KECE321 Communication Systems I

(Haykin Sec. 2.1 - Sec. 2.2 and Ziemer Sec. 2.5)

Lecture #6, March 21, 2012

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From Fourier Series to Fourier Transform

Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi n f_0 t}\, dt \right] e^{j2\pi n f_0 t}$$

where $x(t)$ is periodic signal with period T_0

Now $T_0 \to \infty$ then $nf_0 \to f$, $1/T_0 \to df$, and $\sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty}$

Now $T_0 \to \infty$, which means $f_0 \to 0$

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi n f_0 t} \, dt \right] e^{j2\pi n f_0 t}$$

$\frac{1}{T_0} \to df$, $T_0/2 \to \infty$, $-T_0/2 \to -\infty$, $n f_0 \to f$

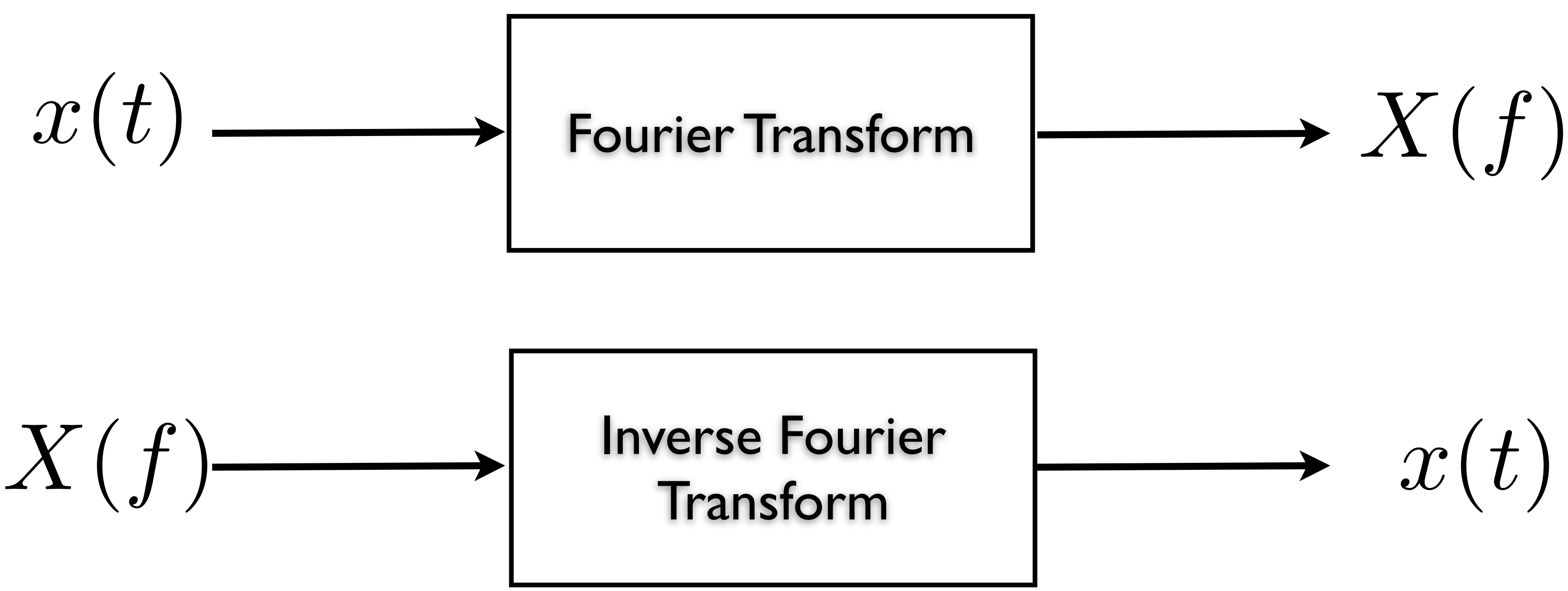
$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt \right] e^{j2\pi f t} \, df$$

$= X(f)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt \qquad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$



- Notations

$$X(f) = \mathcal{F}[x(t)]$$

$$x(t) = \mathcal{F}^{-1}[X(f)]$$

$$x(t) \Longleftrightarrow X(f)$$

Properties of Fourier Transform

- Linearity
- Dilation
- Conjugation rule
- Duality property
- Time shifting property
- Frequency shifting property
- Area property
- Differentiation in the time domain
- Modulation theorem
- Convolution theorem
- Correlation theorem
- Rayleigh's Energy theorem (or Parserval's theorem)

