Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #13

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Outline

- M-ary orthogonal signals
- Optimum decision rule
  - MAP criterion
  - ML criterion
  - Minimum Euclidean distance rule
  - Maximum correlation rule

Technique of BER/SER Calculation

Assume a certain signal was transmitted, say $s_1(t)$

Calculate the conditional error probability, $P_2(e|s_1)$

Check if all the conditional probabilities are equal, that is, $P_2(e|s_1) = P_2(e|s_2)$, Then the average probability of error is

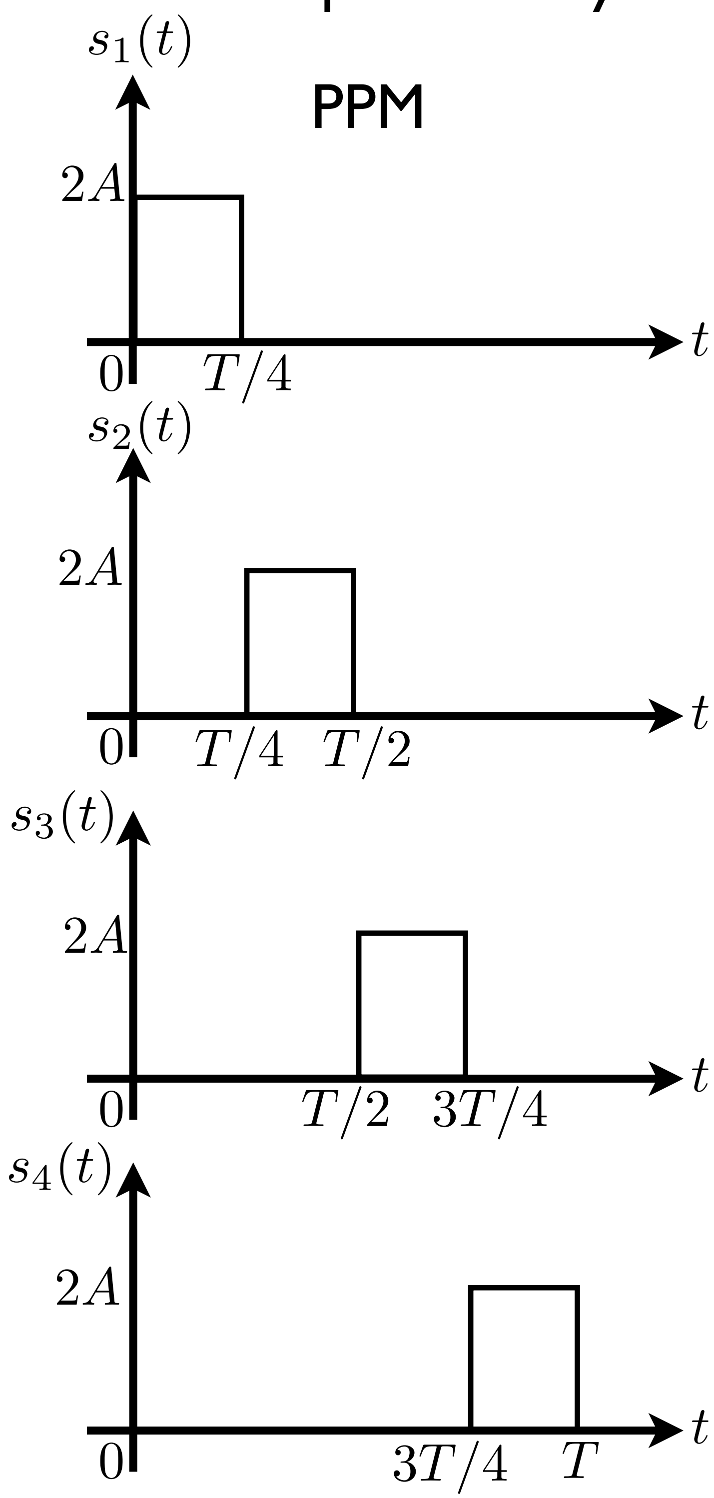
$$P_2 = P_2(e|s_1)P(s_1) + P_2(e|s_2)P(s_2)$$

