Wireless Communications (ITC731)

Lecture Note 12

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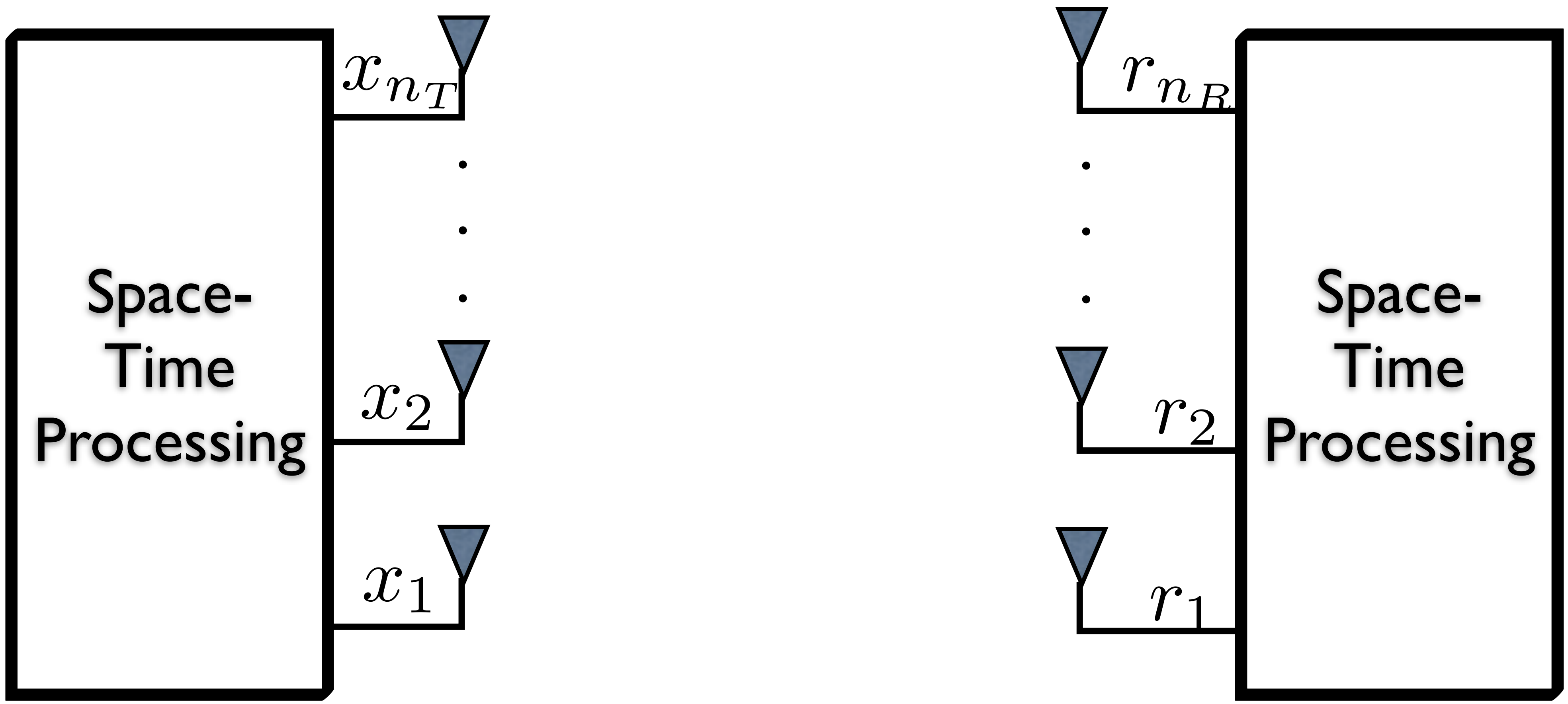
Summary

- MIMO detection schemes
  - ML detection
  - Minimum mean-square error (MMSE) detection
  - Zero-forcing detection
  - Sphere decoding