Linearity

- Let

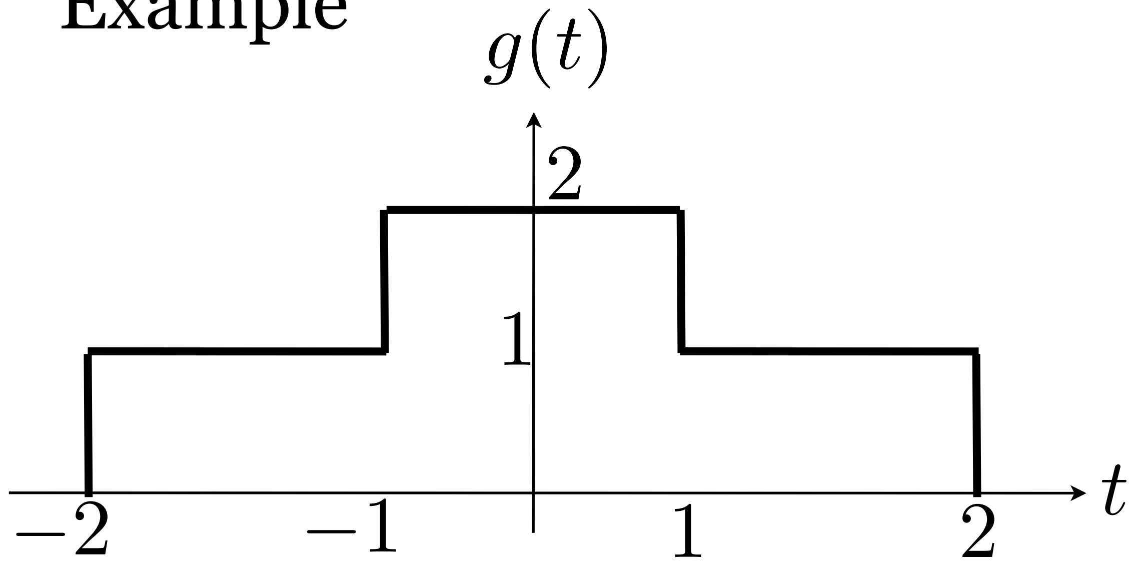
$$g_1(t) \Longleftrightarrow G_1(f)$$

$$g_2(t) \Longleftrightarrow G_2(f)$$

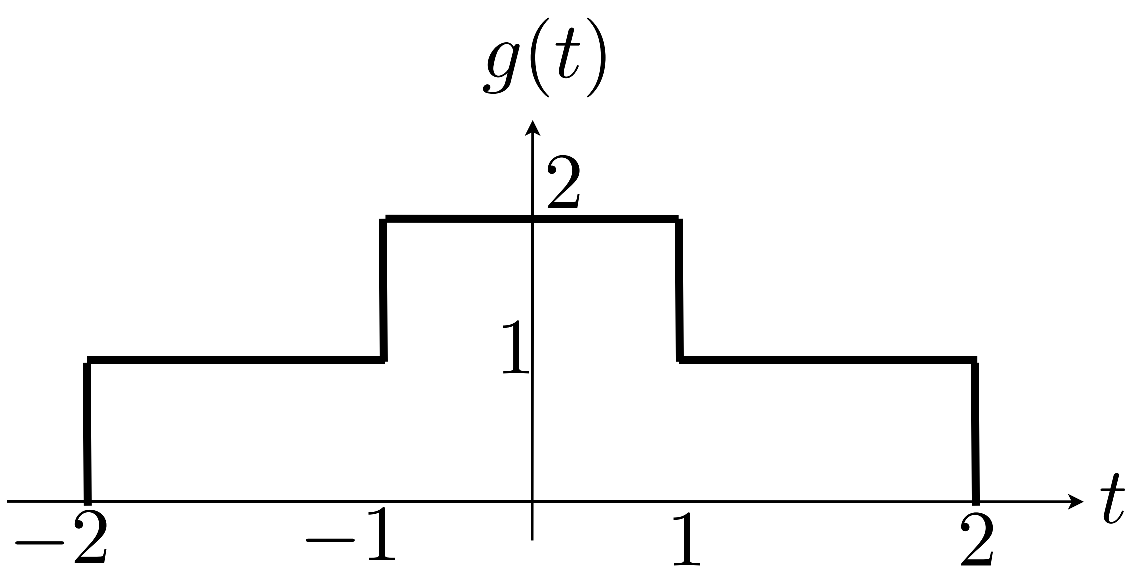
then for all constants c_1 and c_2

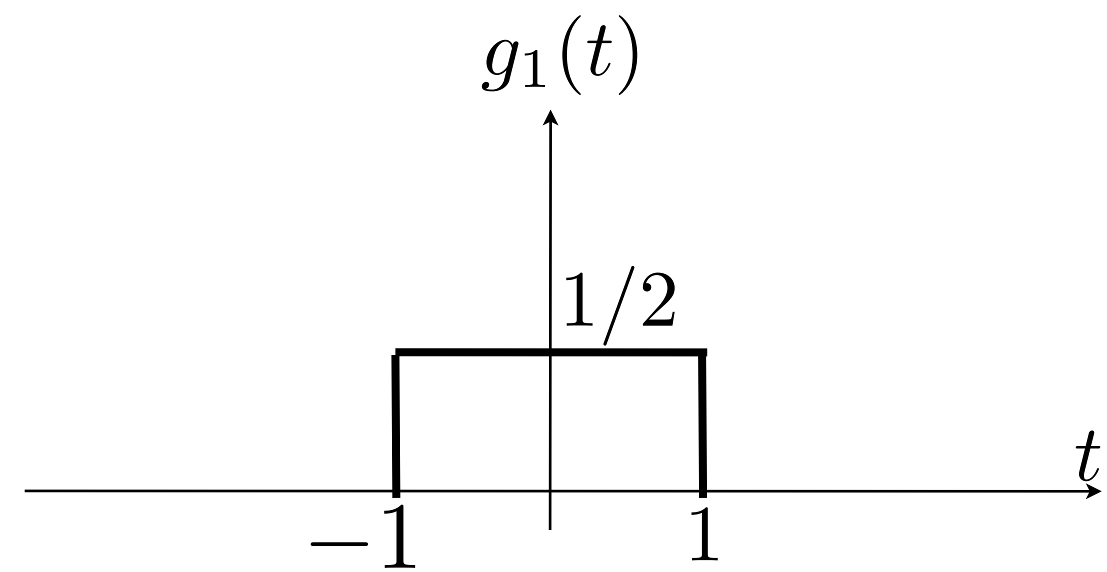
$$c_1 g_1(t) + c_2 g_2(t) \Longleftrightarrow c_1 G_1(f) + c_2 G_2(f)$$

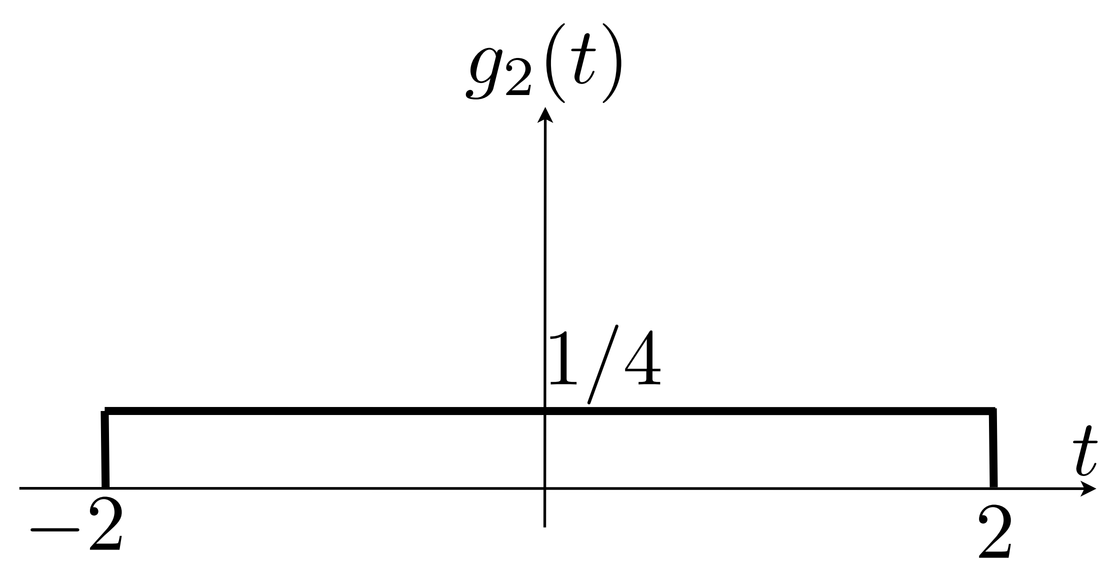
Example



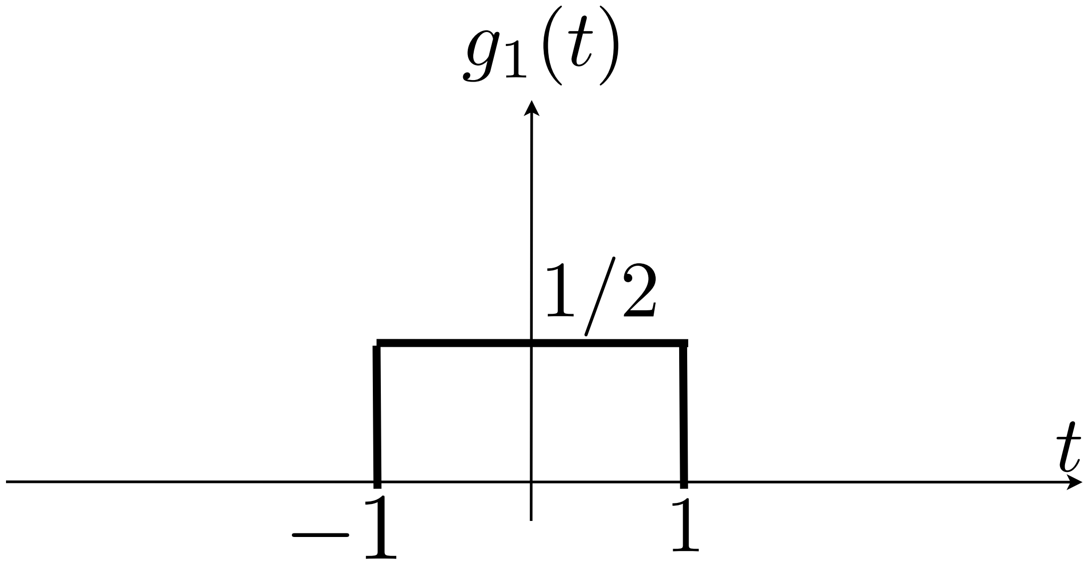
- Find the Fourier transform of $g(t)$.



