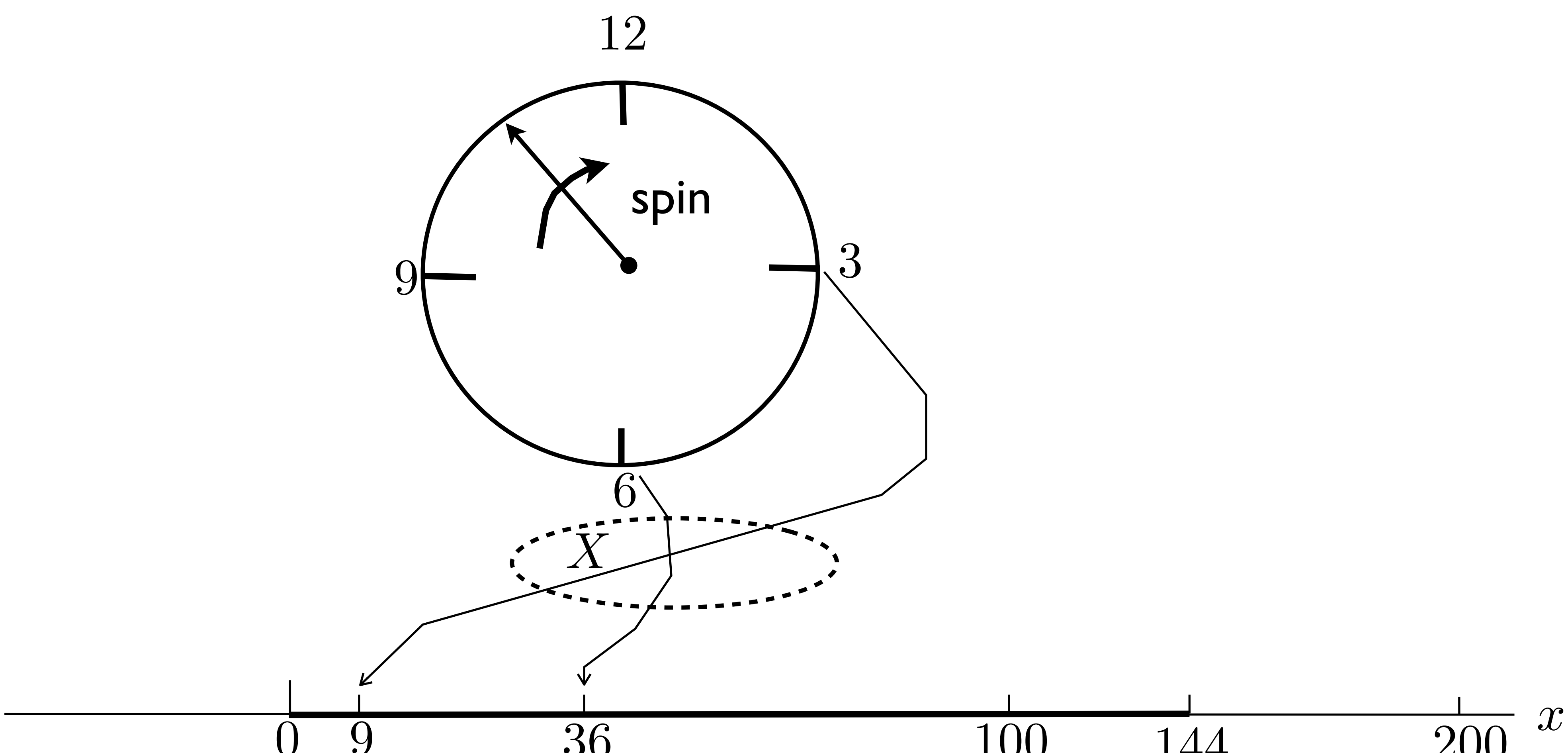
- Represent a random variable by a capital letter such as $W,\ X,\ \text{or}\ Y$ and any particular value of the random variable by a lowercase letter such as $w,\ x,\ \text{or}\ y.$

- Thus, given an experiment defined by a sample space Ω with elements ω, we assign to every ω a real number $X(\omega)$
  - according to some rule and call $X(\omega)$ a random variable.

Example

$$\Omega = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$





Conditions for a Function to be a Random

- First condition
  - The set $\{X \leq x\}$ shall be an event for any real number x .
  - The probability of this event, denoted by $P\{X \leq x\}$ is equal to the sum of the probabilities of all the elementary events corresponding to $\{X \leq x\}$.
- Probabilities of the events $\{X = \infty\}$ and $\{X = -\infty\}$

$$P\{X = \infty\} = 0 \text{ and } P\{X = -\infty\} = 0$$

- Probabilities of the events $\{X \leq \infty\}$

$$P\{X \leq \infty\} = 1$$

Categorization of Random Variables

- Continuous random variable
- Discrete random variable
- Mixed random variable

Bernoulli Trials

- There exist two outcomes in the experiment.
  - Example:
    - binary bit 1 or 0 is generated
    - Head or tail
- Denote each of two outcomes as A and $\bar{A}$

- Repeat experiments N times and A is observed k times out of the N trials.
  - Such repeated experiments are called *Bernoulli trials*.
- Probability

$$P(A) = p \text{ then } P(\bar{A}) = 1 - p$$

- k times out of N trials for the event A
  - one particular sequence is k times of A and $N-k$ times of $\bar{A}$ and its probability is

$$\underbrace{P(A)P(A)\cdots P(A)}_{k \text{ terms}}\underbrace{P(\bar{A})P(\bar{A})\cdots P(\bar{A})}_{N-k \text{ terms}} = p^k(1-p)^{N-k}$$

- Probability that A occurs exactly k times

$$P(A \text{ occurs exactly } k \text{ times}) = \binom{N}{k} p^k (1-p)^{N-k}$$

Distribution Function

- Cumulative distribution function (CDF)