MIMO Channels



$$r_k = h_{k1}x_1 + h_{k2}x_2 + \cdots + h_{kn_T} + n_k$$

$$k = 1, 2, ..., n_R$$

Maximum Likelihood MIMO Detection

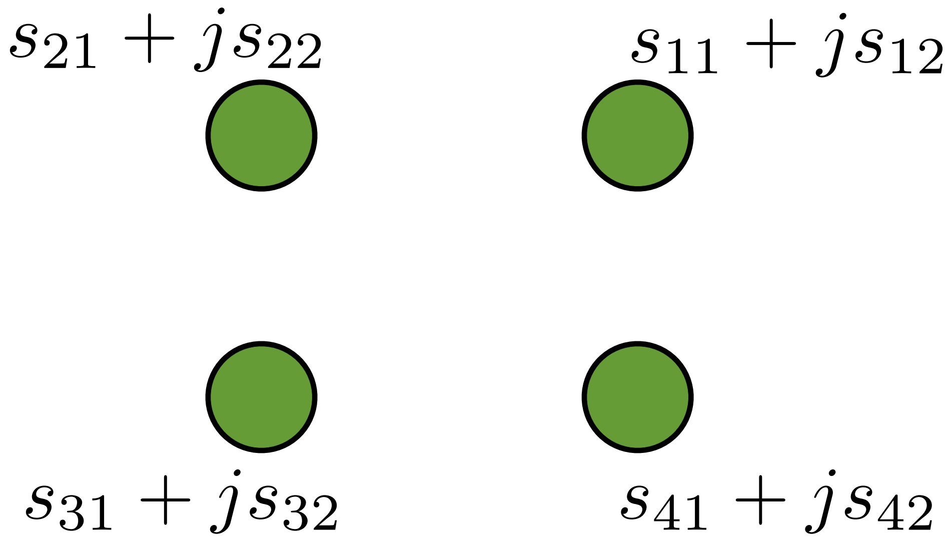
- ML MIMO detection

$$\hat{\mathbf{x}}_{\mathrm{ML}} = \arg\min_{\mathbf{x}\in\mathcal{X}} \left\{ ||\mathbf{r} - \mathbf{H}\mathbf{x}||^2 \right\}$$

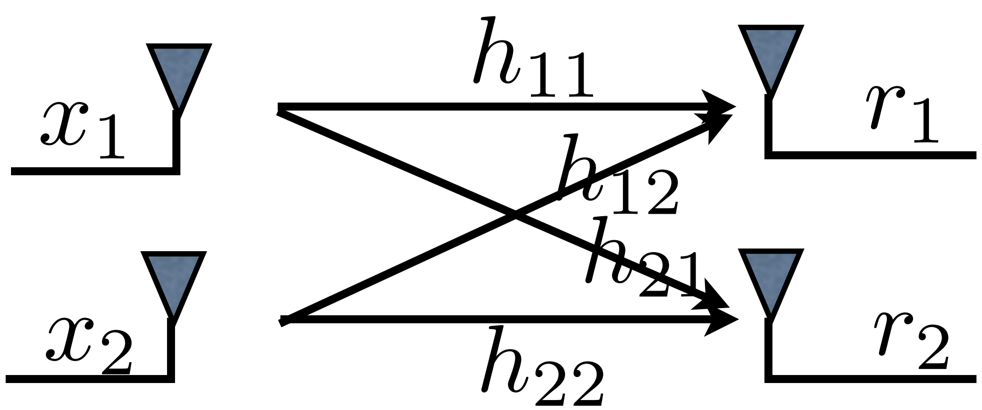
$\mathbf{H}$ are known from the channel estimation!

Example of 2x2 MIMO ML Detection

- 4-QAM case



$$\mathbf{x} \quad \in \quad \mathcal{X} = \mathbf{s}_k = s_{k1} + js_{k2}$$



$$r_1 = h_{11}x_1 + h_{12}x_2 + n_1$$

$$r_2 = h_{21}x_1 + h_{22}x_2 + n_1$$

Possible symbol combinations from two antennas

$$(\mathbf{s}_1, \mathbf{s}_1)$$
$$(\mathbf{s}_1, \mathbf{s}_2)$$
$$\vdots$$
$$(\mathbf{s}_4, \mathbf{s}_4)$$

$$\Rightarrow \quad 4^2 \quad \Rightarrow \quad M^{n_T}$$

where M is the cardinality of symbols.

We assume to know h_{ij} from channel estimation.

Then, calculate the metric as follows:

$$(\mathbf{s}_1, \mathbf{s}_1) \Longrightarrow (r_1 - h_{11}s_1 - h_{12}s_1)^2 + (r_2 - h_{12}s_1 - h_{22}s_1)^2$$

$$(\mathbf{s}_1, \mathbf{s}_2) \Longrightarrow (r_1 - h_{11}s_1 - h_{12}s_2)^2 + (r_2 - h_{12}s_1 - h_{22}s_2)^2$$

$$\vdots$$

$$(\mathbf{s}_4, \mathbf{s}_4) \Longrightarrow (r_1 - h_{11}s_4 - h_{12}s_4)^2 + (r_2 - h_{12}s_4 - h_{22}s_4)^2$$

and choose $\hat{s}_m$ pair to give the minimum metric.