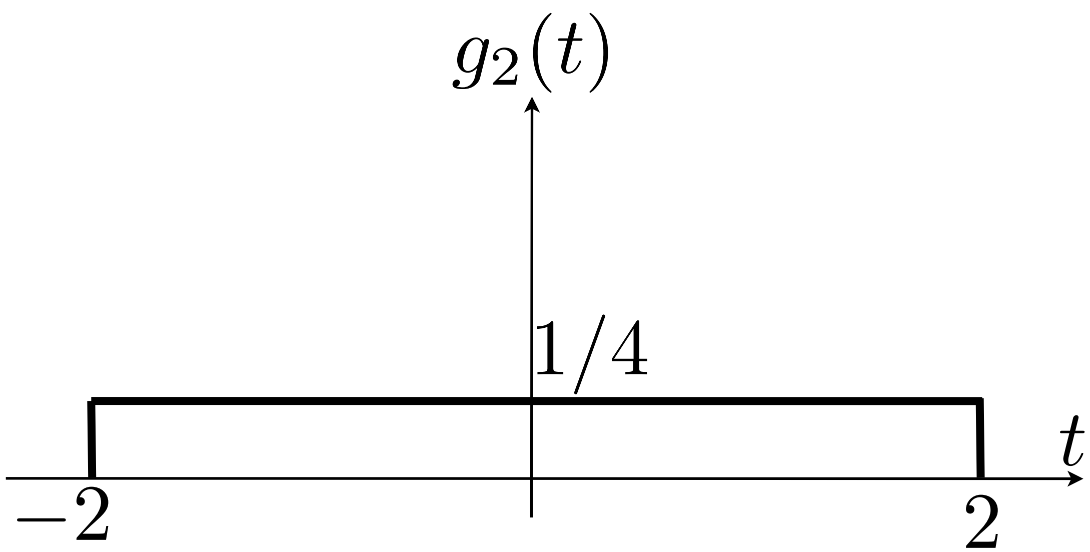




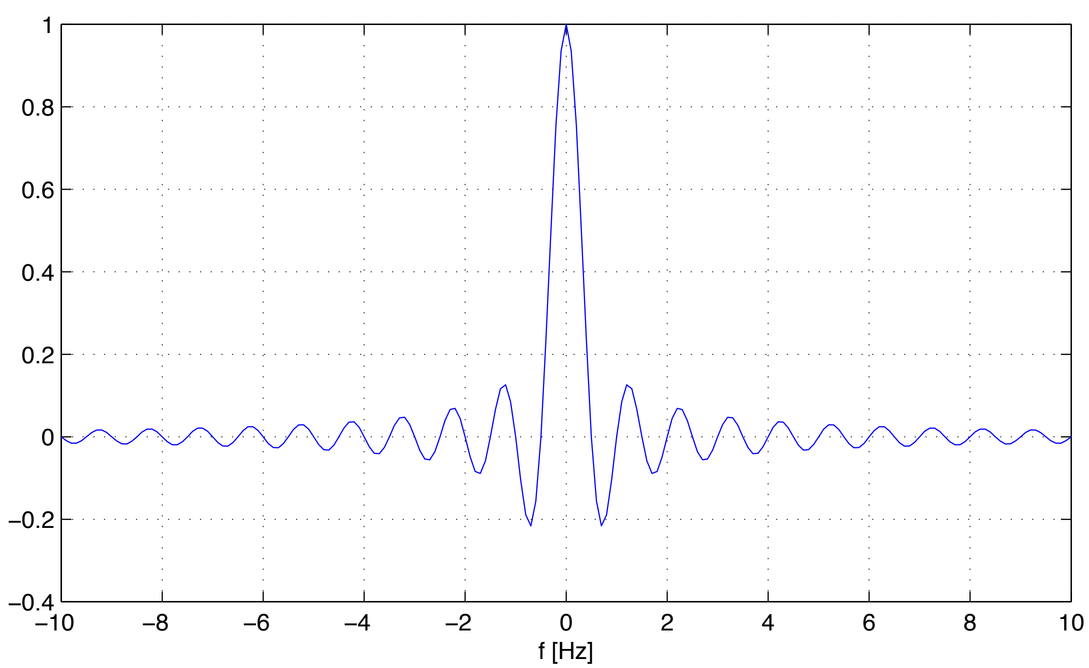
$$g(t) = 2g_1(t) + 4g_2(t)$$

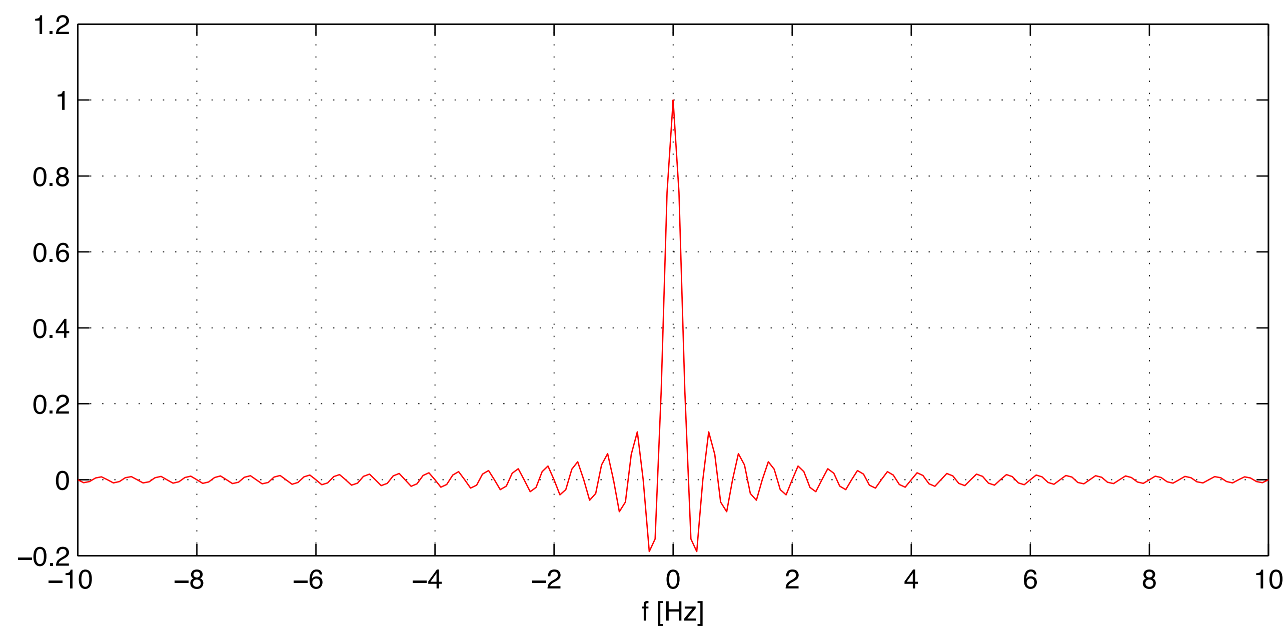


$$\mathcal{F}\left[\frac{1}{2}\text{rect}\left(\frac{t}{2}\right)\right] = \text{sinc}(2f)$$