$$F_X(x) = P\{X \le x\}$$

- Properties of CDF

(1) $F_X(-\infty) = 0$

(2) $F_X(\infty) = 1$

(3) $0 \le F_X(x) \le 1$

(4) $F_X(x_1) \le F_X(x_2), \quad \text{if } x_1 < x_2$

(5) $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

(6) $F_X(x^+) = F_X(x)$

- If the values of x_i, we may write $F_X(x)$

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\}u(x - x_i)$$

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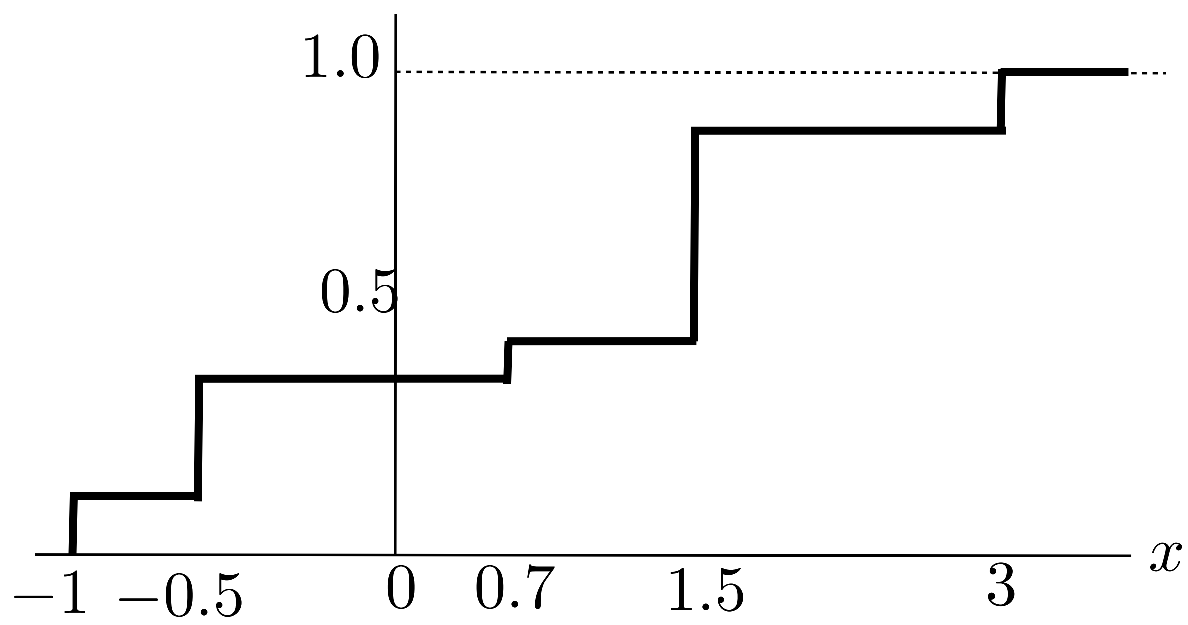
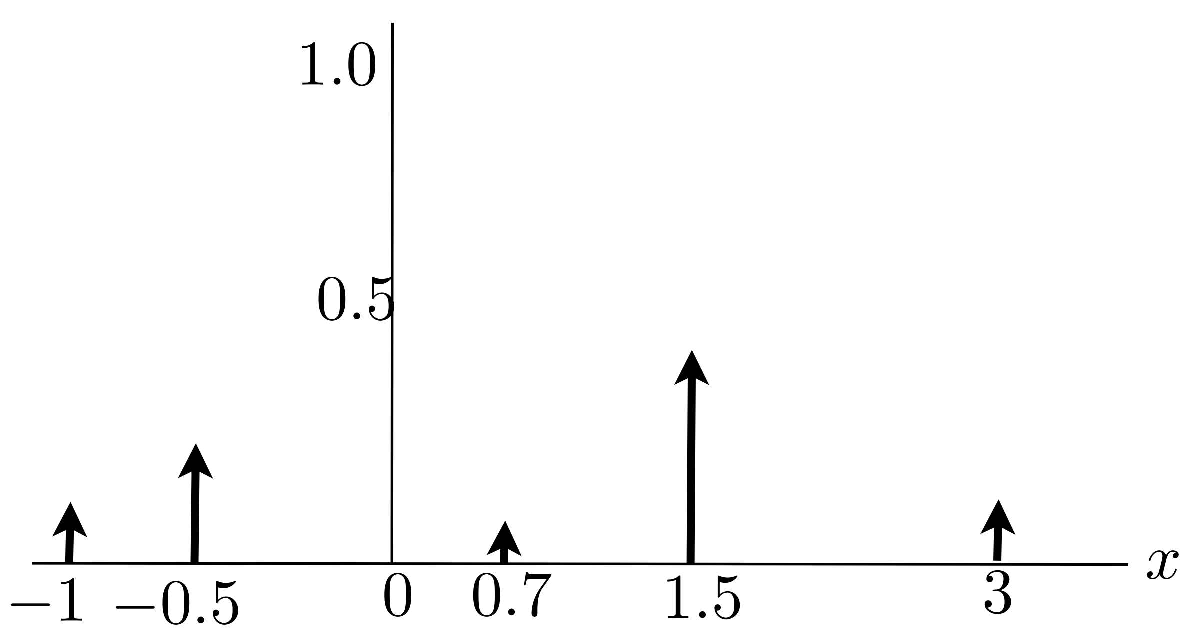


- PDF



Probability Density Function (PDF)

- PDF is defined as the derivative of CDF.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- Properties of PDF

$$\textbf{(1)} \;\; 0 \le f_X(x) \qquad \text{all } x$$

$$\textbf{(2)} \;\; \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\textbf{(3)} \;\; F_X(x) = \int_{-\infty}^{x} f_x(\zeta)\,d\zeta$$