Complexity $\sim$ exponential increasing by the number of antennas

Minimum Mean Square Error (MMSE) Detection

MMSE objective function

$$\min E\left\{(\mathbf{x} - \mathbf{w}^H \mathbf{r})^2\right\}$$

$$\mathbf{w} : n_R \times n_T \text{ matrix}$$

MMSE solution

$$\mathbf{w}^H = \left(\mathbf{H}^H \cdot \mathbf{H} + \frac{1}{\mathrm{SNR}_{N_T}}\right)^{-1} \mathbf{H}^H$$

which is known as *Wiener solution*.

General Solution of MMSE Detection

AWGN channel: $u[n] = d[n] + z[n]$

Fading channel: $u[n] = d[n]h[n] + z[n]$

Estimate $d[n]$ through the filter with the coefficient given as

$$\mathbf{w} = [w_0,\ w_1,\ \cdots, w_{K-1}]^T$$

such that the mean-squared error is minimized.

Definition of error:

$$e[n] = d[n] - y[n]$$

where

$$y[n] = \sum_{k=0}^{K-1} w_k^* u[n-k]$$

and

$$w_k = a_k + jb_k$$

Definition of cost function:

$$J = E\left[e[n]e^*[n]\right] = E[|e[n]|^2]$$

MMSE solution:

$$\mathbf{w}^* = \min_{\mathbf{w}}[J]$$

Solution

$$\begin{aligned} e[n] &= d[n] - y[n] \\ &= d[n] - \sum_{k=0}^{K-1} (a_k - jb_k) u[n-k] \end{aligned}$$

Gradient operator Δ_k:

$$\Delta_k = \frac{\partial}{\partial a_k} + j \frac{\partial}{\partial b_k}, \quad k = 0, 1, ..., K-1$$

Then,

$$\Delta_k J = \frac{\partial J}{\partial a_k} + j \frac{\partial J}{\partial b_k}, \quad k = 0, 1, ..., K-1$$

MMSE solution can be found by solving:

$$\Delta_k J = 0, \qquad \text{for all } k = 0, 1, \cdots, K-1$$

or equivalently

$$\Delta_k J = E\left[\frac{\partial e[n]}{\partial a_k} e^*[n] + \frac{\partial e^*[n]}{\partial a_k} e[n] + j\frac{\partial e[n]}{\partial b_k} e^*[n] + j\frac{\partial e^*[n]}{\partial b_k} e[n]\right]$$

Note that:

$$\begin{aligned} \frac{\partial e[n]}{\partial a_k} &= -u[n-k] & \qquad \frac{\partial e^*[n]}{\partial a_k} &= -u^*[n-k] \\ \frac{\partial e[n]}{\partial b_k} &= ju[n-k] & \qquad \frac{\partial e^*[n]}{\partial a_k} &= -ju^*[n-k]. \end{aligned}$$

$$
\begin{aligned}
\Delta_k J &= E\{-u[n-k]e^*[n] - u^*[n-k]e[n]\} \\
&\quad -E\{u[n-k]e^*[n] - u^*[n-k]e[n]\} \\
&= -2E\left[u[n-k]e^*[n]\right] \\
&= 0
\end{aligned}
$$

or equivalently

$$
E[u[n-k]e^*[n]] = 0, \qquad k = 0, 1, \cdots, K-1
$$

$$
\Rightarrow \quad E\left[u[n-k]\left(d^*[n] - \sum_{l=0}^{K-1} w_l u^*[n-l]\right)\right] = 0
$$

$$
\Rightarrow \quad E\left[u[n-k]d^*[n]\right] = \sum_{l=0}^{K-1} w_l E\left[u[n-k]u^*[n-l]\right]
$$

Wiener-Hopf equation:

$$E\left[u[n-k]d^*[n]\right] = \sum_{l=0}^{K-1} w_l E\left[u[n-k]u^*[n-l]\right]$$

Auto-correlation and Cross-Correlation:

$$\begin{aligned} \phi_{uu}(l-k) &= E[u[n-k]u^*[n-l]] \\ \phi_{ud}(-k) &= E[u[n-k]d^*[n]] \end{aligned}$$

Wiener-Hopf equation:

$$\sum_{l=0}^{K-1} w_k \phi_{uu}(l-k) = \phi_{ud}(-k), \qquad k = 0, 1, \cdots, K-1$$