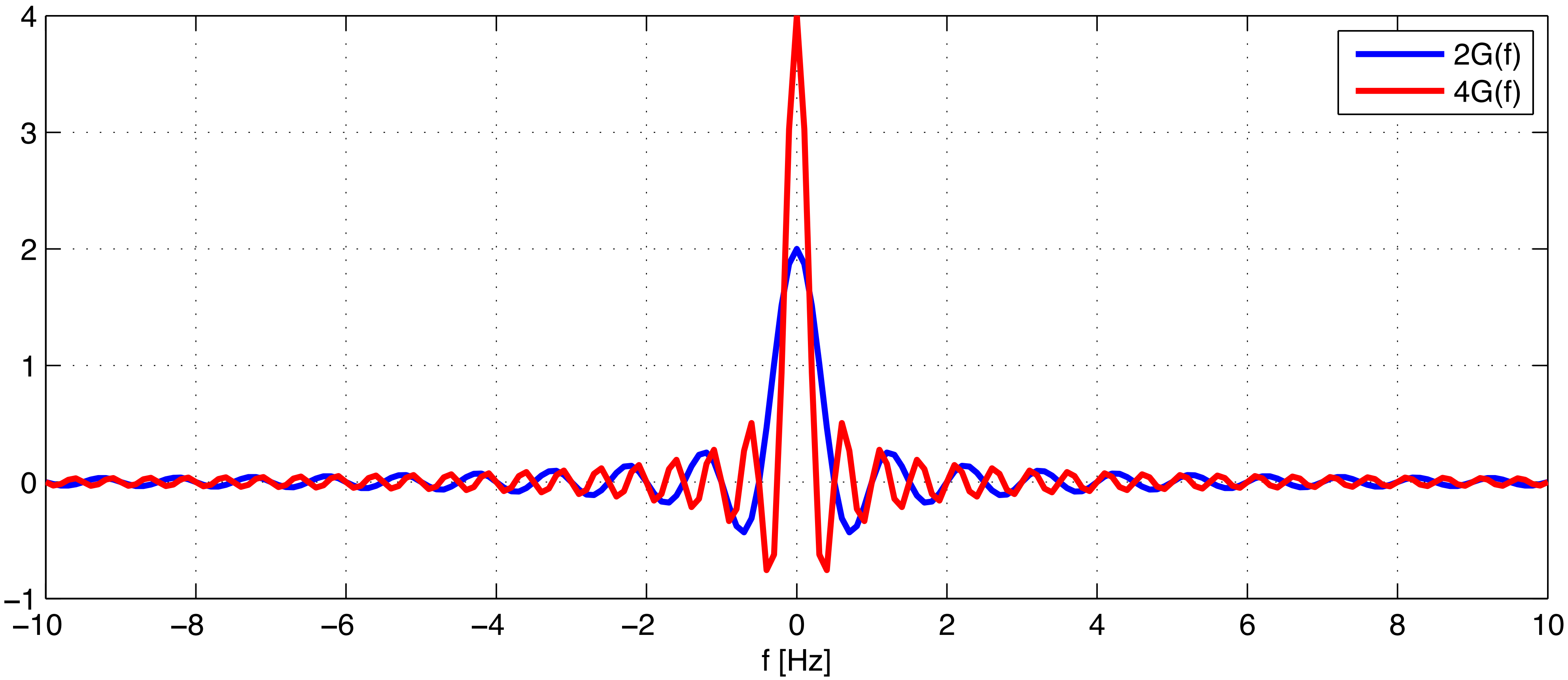


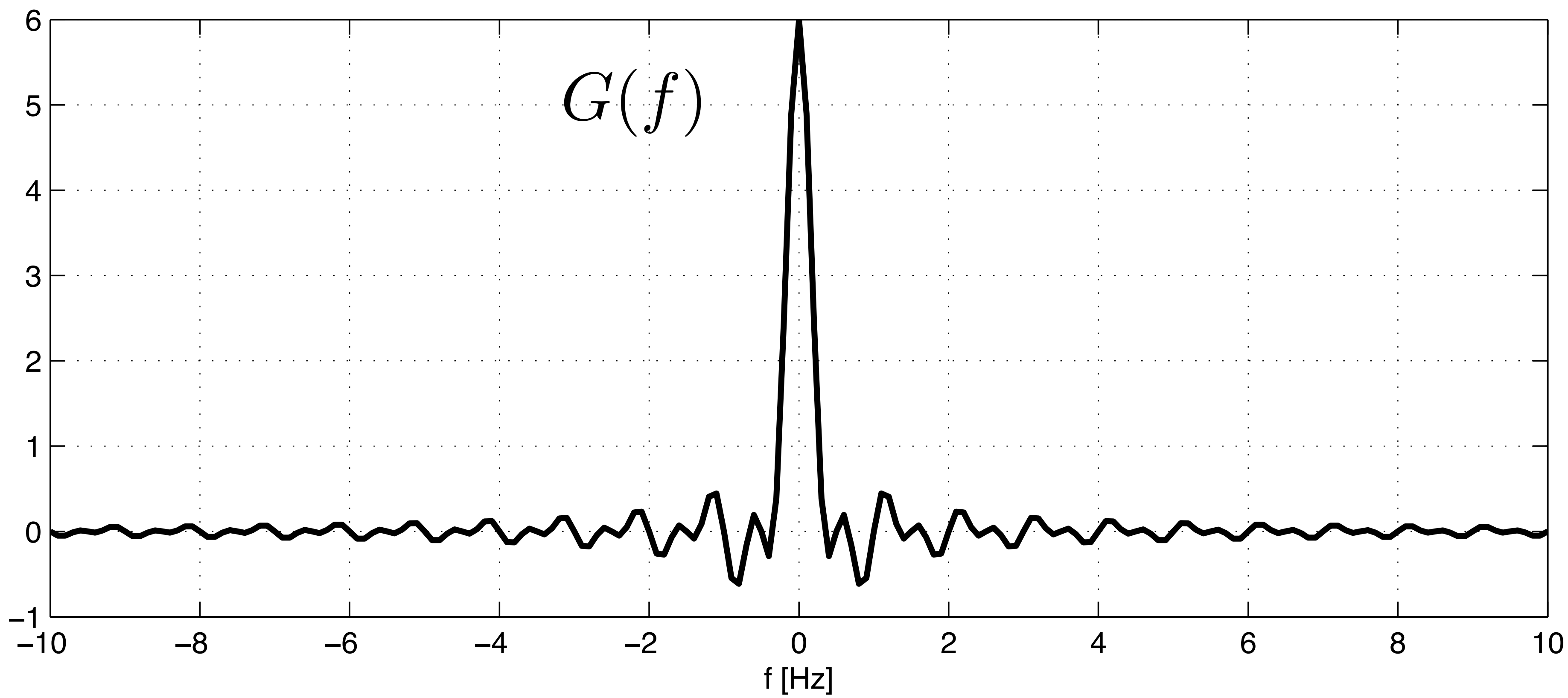
$$\mathcal{F}\left[\frac{1}{4}\text{rect}\left(\frac{t}{4}\right)\right] = \text{sinc}(4f)$$