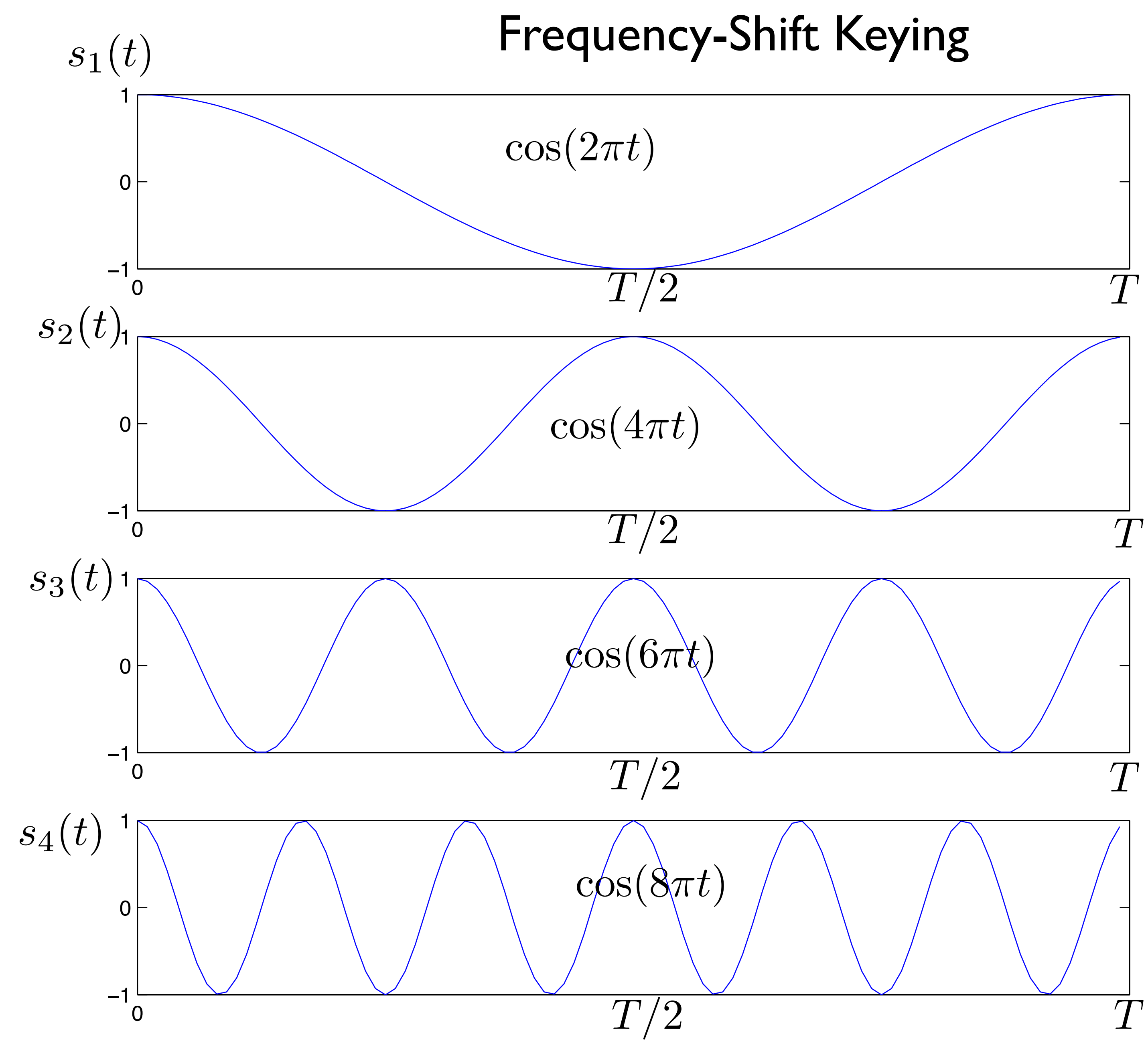
For equally probable case, that is, $P(s_1) = P(s_2) = 1/2$