$$\textbf{(4)} \;\; P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx$$

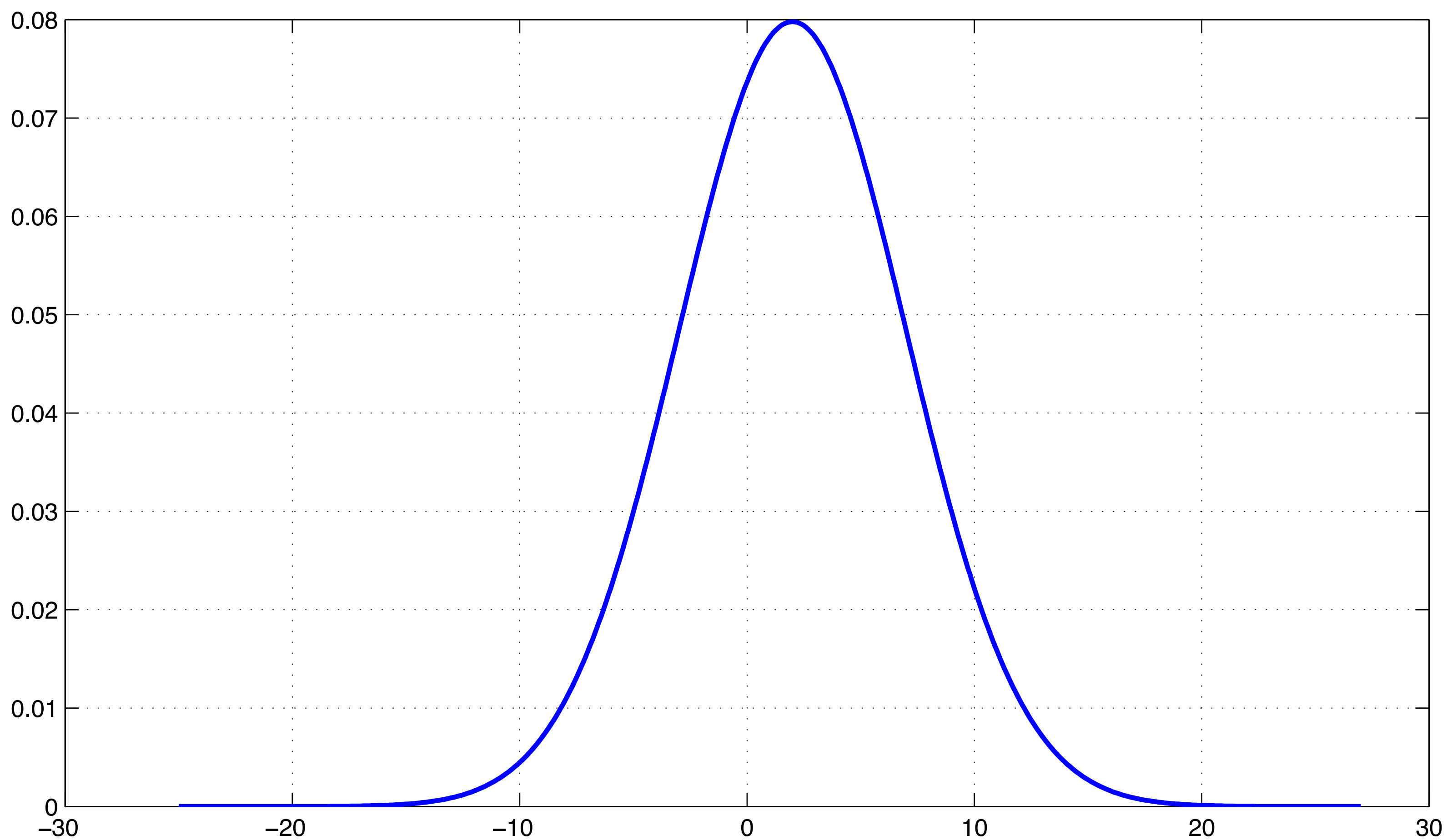
Gaussian Random Variable

- A random variable X is called gaussian if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma_x^2}\right] \qquad \sigma_X > 0 \text{ and } -\infty < m < \infty$$

- Example

$$m = 2 \text{ and } \sigma = 5$$

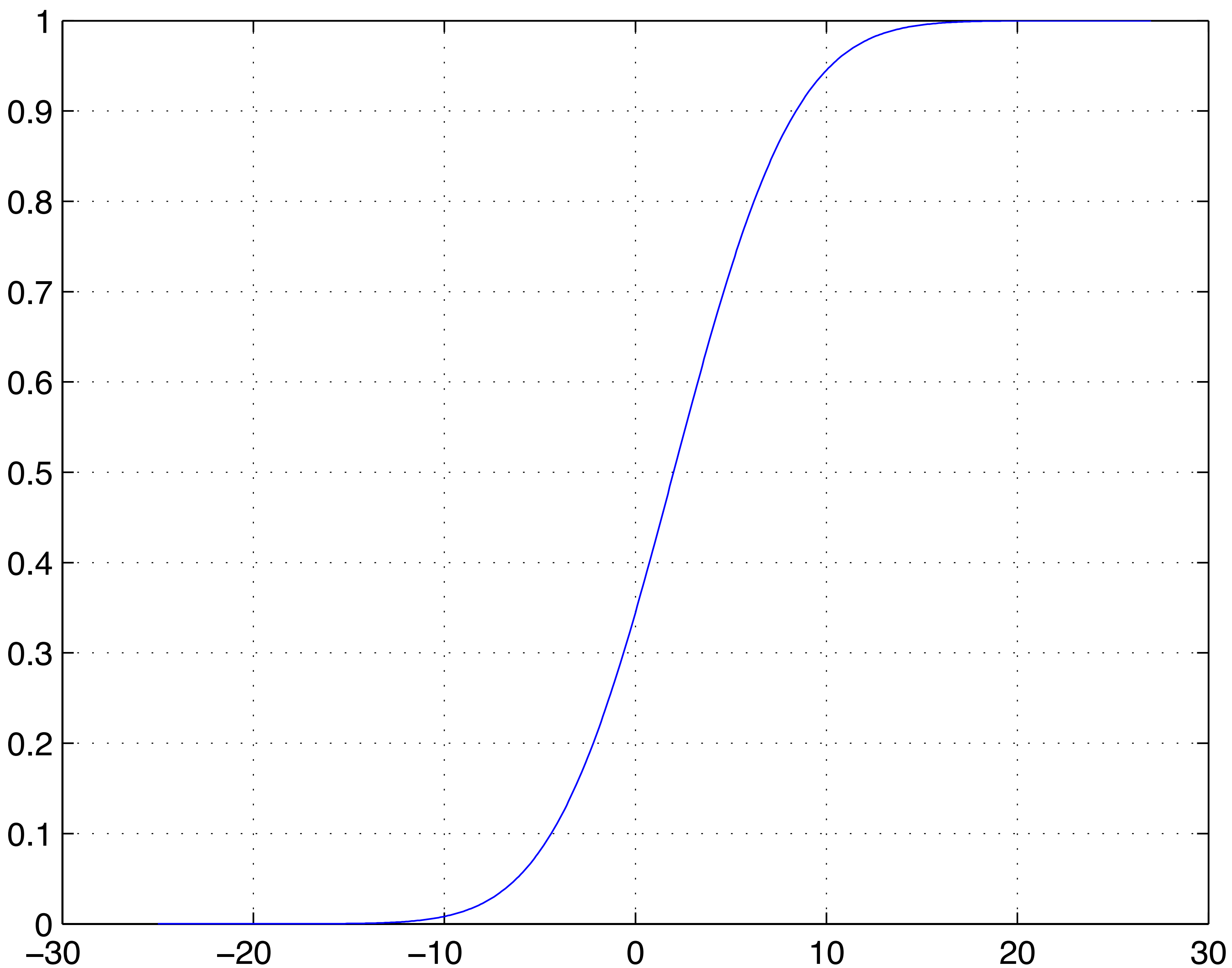


CDF

$$F_X(x) = \Pr[X \le x] = \int_{-\infty}^{x} f_X(\zeta)\, d\zeta = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\zeta - m)^2}{2\sigma_X^2}\right] d\zeta$$

$$= \int_{-\infty}^{\frac{(x-m)}{\sigma_X}} \exp\left[-\frac{t^2}{2}\right] dt = 1 - \int_{\frac{(x-m)}{\sigma_X}}^{\infty} \exp\left[-\frac{t^2}{2}\right] dt$$