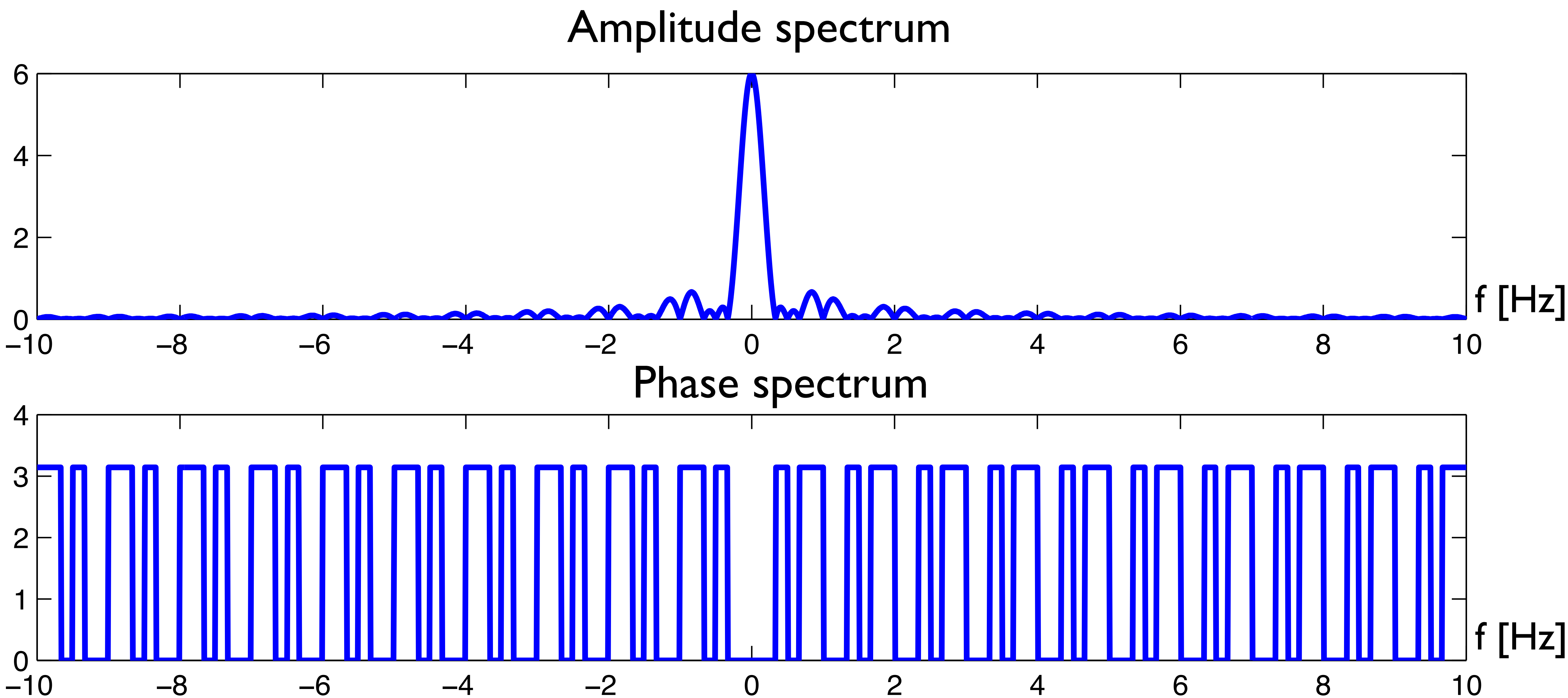




$$g(t) = 2g_1(t) + 4g_2(t) \Longleftrightarrow G(f) = 2G_1(f) + 4G_2(f)$$






Amplitude spectrum
6
4
2
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
f [Hz]
Phase spectrum
4
3
2
1
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
f [Hz]

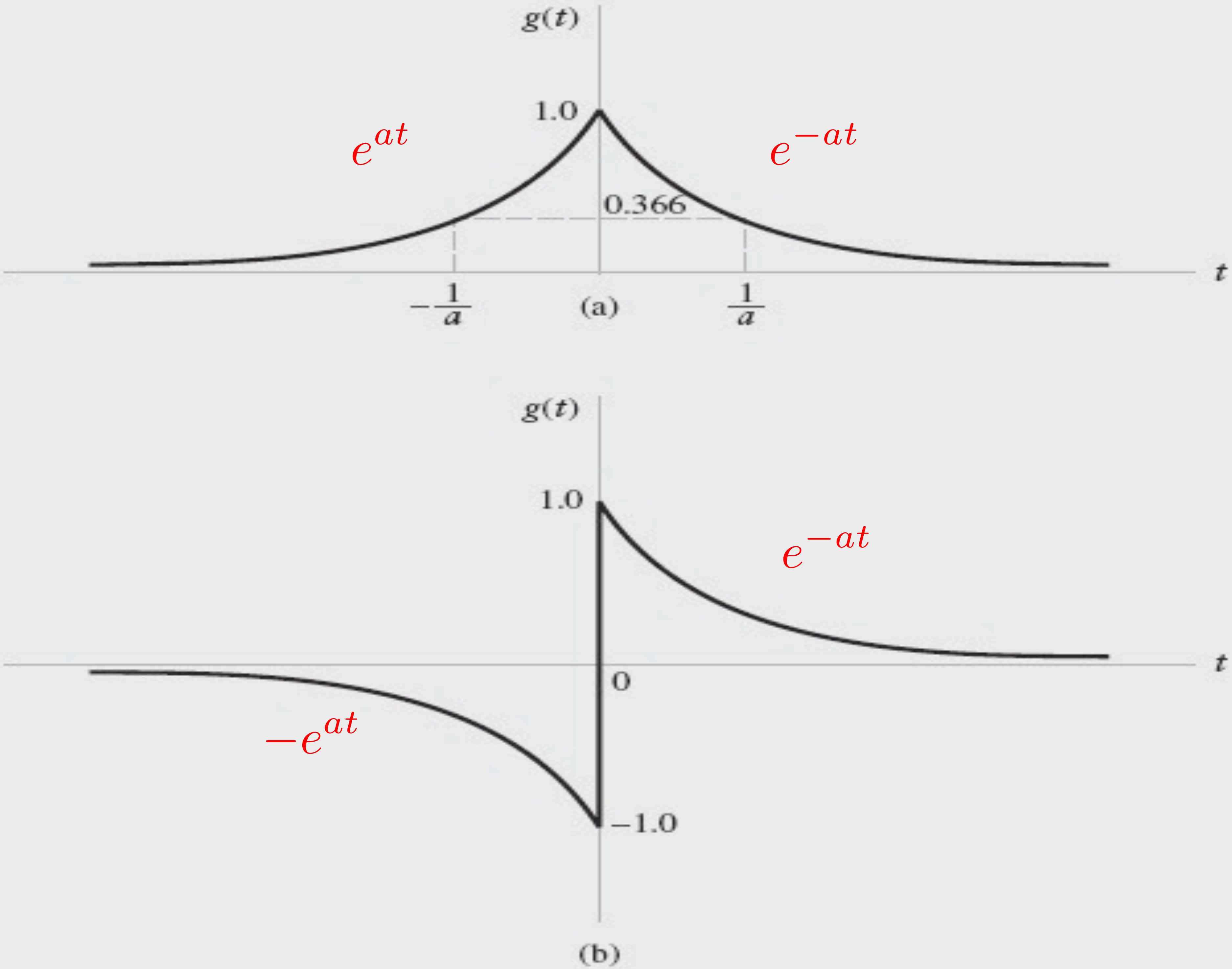


FIGURE 2.6 (*a*) Double-exponential pulse (symmetric). (*b*) Another double-exponential pulse (odd-symmetric).

EXAMPLE 2.3 Combinations of Exponential Pulses

Consider a *double exponential pulse* (defined by (see Fig. 2.6(*a*))

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$
$$= \exp(-a|t|) \qquad (2.15)$$

This pulse may be viewed as the sum of a truncated decaying exponential pulse and a truncated rising exponential pulse. Therefore, using the linearity property and the Fourier-transform pairs of Eqs. (2.12) and (2.13), we find that the Fourier transform of the double exponential pulse of Fig. 2.6(*a*) is

$$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f}$$
$$= \frac{2a}{a^2 + (2\pi f)^2}$$

We thus have the following Fourier-transform pair for the double exponential pulse of Fig. 2.6(*a*):

$$\exp(-a|t|) \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2} \qquad (2.16)$$

[Ref: Haykin & Moher, Textbook]

Dilation Property

$$g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

Reflection Property

$$g(-t) \longleftrightarrow G(-f)$$

Conjugation Rule

$$g^*(t) \longleftrightarrow G^*(-f)$$

Duality Property

If $\quad g(t) \longleftrightarrow G(f)$

then $\quad G(t) \longleftrightarrow g(-f)$

Example of Duality Property: Sinc Pulse

- We have the following pair of the Fourier transform:

$$g(t) = A\,\mathrm{sinc}(2Wt) \longleftrightarrow G(f) = \frac{A}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$$