$$P_2 = \frac{1}{2}(P_2(e|s_1) + P_2(e|s_2)) = P_2(e|s_1)$$

M-ary Orthogonal Signals

- Example of 4-ary orthogonal signal waveforms





- Orthogonality condition for $\{s_m(t)\}_{m=1}^{M}$

$$\int_0^T s_i(t)s_j(t)\,dt = 0, \quad i \neq j$$

- Signal waveform expression

$$s_m(t) = \sqrt{\mathcal{E}_s}\psi_m(t), \quad m = 1, 2, \ldots, M$$

  - For PPM,

$$\psi_m(t) = g_T\left(t - \frac{(m-1)T}{M}\right), \quad \frac{(m-1)T}{M} \leq t \leq \frac{mT}{M}$$

  - For frequency shift keying,

$$\psi_m(t) = \sqrt{\frac{2}{T}}\cos(2\pi mt), \quad m = 1, 2, \ldots, M$$

- Dimensionality of M-ary orthogonal signals
  - Dimensionality is M
- Energy

$$\int_0^T s_m^2(t)\,dt = \mathcal{E}_s \int_0^T \psi_m^2(t)\,dt = \mathcal{E}_s, \quad \text{all } m$$

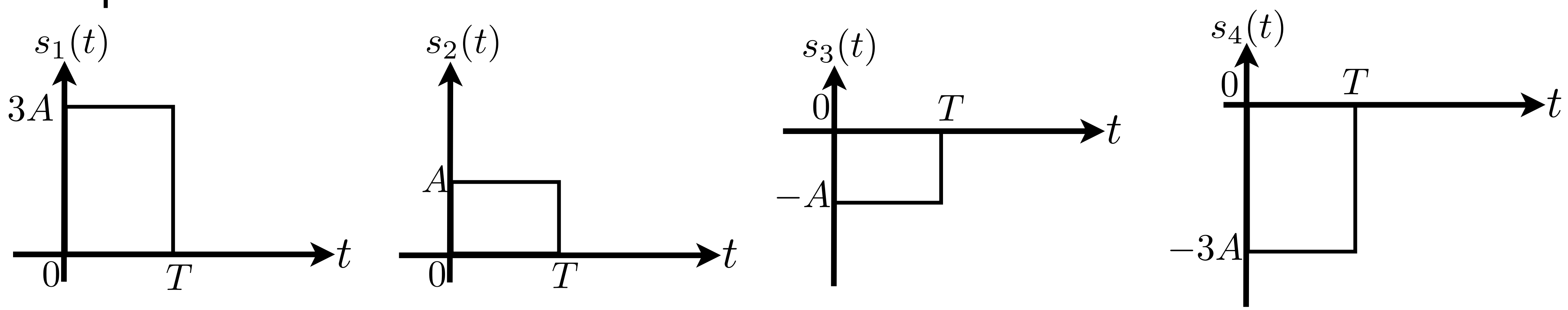
- Geometrical expression

$$\begin{aligned} \mathbf{s}_1 &= (\sqrt{\mathcal{E}_s}, 0, 0, \ldots, 0) \\ \mathbf{s}_2 &= (0, \sqrt{\mathcal{E}_s}, 0, \ldots, 0) \\ &\vdots \\ \mathbf{s}_M &= (0, 0, 0, \ldots, \sqrt{\mathcal{E}_s}) \end{aligned}$$

- Euclidean distance between M signal vectors are mutually equidistant, i.e.,

$$d_{mn} = \sqrt{||\mathbf{s}_m - \mathbf{s}_n||^2} = \sqrt{2\mathcal{E}_s}, \quad \text{for all } m \neq n$$