$$= 1 - Q\left(\frac{x-m}{\sigma_X}\right)$$



where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

Some Special Functions

- Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt$$

- Error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \, dt$$

  - Properties of error function

    - symmetry relation: $\mathrm{erf}(-x) = -\mathrm{erf}(x)$

    - As x approaches infinity, $\mathrm{erf}(x)$ approaches unity; that is,

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2} \, dt = 1$$

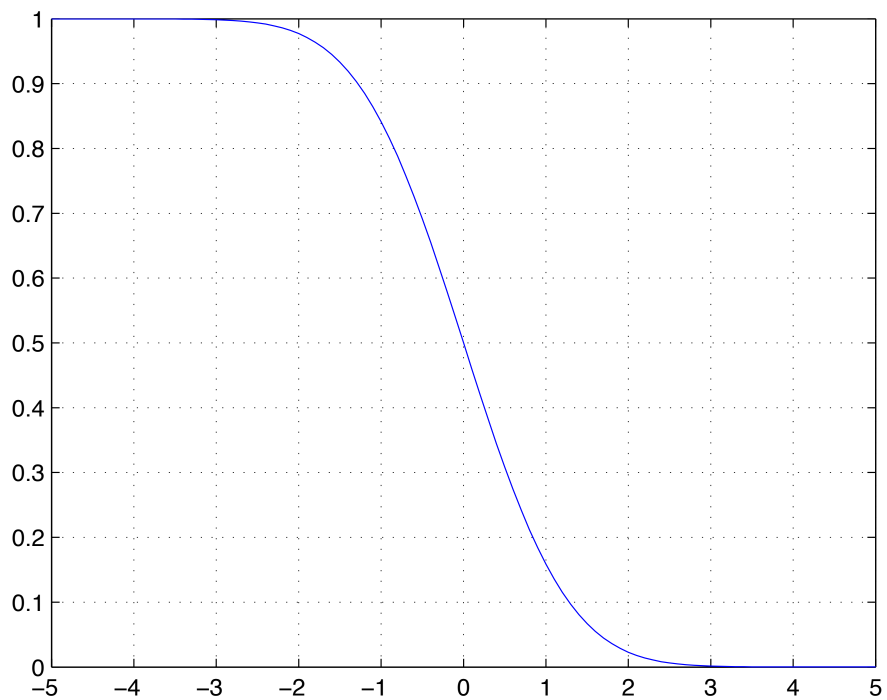
    - Complementary error function

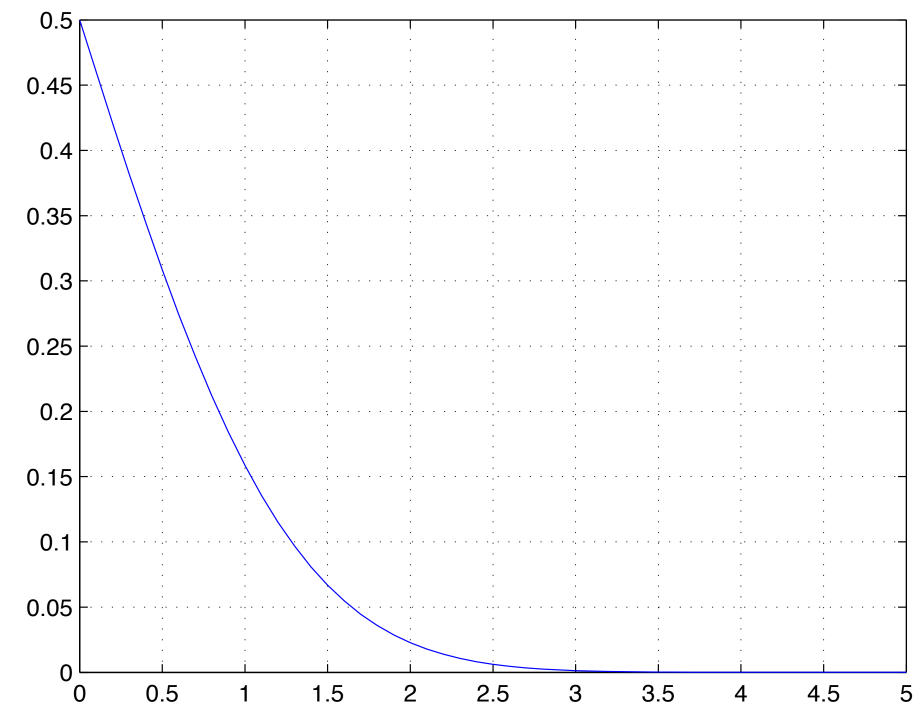
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \, dt = 1$$

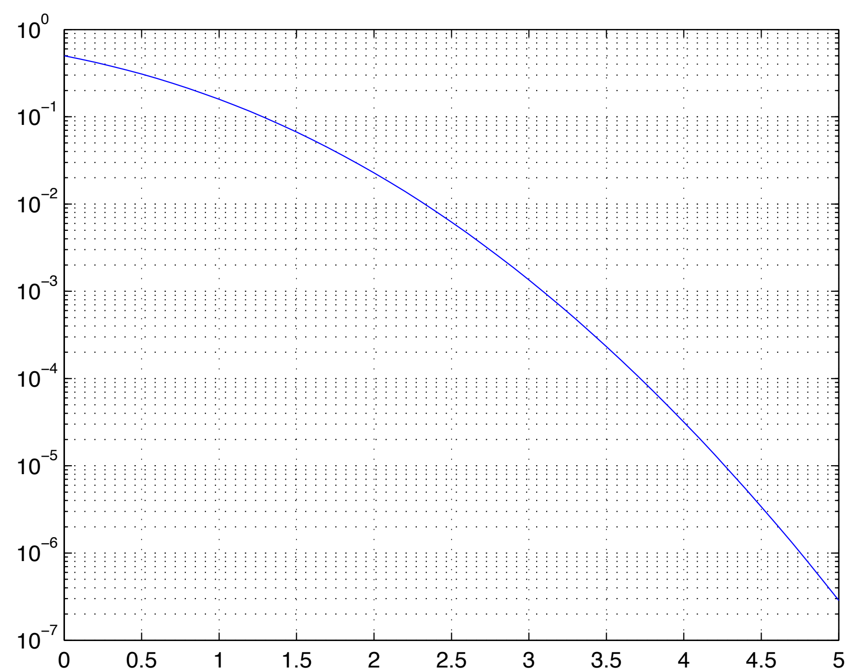
Relation between Q and erfc functionss

$$
\begin{aligned}
Q(x) &= \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \\
\mathrm{erfc}(x) &= 2Q(\sqrt{2}x)
\end{aligned}
$$