- Then, if the time function, given as

$$\frac{A}{2W}\mathrm{rect}\left(\frac{t}{2W}\right) \longleftrightarrow A\mathrm{sinc}(-2Wf) = A\mathrm{sinc}(2Wf)$$

Time Shifting Property

If $$g(t) \longleftrightarrow G(f)$$

then $$g(t - t_0) \longleftrightarrow G(f)\exp(-j2\pi f t_0)$$

Frequency Shifting Property

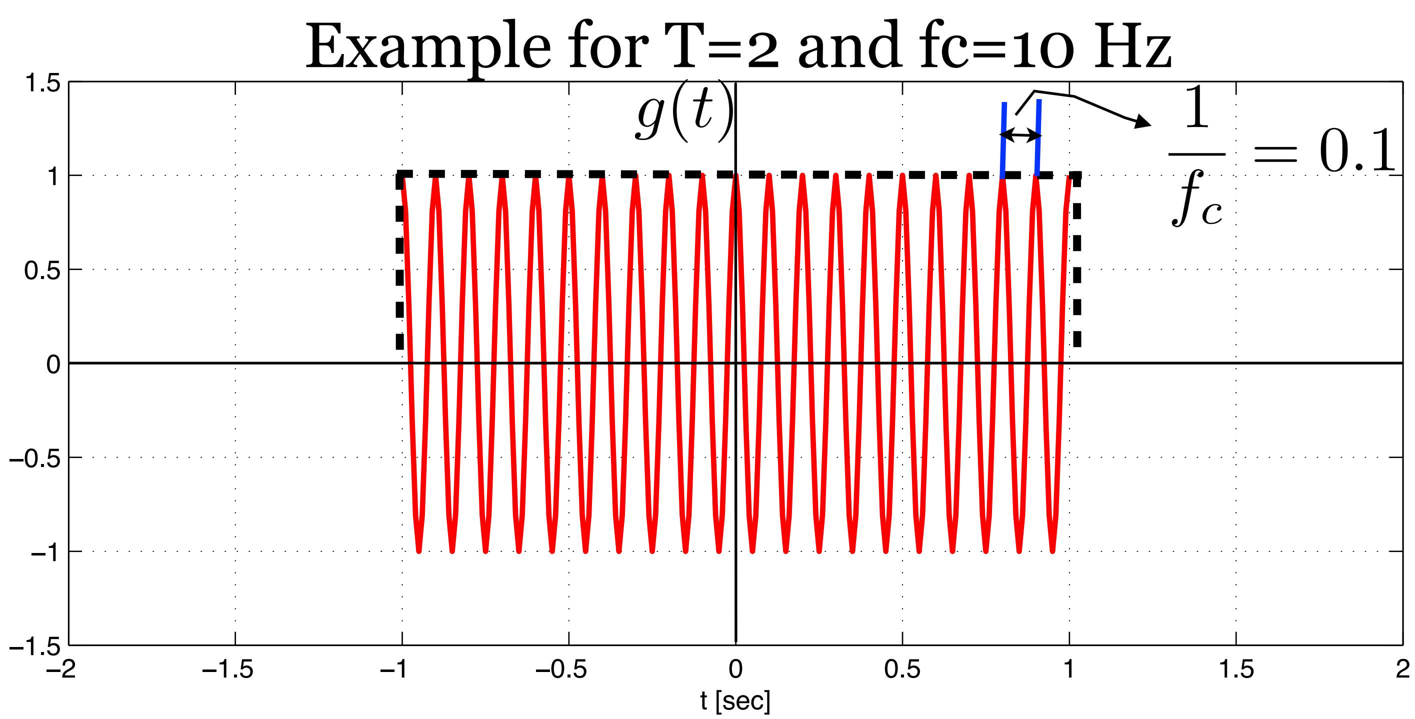
If $$g(t) \longleftrightarrow G(f)$$

then $$\exp(j2\pi f_c t)g(t) \longleftrightarrow G(f - f_c)$$

Example of Frequency Shifting Property

$$s(t) = \text{rect}\left(\frac{t}{T}\right) \quad \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} \quad S(f) = T\text{sinc}(Tf)$$

$$g(t) = s(t) \cdot \cos(2\pi f_c t) \longrightarrow G(f) = ?$$



$$\cos(2\pi f_c t) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$$

Euler's formula

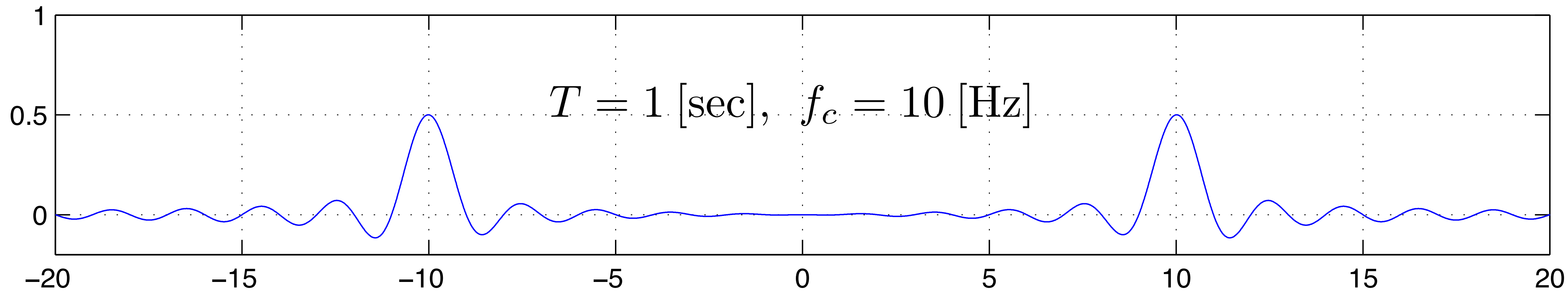
Frequency shift property

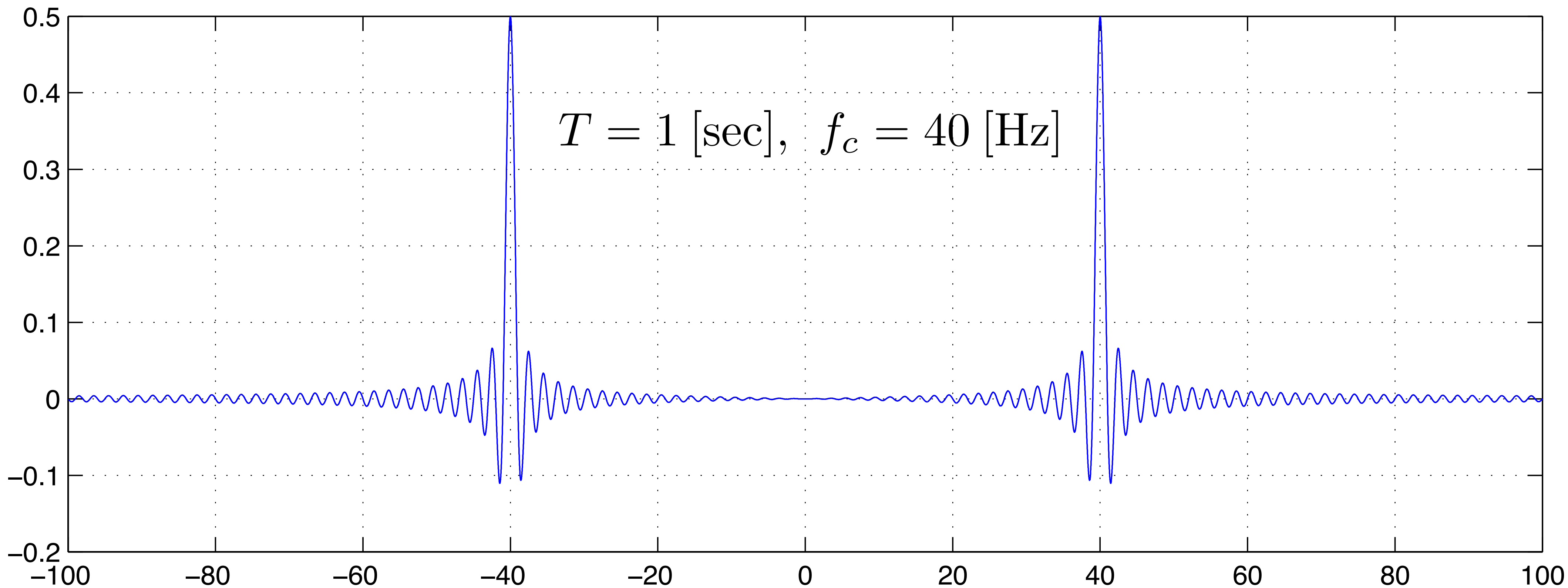
$$\exp(j2\pi f_c t) g(t) \longleftrightarrow G(f - f_c)$$