Example of 4-PAM



- Average energy

$$\mathcal{E}_{av} = 5A^2T = 5d^2 \qquad \text{where } d^2 = A^2T$$

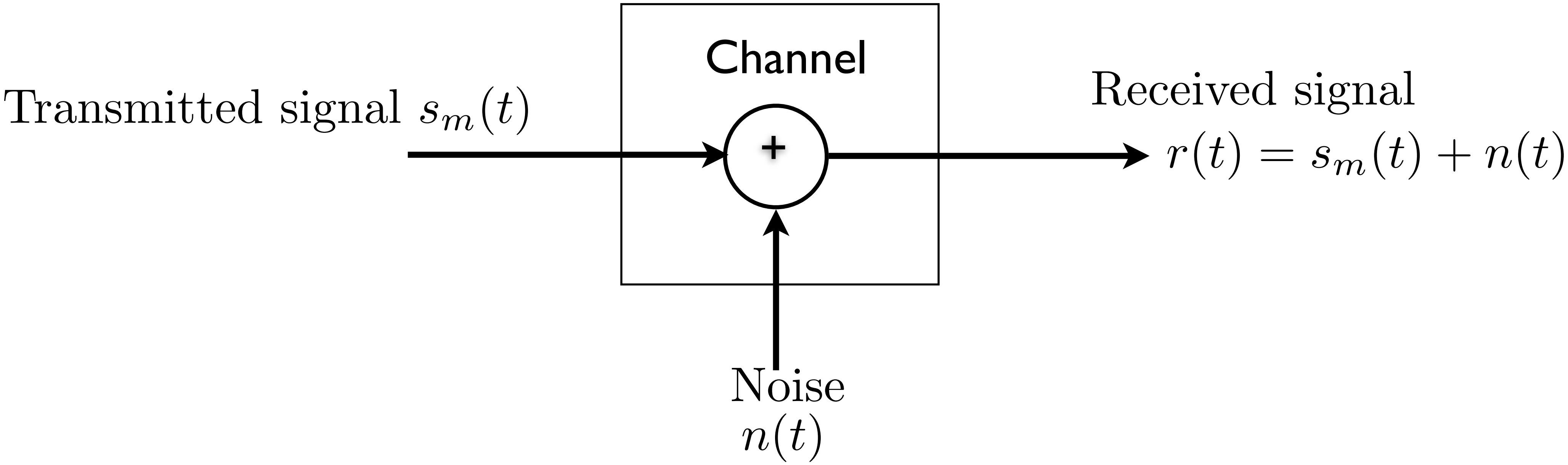
Optimum Receiver for M-ary Signals in AWGN

- Received signal over AWGN during the time interval $0 \le t \le T$

$$r(t) = s_m(t) + n(t), \quad 0 \le t \le T, \quad m = 1, 2, \ldots, M$$

where $M = 2^k$

- $n(t)$ is AWGN with PSD $S_n(f) = \dfrac{N_0}{2}$ [W/Hz]