Q-function Plot







semilog plot

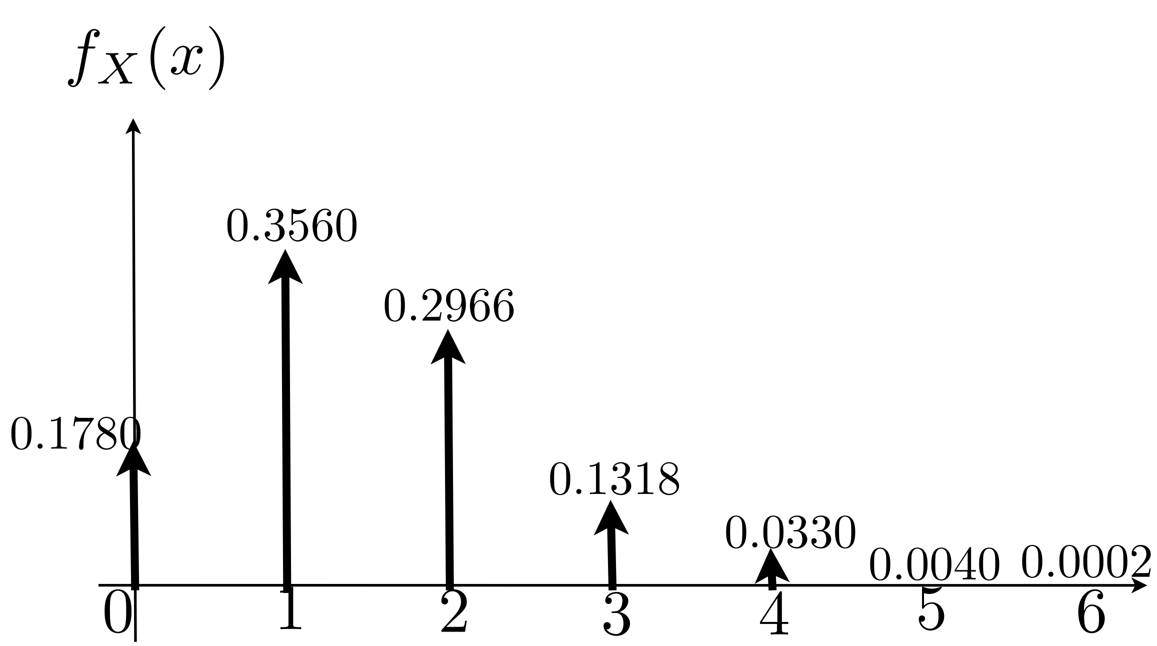
Binomial Distribution and Density

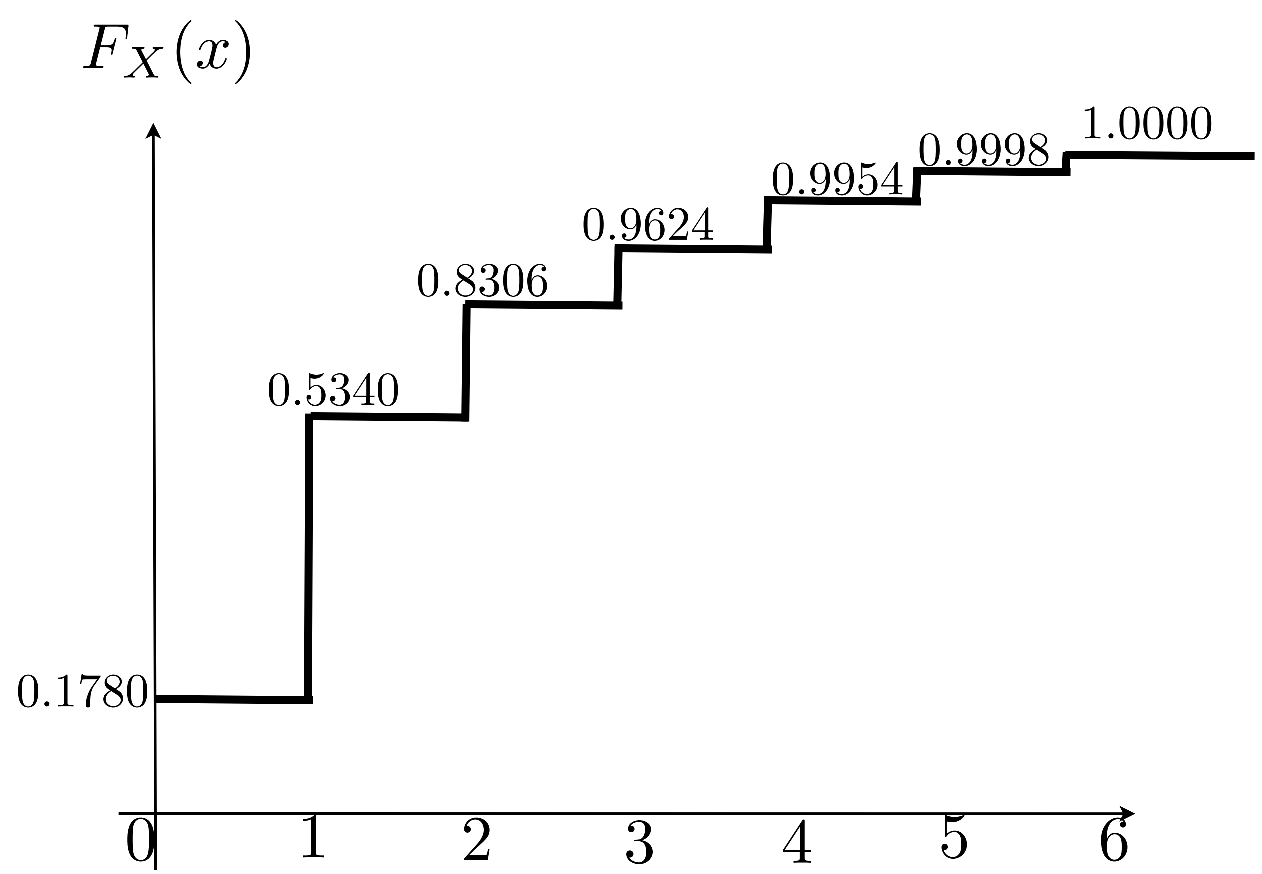
- Let $0 < p < 1, \text{ and } N = 1, 2, \ldots$. Then,
  - PDF