$$g(t) = s(t) \cdot \cos(2\pi f_c t) = \frac{\mathrm{rect}\left(\frac{t}{T}\right)}{2} e^{j2\pi f_c t} + \frac{\mathrm{rect}\left(\frac{t}{T}\right)}{2} e^{-j2\pi f_c t}$$

$$G(f) = \frac{1}{2} S(f - f_c) + \frac{1}{2} S(f + f_c)$$

$$= \frac{T}{2}\mathrm{sinc}[T(f - f_c)] + \frac{T}{2}\mathrm{sinc}[T(f + f_c)]$$


$T = 1$ [sec], $f_c = 10$ [Hz]
1
0.5
0
–20
–15
–10
–5
0
5
10
15
20


$T = 1$ [sec], $f_c = 40$ [Hz]
0.5
0.4
0.3
0.2
0.1
0
–0.1
–0.2
–100
–80
–60
–40
–20
0
20
40
60
80
100

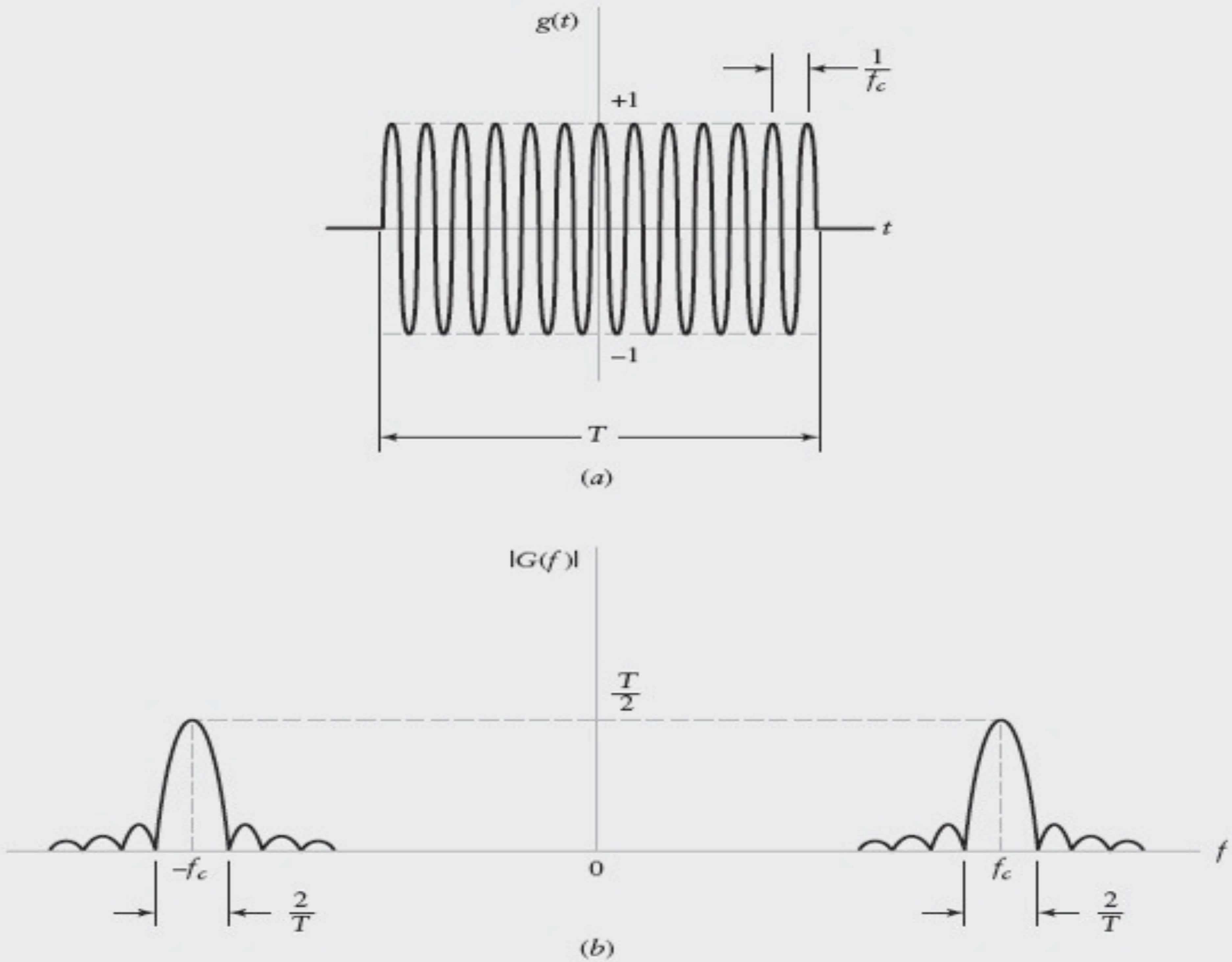


FIGURE 2.9 (a) RF pulse of unit amplitude and duration T. (b) Amplitude spectrum.

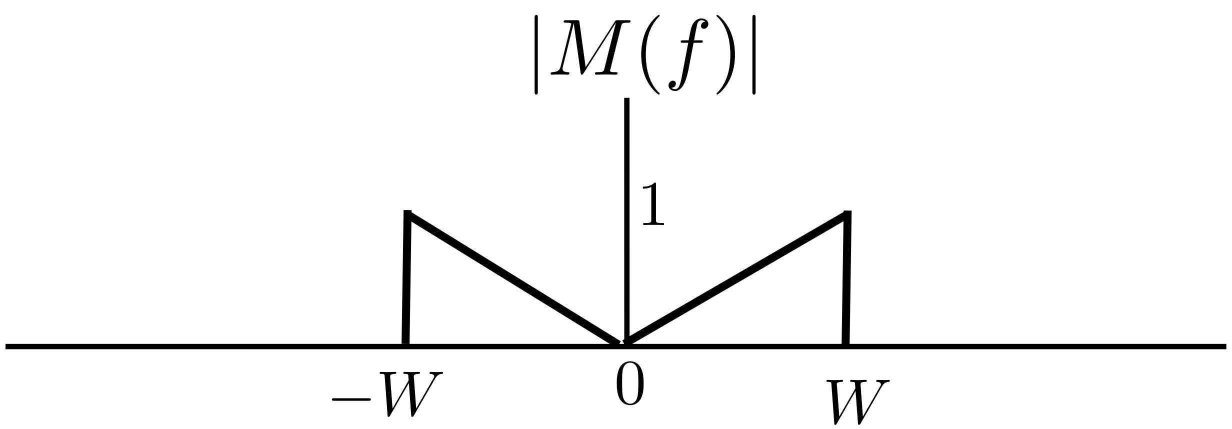
[Ref: Haykin & Moher, Textbook]