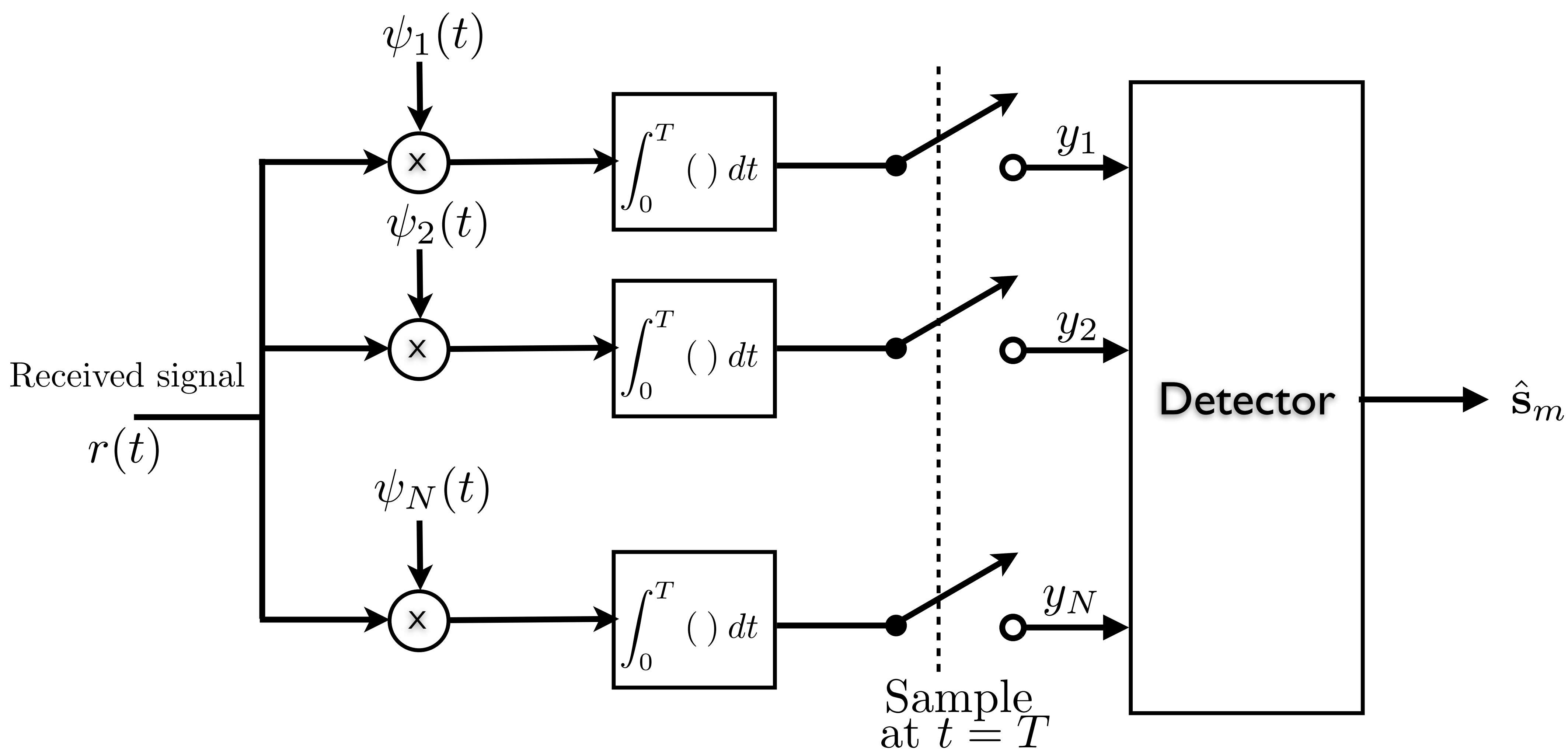
- Geometrical representation of M-ary signals

$$s_m(t) = \sum_{k=1}^{N} s_{mk}\psi_k(t), \quad 0 \le t \le T, \quad m = 1, 2, \ldots, M$$

where $\{s_{mk}\}$ are the coordinates of the signal vector

$$\mathbf{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN}), \quad m = 1, 2, \ldots, M$$

- Receiver with correlation type demodulator



- Correlator output at the end of the signal interval

$$\begin{aligned} y_k &= \int_0^T r(t)\psi_k(t)\,dt \\ &= \int_0^T [s_m(t) + n(t)]\psi_k(t)\,dt \\ &= \int_0^T \left(\sum_{n=1}^{N} s_{mn}\psi_n(t)\right)\psi_k(t)\,dt + \int_0^T n(t)\psi_k(t)\,dt \\ &= \sum_{n=1}^{N} s_{mn}\int_0^T \psi_n(t)\psi_k(t)\,dt + \int_0^T n(t)\psi_k(t)\,dt \\ &= s_{mk} + n_k \qquad \text{for } k = 1, 2, \ldots, N \end{aligned}$$

where $n_k = \int_0^T n(t)\psi_k(t)\,dt$

- In vector form we can express the output signal of the demodulator as

$$\mathbf{y} = \mathbf{s}_m + \mathbf{n}$$

- Statistics of noise

$$n_k = \int_0^T n(t)\psi_k(t)\,dt$$ is Gaussian. Hence, we need to find the mean and variance for PDF.