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

  - CDF

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k)$$





Uniform Distribution and Density

- PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$$

- CDF

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & b \leq x \end{cases}$$

Uniform Distribution and Density

- PDF

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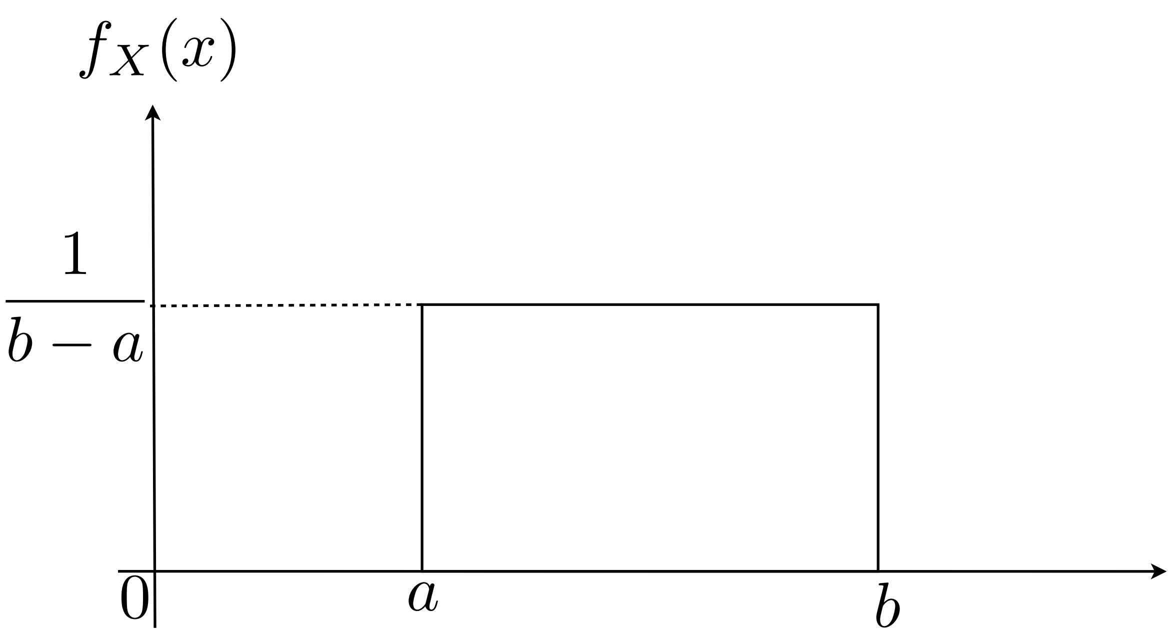
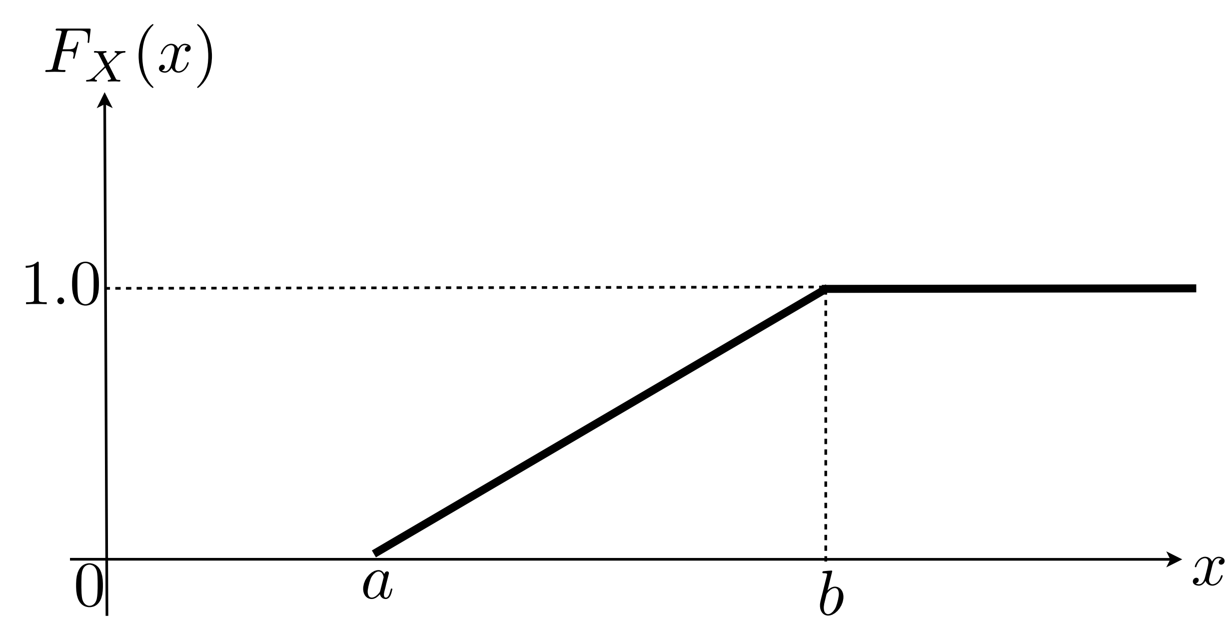




Uniform Random Samples in Matlab

- In matlab, "rand(N)" generates the N random samples distributed uniformly between zero and one.

- For example,

```
u=rand(100);        % generates 10 uniformly distributed random samples

u =    0.8147    0.9058    0.1270    0.9134    0.6324    0.0975    0.2785    0.5469    0.9575    0.9649
```

- Binary random sample generation with probability of half for zero and one, respectively.

```
u=rand(1,1);

if (u<0.5)

        b=0;
```