Let us define:

$$\mathbf{u} = [u[n], u[n-1], \cdots, u[n-K+1]]^T$$

and define the correlation matrix R as

$$R = \begin{bmatrix} \phi_{uu}(0) & \phi_{uu}(1) & \cdots & \phi_{uu}(K-1) \\ \phi^*_{uu}(1) & \phi_{uu}(0) & \cdots & \phi_{uu}(K-2) \\ & & \vdots & \\ \phi^*_{uu}(K-1) & \phi^*_{uu}(K-2) & \cdots & \phi_{uu}(0) \end{bmatrix}$$

and

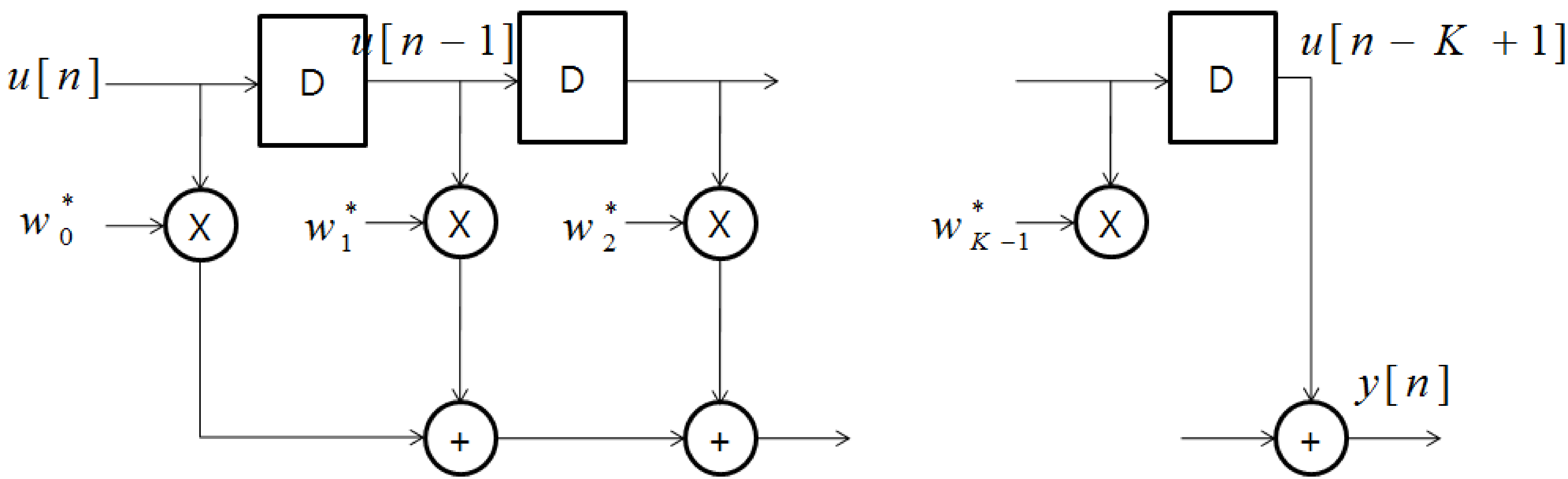
$$\begin{aligned} T &= E[\underline{\mathrm{u}}[n]d^*[n]] \\ &= [\phi_{ud}(0), \phi_{ud}(-1), \cdots, \phi_{ud}(1-K)]^T \end{aligned}$$

then, Wiener-Hopf equation can be rewritten as

$$R\mathbf{w} = T$$

Finally, the Wiener solution is given as

$$\mathbf{w} = R^{-1}T$$



Let us apply Wiener solution to MIMO detection:

$$\hat{x_1} = \sum_{k=1}^{N_R} w_k^{1*} r_k = \mathbf{w}^{1H} \cdot \underline{\mathrm{r}}$$

$$\hat{x_2} = \sum_{k=1}^{N_R} w_k^{2*} r_k = \mathbf{w}^{2H} \cdot \underline{\mathrm{r}}$$

$$\vdots$$

$$x_{\hat{N_T}} = \sum_{k=1}^{N_R} w_k^{N_R*} r_k = \mathbf{w}^{N_R H} \cdot \underline{\mathrm{r}}$$

where $\mathbf{w}^l = [w_1^l, w_2^l, \cdots, w_{N_R}^l]^T$

Then, from the Wiener solution, $\mathbf{w}^l$ is given as

$$\mathbf{w}^l = R^{-1} T_l$$

Now let us find R and T. Note that $R = E[\mathbf{r}\mathbf{r}^H]$

which is $N_R \times N_R$ matrix given as

$$R = \begin{bmatrix} E[r_1 r_1^*] & E[r_1 r_2^*] & \cdots & E[r_1 r_{N_R}^*] \\ & & \vdots & \\ E[r_{N_R} r_1^*] & E[r_{N_R} r_2^*] & \cdots & E[r_{N_R} r_{N_R}^*] \end{bmatrix}$$

$$\begin{aligned} \text{where} \quad r_k &= h_{k1}x_1 + h_{k2}x_2 + \cdots + h_{kN_T}x_{N_T} + n_k \\ &= \mathbf{h}_k^T \underline{\mathrm{x}} + n_k \end{aligned}$$