- Mean

$$E[n_k] = \int_0^T E[n(t)]\psi_k(t)\,dt = 0$$

- Covariance

$$\begin{aligned} E[n_k n_j] &= \int_0^T \int_0^T E[n(t)n(\tau)]\psi_k(t)\psi_j(t)\,dtd\tau \\ &= \int_0^T \int_0^T \frac{N_0}{2}\frac{N_0}{2}\delta(t-\tau)\psi_k(t)\psi_j(\tau)\,dtd\tau \\ &= \frac{N_0}{2}\int_0^T \psi_k(t)\psi_j(t)\,dt \\ &= \frac{N_0}{2}\delta_{jk} \end{aligned}$$

where $$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

- Joint PDF of noise

$$f(\mathbf{n}) = \prod_{l=1}^{N} f(n_l) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{l=1}^{N} \frac{n_l^2}{N_0}}$$

- Statistic of the output signal at the demodulator

$y_k = s_{mk} + n_k$ which is also Gaussian given s_{mk}.

  - Mean

    $$E[y_k] = E[s_{mk} + n_k] = s_{mk}$$

  - Variance

    $$\text{var}[y_k] = N_0/2$$

  - Conditional PDF of $\mathbf{y} = (y_1, y_2, \ldots, y_N)$

$$\begin{aligned} f(\mathbf{y}|\mathbf{s}_m) &= \prod_{k=1}^{N} f(y_k|s_{mk}) \\ &= \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} (y_k - s_{mk})^2 / N_0\right] \\ &= \frac{1}{(\pi N_0)^{N/2}} \exp[-||\mathbf{y} - \mathbf{s}_m||^2 / N_0], \quad m = 1, 2, \ldots, M. \end{aligned}$$

Example of 4-PAM

- Received signal

$$r(t) = s_m(t) + n(t)$$

- Output of the demodulator

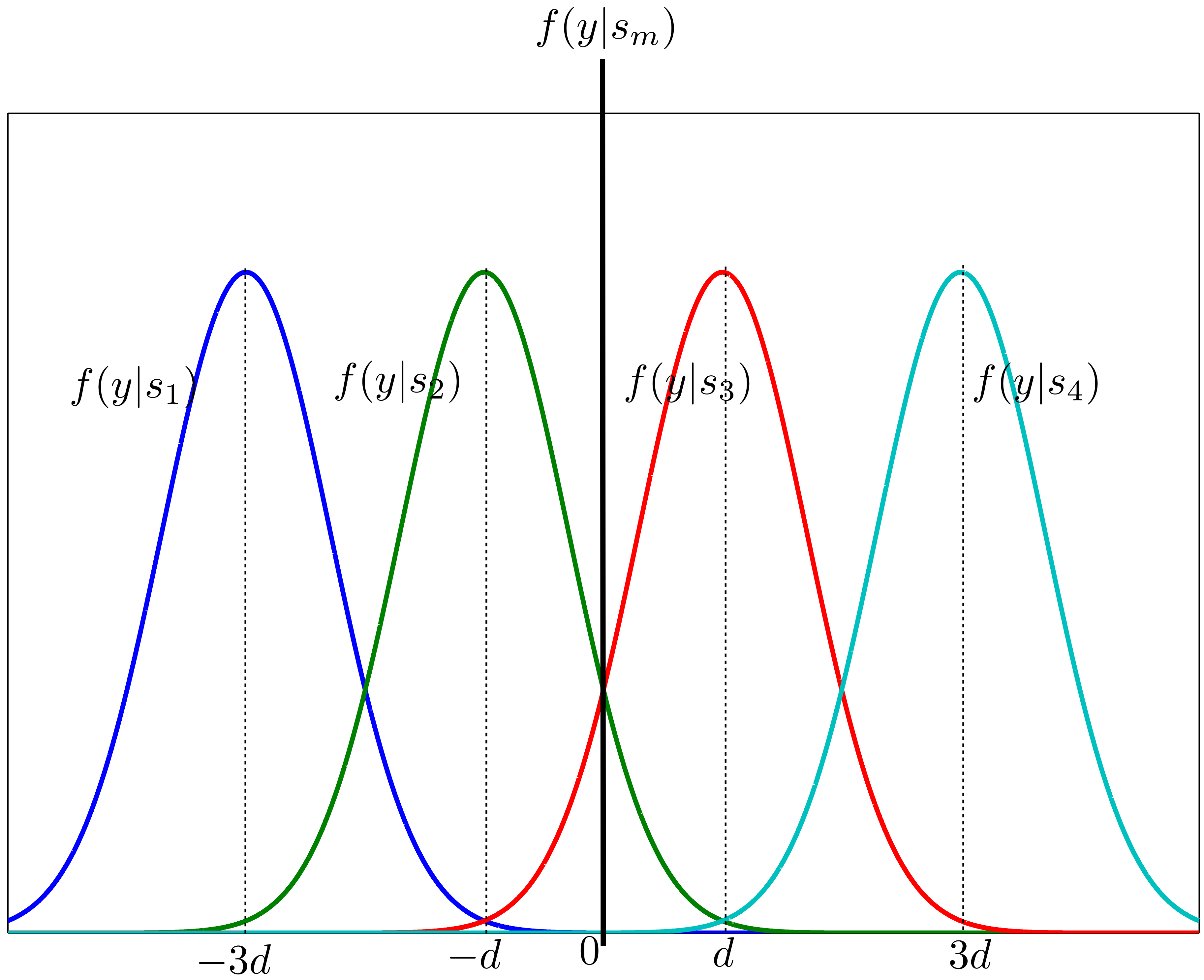
$$y(T) = \int_0^T r(t)\psi(t)\,dt = \int_0^T [s_m(t) + n(t)]\psi(t)\,dt = s_m + n,$$