Exponential Distribution and Density

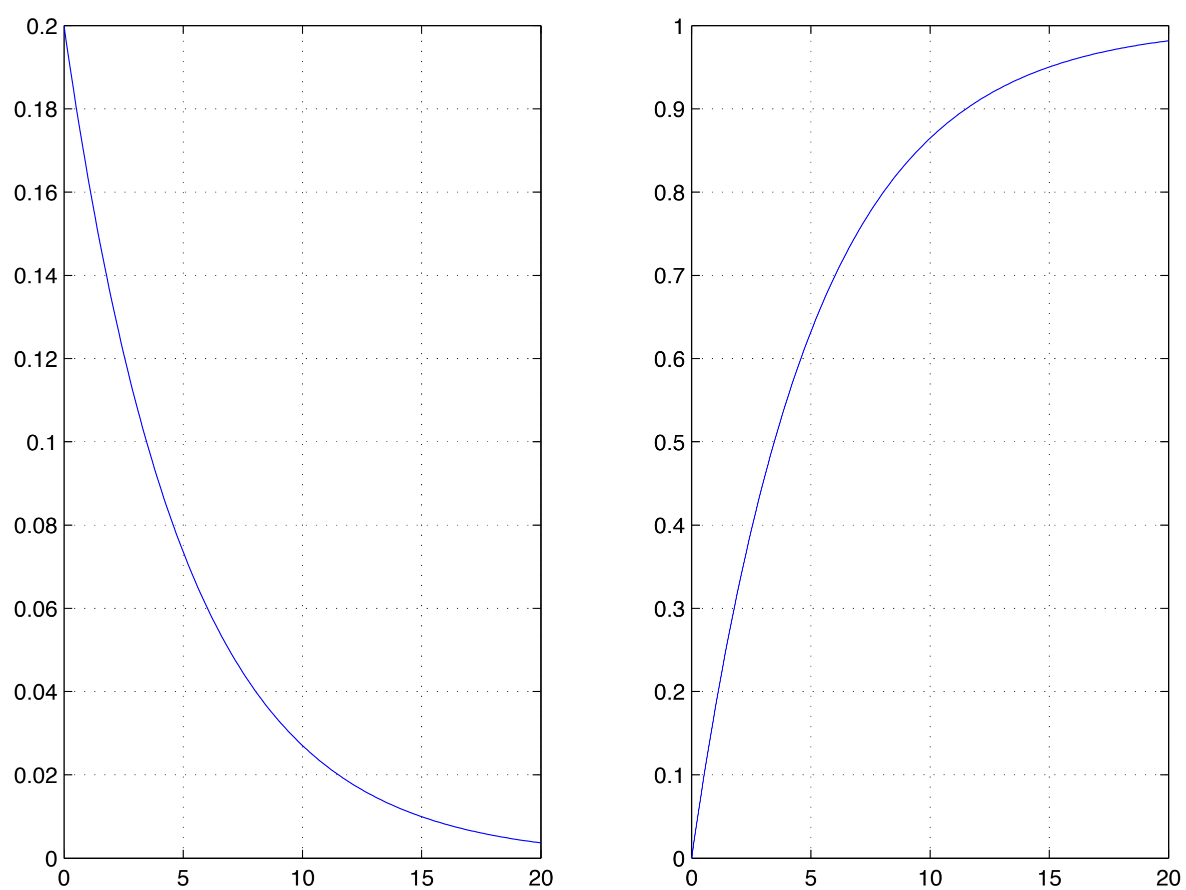
- PDF

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad \text{for } x \geq 0$$

- CDF

$$F_X(x) = 1 - e^{-\frac{x}{\lambda}} \quad \text{for } x \geq 0$$

- Example for $\lambda = 5$



Rayleigh Distribution and Density

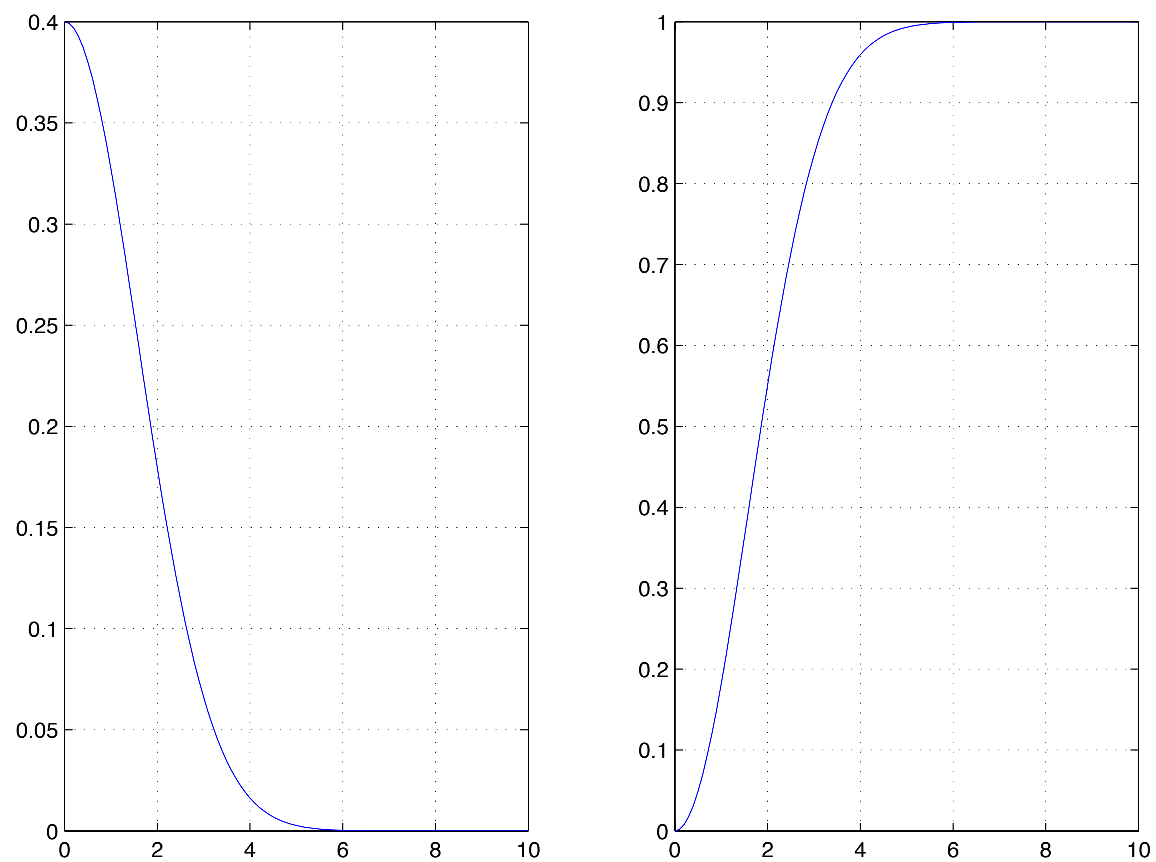
- PDF

$$f_X(x) = \frac{2}{\lambda} x e^{-\frac{x^2}{\lambda}} \qquad \text{for } x \geq 0$$

- CDF

$$F_X(x) = 1 - e^{-\frac{x^2}{\lambda}} \quad \text{for } x \geq 0$$

- Example for $\lambda = 5$



Conditional Distribution and Density Function

- Conditional probability of A

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Conditional distribution

$$F_X(x|B) = P\{X \leq x|B\} = \frac{P\{X \leq x \cap B\}}{P(B)}$$

  - where $$P\{X \leq x \cap B\} = \{X \leq x\} \cap B$$

- Properties of conditional distribution

$$\textbf{(1)} \ \ F_X(-\infty|B) = 0$$

$$\textbf{(2)} \ \ F_X(\infty|B) = 1$$

$$\textbf{(3)} \ \ 0 \leq F_X(X|B) \leq 1$$

$$\textbf{(4)} \ \ F_X(x_1|B) \leq F_X(x_2|B) \qquad \text{if } x_1 \leq x_2$$

$$\textbf{(5)} \ \ P\{x_1 < X \leq x_2|B\} = F_X(x_2|B) - F_X(x_1|B)$$

$$\textbf{(6)} \ \ F_X(x^+|B) = F_X(x|B)$$

- Conditional Density

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

- Properties of conditional density

(1) $f_X(x|B) \geq 0$

(2) $\int_{-\infty}^{\infty} f_X(x|B)\, dx = 1$

(3) $F_X(x|B) = \int_{-\infty}^{x} f_X(\zeta|B)\, d\zeta$

(4) $P(x_1 < X \leq x_2|B) = \int_{x_1}^{x_2} f_X(x|B)\, dx$

Example

- Define discrete random variable X to have values $x_1 = 1, x_2 = 2,$ and $x_3 = 3$ for selecting red, green, and blue balls, respectively.