Hence,

$$
\begin{aligned}
E[r_k r_l^*] &= E[(h_{k1}x_1 + \cdots + h_k x_{N_T} + n_k) \\
&\quad \times (h_{l1}^* x_1^* + \cdots + h_{lN_T}^* x_{N_T}^* + n_l^*)] \\
&= E[h_{k1}h_{l1}^*|x_1|^2 + h_{k2}h_{l2}^*|x_2|^2 + \cdots + h_{kN_T}h_{lN_T}^*|x_{N_T}|^2 \\
&\quad + h_{k1}h_{l2}^* x_1 x_2^* + \cdots + h_{kN_T}h_{lN_T-1}x_{N_T-1}^* + n_k n_l^*]
\end{aligned}
$$

Since $E[x_1 x_2^*] = 0$ and h_k are known

$$
E[r_k r_l^*] = h_{k1}h_{l1}^* E[|x_1|^2] + \cdots + h_{kN_T}h_{lN_T}^* E[|x_{N_T}|^2] + \sigma^2\delta(k-l)
$$

Note that $E[|x_l|^2] = P$ is the signal power for all k.

Then

$$
\begin{aligned}
E[r_k r_l^*] &= P\left(h_{k1}h_{l1}^* + \cdots + h_{kN_T}h_{lN_T}^*\right) \\
&= P\left[h_{l1}^*, h_{l2}^*, \cdots, h_{lN_T}^*\right] \cdot \begin{bmatrix} h_{k1} \\ h_{k2} \\ \vdots \\ h_{kN_T} \end{bmatrix} + \sigma^2\delta(k-l) \\
&= P(\mathbf{h}_l^H \cdot \mathbf{h}_k) + \sigma^2\delta(k-l)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
T_l &= E[\mathbf{r} \cdot x_l^*] \\
&= E \begin{bmatrix} r_1 \cdot x_l^* \\ r_2 \cdot x_l^* \\ \vdots r_{N_R} \cdot x_l^* \end{bmatrix} \\
&= E \begin{bmatrix} h_{l1} P \\ h_{l2} P \\ \vdots \\ h_{lN_T} P \end{bmatrix} \\
&= P\mathbf{h}_l,
\end{aligned}
$$

where we note that $r_k = h_{k1}x_1 + h_{k2}x_2 + \cdots + h_{kN_T}x_{N_T}$.

Now the MMSE solution is

$$
\begin{aligned}
\mathbf{w}_l &= R^{-1}T_l \\
&= \left(\frac{1}{P} \begin{bmatrix} \mathbf{h}_1^H\mathbf{h}_1 & \mathbf{h}_2^H\mathbf{h}_1 & \cdots & \mathbf{h}_{N_R}^H\mathbf{h}_1 \\ \mathbf{h}_1^H\mathbf{h}_2 & \mathbf{h}_2^H\mathbf{h}_2 & \cdots & \mathbf{h}_{N_R}^H\mathbf{h}_2 \\ & \vdots & & \\ \mathbf{h}_1^H\mathbf{h}_{N_R} & \mathbf{h}_2^H\mathbf{h}_{N_R} & \cdots & \mathbf{h}_{N_R}^H\mathbf{h}_{N_R} \end{bmatrix} + \sigma^2 \right)^{-1} P\mathbf{h}_l \\
&= \left(\frac{1}{P} \begin{bmatrix} \mathbf{h}_1^H\mathbf{h}_1 & \mathbf{h}_2^H\mathbf{h}_1 & \cdots & \mathbf{h}_{N_R}^H\mathbf{h}_1 \\ \mathbf{h}_1^H\mathbf{h}_2 & \mathbf{h}_2^H\mathbf{h}_2 & \cdots & \mathbf{h}_{N_R}^H\mathbf{h}_2 \\ & \vdots & & \\ \mathbf{h}_1^H\mathbf{h}_{N_R} & \mathbf{h}_2^H\mathbf{h}_{N_R} & \cdots & \mathbf{h}_{N_R}^H\mathbf{h}_{N_R} \end{bmatrix} + \frac{\sigma^2}{P} \right)^{-1} \mathbf{h}_l
\end{aligned}
$$