where $n \sim \mathcal{N}(0, N_0/2)$

- PDF of $y(T)$

$$f(y|s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1, 2, \ldots, M$$

where $s_m = (2m - 1 - M)d.$



$f(y|s_m)$
$f(y|s_1)$
$f(y|s_2)$
$f(y|s_3)$
$f(y|s_4)$
$-3d$
$-d$
0
d
$3d$

Example of 4-PPM

- The signal vector

$$\mathbf{s}_1 = (\sqrt{\mathcal{E}_s}, 0, 0, 0)$$

- Received signal vector under $s_1(t)$

$$\mathbf{y} = \mathbf{s}_1 + \mathbf{n} = \left(\sqrt{\mathcal{E}_s} + n_1, n_2, n_3, n_4\right)$$

- Joint conditional PDF

$$f(\mathbf{y}|\mathbf{s}_1) = f(y_1, y_2, y_3, y_4|\mathbf{s}_1) = \frac{1}{(\pi N_0)^2} \exp\left[-\frac{(y_1 - \sqrt{\mathcal{E}_s})^2 + y_2^2 + y_3^2 + y_4^2}{N_0}\right]$$

Optimum Detector

- Posteriori probability

$$P(\text{signal } \mathbf{s}_m \text{ was transmitted } |\mathbf{y})$$

- Optimum decision rule
  - Select the signal corresponding to the maximum set of posteriori probabilities:

$$\text{choose } m \text{ such that } \{P(\mathbf{s}_m|\mathbf{y})\}_{m=1}^{M} \text{ is maximum}$$

  - which is called "*maximum a posteriori (MAP)*" criterion.

- Bays' rule

$$P(\mathbf{s}_m|\mathbf{y}) = \frac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}$$

  - where $P(\mathbf{s}_m)$ is called "a priori probability".

- MAP criterion

$$\text{choose } m \text{ such that } \{P(\mathbf{s}_m|\mathbf{y})\}_{m=1}^{M} \text{ is maximum}$$
$$= \text{choose } m \text{ such that } \{\tfrac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}\}_{m=1}^{M} \text{ is maximum}$$

  - For equally probable case, that is, $P(\mathbf{s}_m) = \dfrac{1}{M}$, MAP criterion becomes

$$\text{choose } m \text{ such that } \{f(\mathbf{y}|\mathbf{s}_m)\}_{m=1}^{M} \text{ is maximum}$$

    - which is called "maximum likelihood (ML)" criterion.