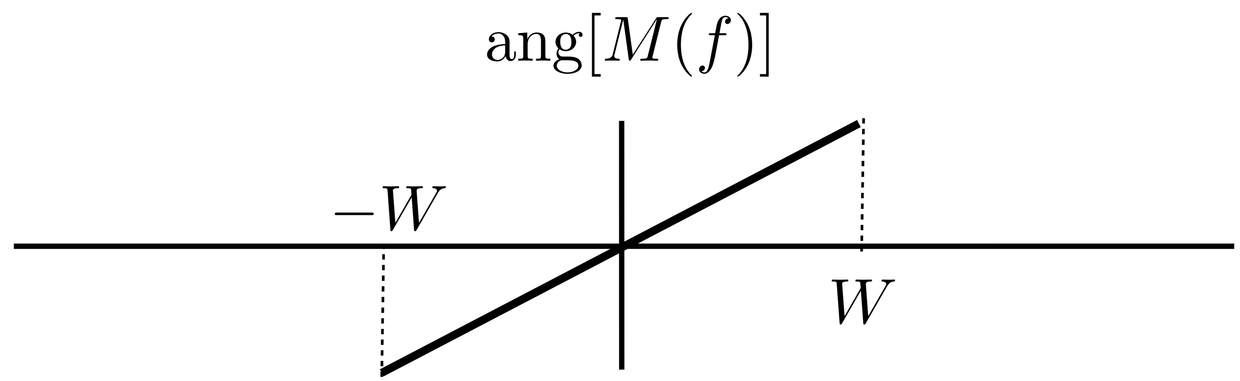
- In the special case of $f_c T >> 1$, that is, the frequency f_c is large compared to the reciprocal of the pulse duration T - we may use the approximate result

$$G(f) = \begin{cases} \frac{T}{2}\text{sinc}[T(f - f_c)], & f > 0 \\ 0, & f = 0, \\ \frac{T}{2}\text{sinc}[T(f + f_c)], & f < 0 \end{cases}$$

Example 2 for frequency shift property of FT

Given the amplitude and phase spectrum for the signal $m(t)$ as follows:



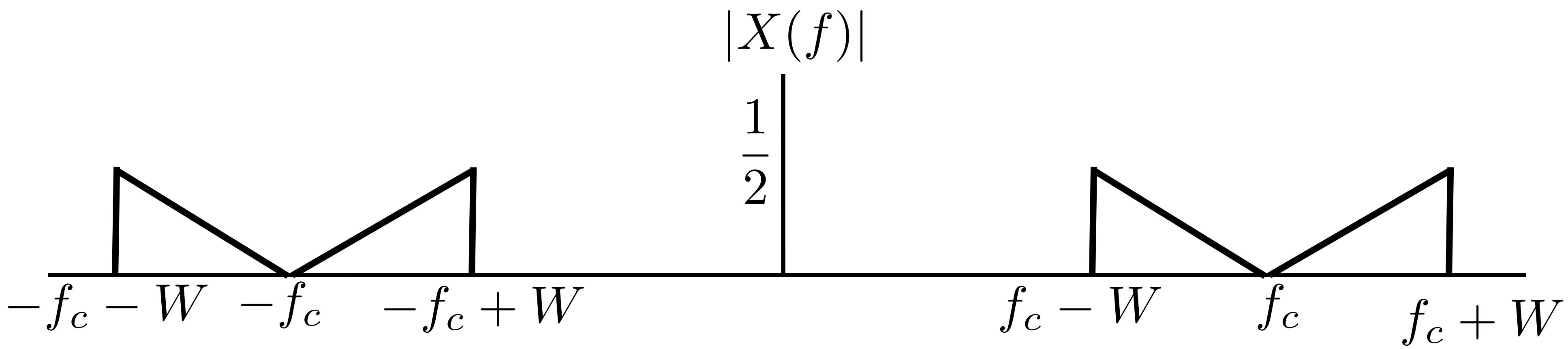


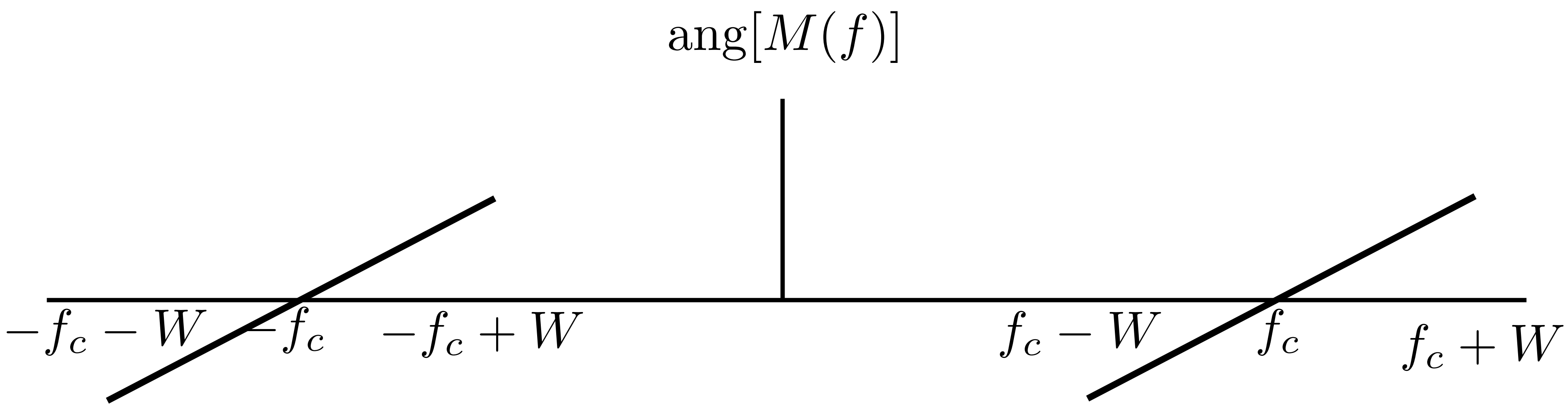
Draw the amplitude and phase spectrum of $x(t) = m(t)\cos(2\pi f_c t)$ for $f_c >> W$

Solution:

$$x(t) = \frac{m(t)}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$$

$$\Rightarrow \quad X(f) = \frac{M(f-f_c)}{2} + \frac{M(f+f_c)}{2}$$


$|X(f)|$
$\frac{1}{2}$
$-f_c - W$
$-f_c$
$-f_c + W$
$f_c - W$
f_c
$f_c + W$


$\text{ang}[M(f)]$
$-f_c - W$
$-f_c$
$-f_c + W$
$f_c - W$
f_c
$f_c + W$

Example 3 for frequency shift property of FT

Given $x(t) = m(t)\cos(2\pi f_c t)$ and signal $y(t)$ is given as

$$y(t) = x(t) \cdot \cos(2\pi f_c t)$$

Draw the amplitude spectrum of $y(t)$

Solution:

$$\begin{aligned} y(t) &= m(t)\cos(2\pi f_c t) \cdot \cos(2\pi f_c t) \\ &= m(t)\cos^2(2\pi f_c t) \\ &= m(t)\left[\frac{1+\cos(4\pi f_c t)}{2}\right] \\ &= \frac{m(t)}{2} + \frac{m(t)\cos(4\pi f_c t)}{2} \\ &= \frac{m(t)}{2} + \frac{m(t)e^{j4\pi f_c t}}{4} + \frac{m(t)e^{-j4\pi f_c t}}{4} \end{aligned}$$

$$\cos(4\pi f_c t) = \frac{e^{j4\pi f_c t} + e^{-j4\pi f_c t}}{2}$$

$$Y(f) = \mathcal{F}\left[y(t)\right]$$

$$Y(f) = \frac{M(f)}{2} + \frac{M(f - f_c)}{4} + \frac{M(f + f_c)}{4}$$