We can rewrite the optimal weight vector $\mathbf{w}$:

$$\mathbf{w} = \begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_{N_R} \\ | & | & & | \end{bmatrix}$$

$$= \mathbf{H}\left(\mathbf{H}\cdot\mathbf{H}^H + \frac{1}{\mathrm{SNR}_{N_T}}\right)^{-1}$$

Equivalently, we have

$$\mathbf{w}^H = \left(\mathbf{H}^H\cdot\mathbf{H} + \frac{1}{\mathrm{SNR}_{N_T}}\right)^{-1}\mathbf{H}^H$$

Zero-Forcing MIMO Detection

Received signal signal:

$$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

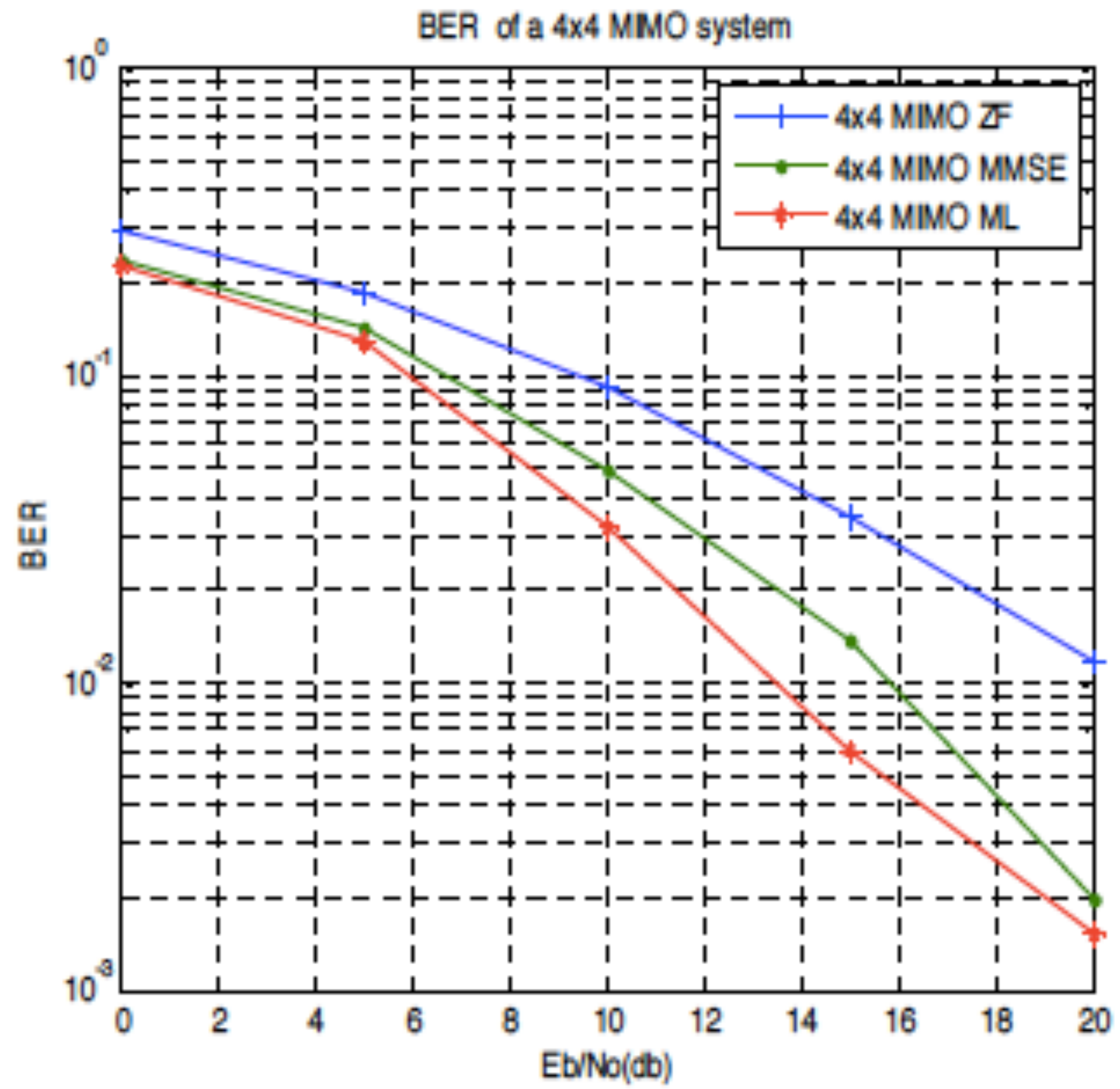
Zero-forcing detection should satisfy:

$$\mathbf{w}^H \mathbf{r} = \mathbf{x} + \tilde{n}$$

We can show that the ZF solution is

$$\mathbf{w}^H = (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H$$

Performance Comparison



Complexity of ML MIMO Detection

- 64-QAM constellation

$$\text{QAM alphabet set}=\mathcal{X} = \{-7, -5, -3, -1, 1, 3, 5, 7\}$$

- ML equation

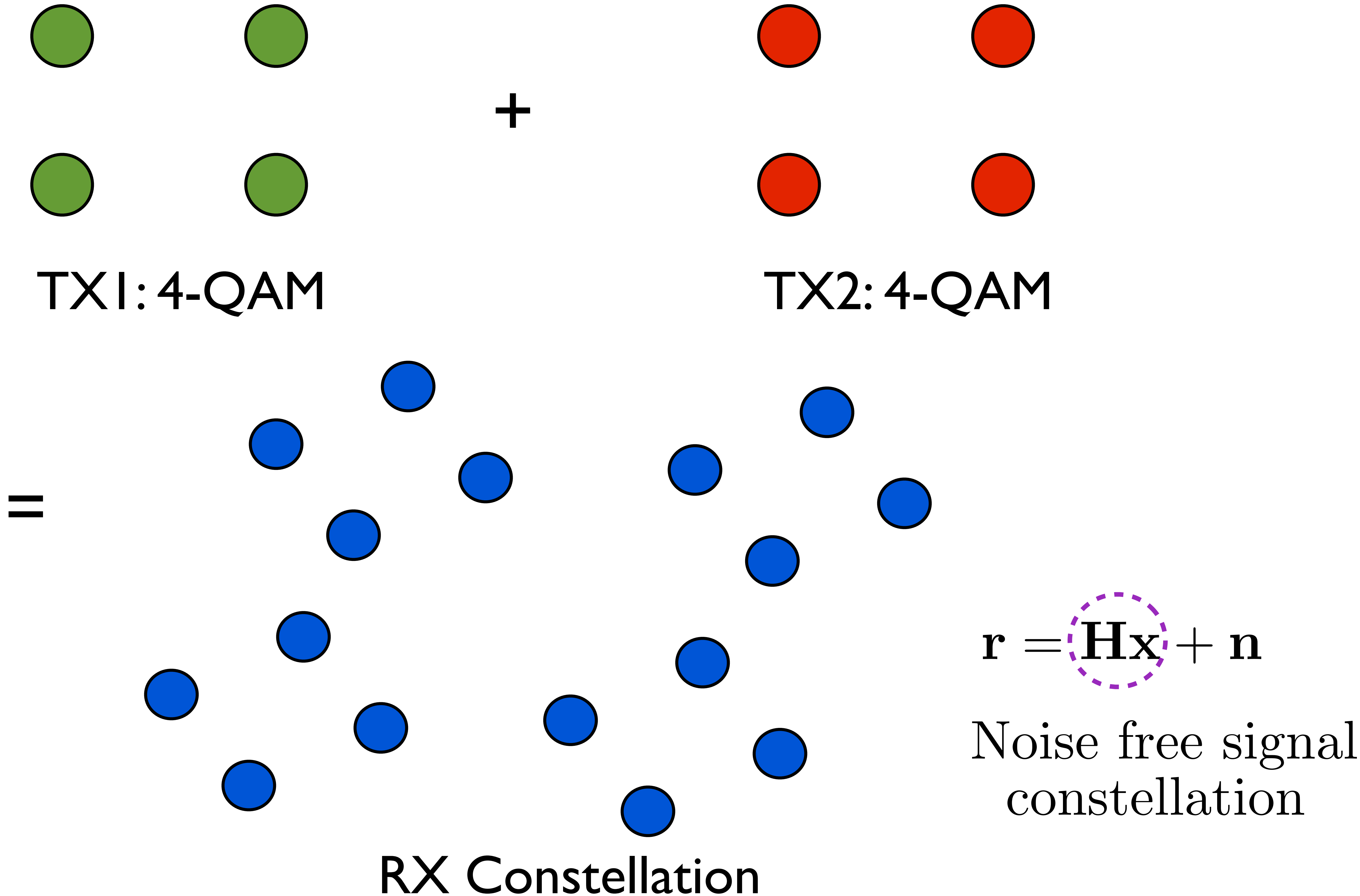
$$\hat{\mathbf{x}}_{\text{ML}} = \arg\min_{\mathbf{x}\in\mathcal{X}} \{||\mathbf{r} - \mathbf{H}\mathbf{x}||^2\}$$