- Definition
  - likelihood function: $f(\mathbf{y}|\mathbf{s}_m)$
  - Log-likelihood function: $\ln f(\mathbf{y}|\mathbf{s}_m)$

MAP criterion

$$\text{choose } m \text{ such that } \{P(\mathbf{s}_m|\mathbf{y})\}_{m=1}^{M} \text{ is maximum}$$

$$= \text{choose } m \text{ such that } \{\tfrac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}\}_{m=1}^{M} \text{ is maximum}$$

for equally probable case => $= \text{choose } m \text{ such that } f(\mathbf{y}|\mathbf{s}_m)$

$$= \text{choose } m \text{ such that } \ln f(\mathbf{y}|\mathbf{s}_m)$$

4-PAM case

- Likelihood function

$$f(y|s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1, 2, \ldots, M$$

- Log-Likelihood function

$$\ln f(y|s_m) = -\frac{1}{2}\log(\pi N_0) - \frac{(y-s_m)^2}{N_0}, \quad m = 1, 2, \ldots, M$$

- ML criterion

$$\max_m \left[-\frac{1}{2}\log(\pi N_0) - \frac{(y-s_m)^2}{N_0}\right], \quad m = 1, 2, \ldots, M$$

$$= \max_m \left[-\frac{(y-s_m)^2}{N_0}\right], \quad m = 1, 2, \ldots, M$$

$$= \min_m \left[(y-s_m)^2\right], \quad m = 1, 2, \ldots, M$$

$$= \min_m ||y-s_m||, \quad m = 1, 2, \ldots, M$$

- Generally, the output of the demodulator over AWGN channel can be written as

$$y_k = s_{mk} + n_k, \quad k = 1, 2, \ldots, N$$

dimension

- Its likelihood function is given as

$$f(y_k | s_{mk}) = \frac{1}{\sqrt{\pi N_0}} e^{-(y - s_{mk})^2 / N_0}, \quad m = 1, 2, \ldots, M$$

- Joint likelihood function

$$f(\mathbf{y} | \mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{k=1}^{N} (y - s_{mk})^2 / N_0}, \quad m = 1, 2, \ldots, M$$

- Log-likelihood function

$$\ln f(\mathbf{y} | \mathbf{s}_m) = -\frac{N}{2} \ln(\pi N_0) - \ln \sum_{k=1}^{N} \frac{(y - s_{mk})^2}{N_0}, \quad m = 1, 2, \ldots, M$$

- ML criterion

$$\min_{m} (y - s_{mk})^2, \quad m = 1, 2, \ldots, M$$

  - which is called *minimum (Euclidean) distance rule*

- Optimum decision rule
  - MAP criterion becomes ML criterion for equally probable case.
  - ML criterion can be reduced to minimum Euclidean distance rule over AWGN channels.
- Calculation of Euclidean distance

$$
\begin{aligned}
D(\mathbf{y}, \mathbf{s}_m) &= \sum_{n=1}^{N} y_n^2 - 2\sum_{n=1}^{N} y_n s_{mn} + \sum_{n=1}^{N} s_{mn}^2 \\
&= ||\mathbf{y}||^2 - 2\,\mathbf{y}\cdot\mathbf{s}_m + ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M
\end{aligned}
$$

  - Minimum distance rule choose $\mathbf{s}_m$ to give the minimum distance metric which is equivalent to choose minimum value of the metric given as

$$D'(\mathbf{y}, \mathbf{s}_m) = -2\,\mathbf{y}\cdot\mathbf{s}_m + ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M$$

or choose the maximum distance metric given as

$$C(\mathbf{y}, \mathbf{s}_m) = 2\,\mathbf{y}\cdot\mathbf{s}_m - ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M$$

Correlation metric

$$C(\mathbf{y}, \mathbf{s}_m) = 2\,\mathbf{y} \cdot \mathbf{s}_m - ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M$$

- We choose s_m which gives maximum correlation metric.
- If all the signals have equal energy, that is, $||\mathrm{s}_m||^2 = \mathcal{E}_s$, for all *m*
  - we can just neglect the term $||\mathrm{s}_m||^2$.

Summary of Optimum Decision Rule

- Optimum decision rule is MAP criterion.
- MAP criterion is equivalent to ML criterion for equally probable case.
- ML criterion is equivalent to minimum Euclidean distance rule over AWGN channels.
- Minimum Euclidean distance rule is equivalent to maximum correlation rule.