  B_1, or B_2

x_i	Ball color	Box		
		1	2	Total
1	Red	5	80	85
2	Green	35	60	95
3	Blue	60	10	70
Totals		100	150	250

$$P(X=1|B=B_1)=\frac{5}{100},\ P(X=1|B=B_2)=\frac{80}{150},$$
$$P(X=2|B=B_1)=\frac{35}{100},\ P(X=2|B=B_2)=\frac{60}{150},$$
$$P(X=3|B=B_1)=\frac{60}{100},\ P(X=3|B=B_2)=\frac{10}{150}$$

- Conditional PDF $$f_X(x|B)=\frac{5}{100}\delta(x-1)+\frac{35}{100}\delta(x-2)+\frac{60}{100}\delta(x-3)$$

Conditional CDF

$$F_X(x|B) = \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3)$$

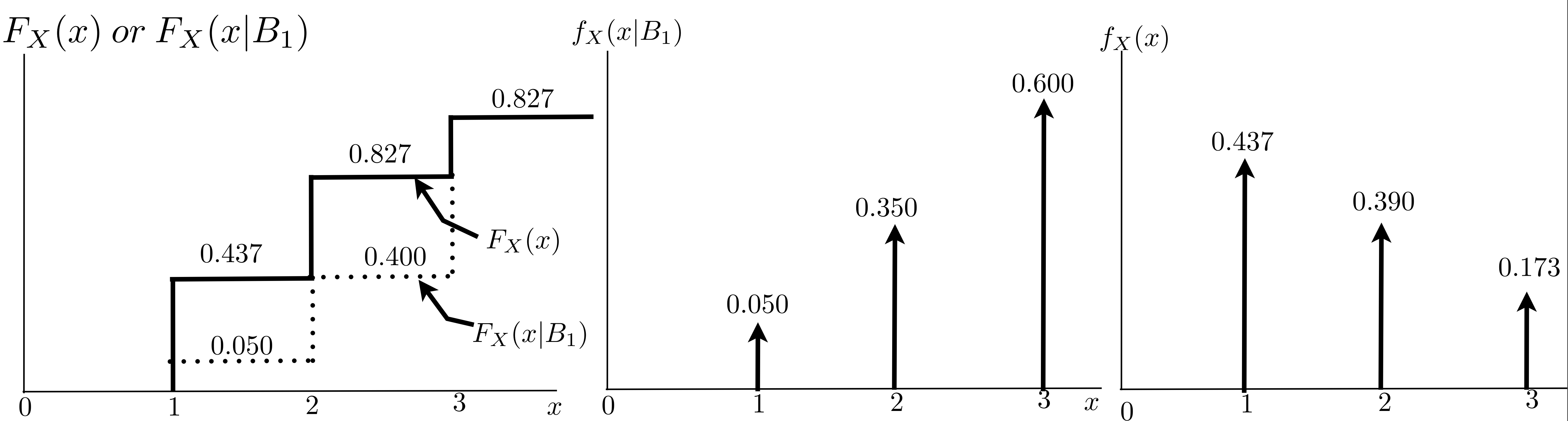
- Determine the probabilities of $P(X = 1), P(X = 2),$ and, $P(X = 3)$.

Total probability theorem

$$P(X=1) = P(X=1|B_1)P(B_1) + P(X=2|B_2)P(B_2) = \frac{5}{100}\cdot\frac{2}{10} + \frac{80}{150}\cdot\frac{8}{10} = 0.437$$

$$P(X=2) = \frac{35}{100}\cdot\frac{2}{10} + \frac{60}{150}\cdot\frac{8}{10} = 0.390$$

$$P(X=2) = \frac{60}{100}\cdot\frac{2}{10} + \frac{10}{150}\cdot\frac{8}{10} = 0.173$$



Conditional CDF/PDF

- Event B can be also random variable. For example,

$$B = \{X \le b\}$$

- Conditional CDF

$$F_X(x|X \le b) = P(X \le x | X \le b) = \frac{P(X \le x \cap X \le b)}{P(X \le b)}$$

- If $b \le x$, the event $\{X \le b\}$ is the subset of the event $\{X \le x\}$. Then, we have

$$F_X(x|X \le b) = \frac{P(X \le x \cap X \le b}{P(X \le b)} = \frac{P(X \le b)}{P(X \le b)} = 1$$

- On the other hand, when $x < b$, $\{X \le x\}$ is the subset of $\{X \le b\}$. In this case, we have

$$F_X(x|X \le b) = \frac{P(X \le x \cap X \le b)}{P(X \le b)} = \frac{P(X \le x)}{P(X \le b)} = \frac{F_X(x)}{F_X(b)}$$

- By combining those two cases

$$F_X(x|X \le b) = \begin{cases} \frac{F_X(x)}{F_X(b)}, & x < b \\ 1 & x \ge b \end{cases}$$

- The PDF can be obtained by taking the derivative of the CDF with respect to x.

$$f_X(x|X \le b) = \begin{cases} \frac{f_X(x)}{F_X(b)} = \frac{f_X(x)}{\int_{-\infty} b f_X(x)\, dx}, & x < b \\ 0 & x \ge b \end{cases}$$

- Remark

  - From our assumptions that the conditioning event has nonzero probability, we have

$$0 < F_X(b) \le 1$$

  - Hence, the conditional distribution is never smaller than the ordinary distribution function:

$$F_X(x|X \le b) \ge F_X(x)$$

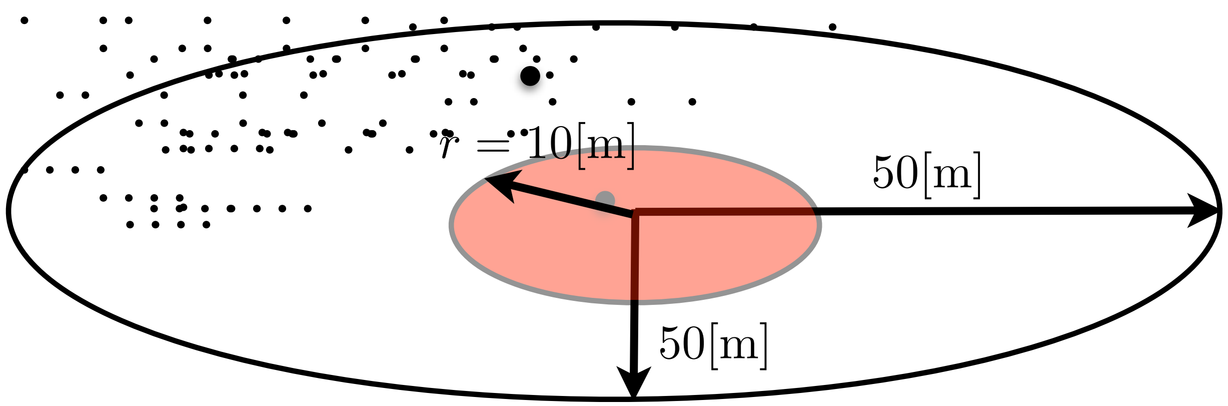
    - from which we also have

$$f_X(x|X \le b) \ge f_X(x), \qquad x < b$$

Example

- The radial "miss-distance" of landings from parachuting sky drivers, as measured from a target center, is Rayleigh random variable.

$$F_X(x) - [1 - e^{-x^2/800}]u(x)$$



- Find the probability of a parachuting hitting the bull's eye (circle in red) given that the landing is on the target.

$$P(\text{bull's eye}|\text{landing on target}) = \frac{F_X(10)}{F_X(50)} = \frac{1 - e^{-100/800}}{1 - e^{-2500/800}} = 0.1229$$