- Total number of possibilities for

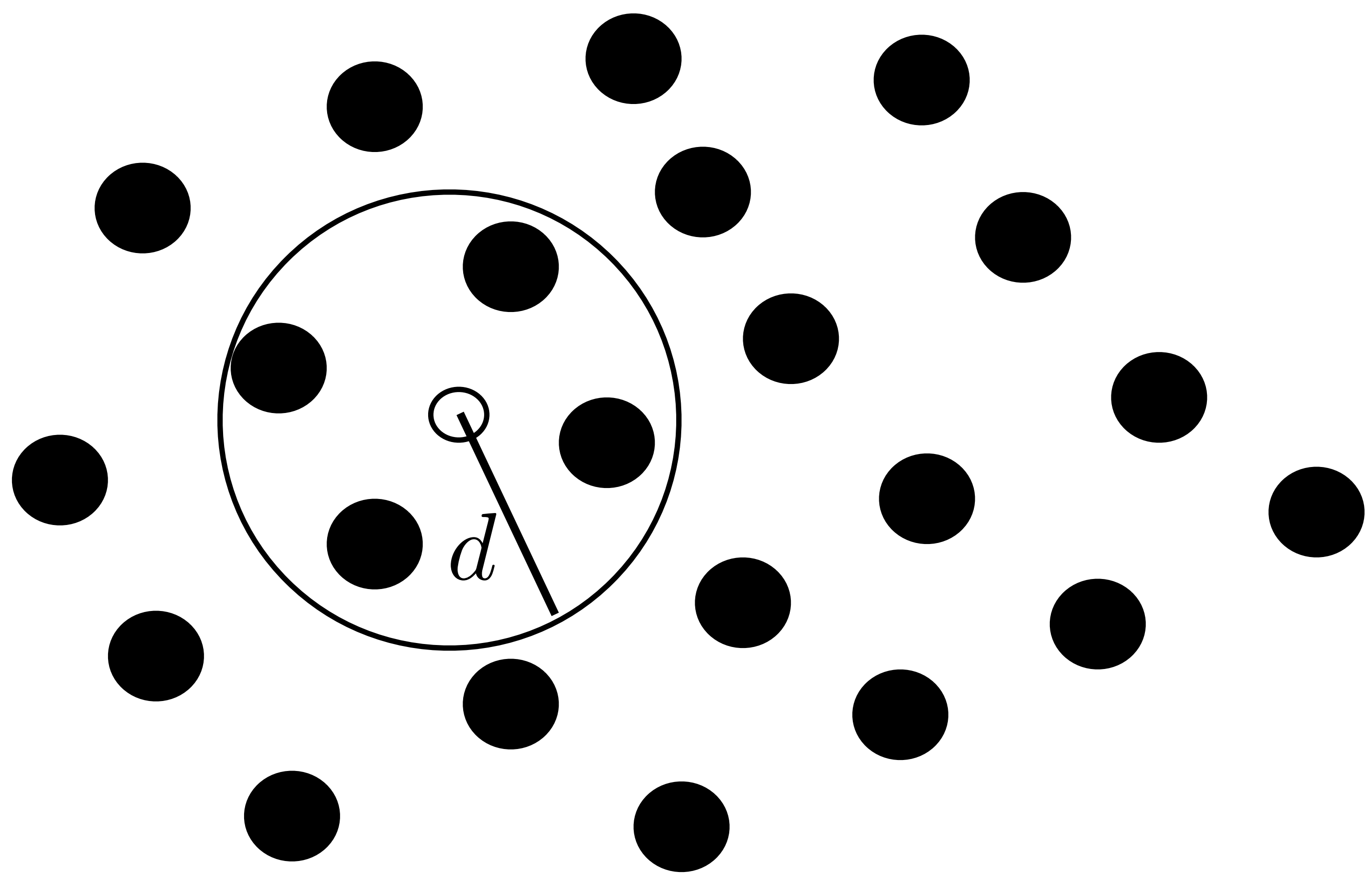
$$\hat{\mathbf{x}} = 8^8 = 16777216$$

Lattice Structure

- Each receive antenna observes the random sum of n_R transmit signals.
- If the transmitted signal constellation contain M points (M-QAM, M-PSK)
  - Received noise free constellation M^{n_R} points.
  - Exponential increase in constellation size.


+
TX1: 4-QAM
TX2: 4-QAM
=
RX Constellation
$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{n}$
Noise free signal constellation

Sphere Decoding



$$\hat{\mathbf{x}} = \arg\left\{\min_{x\in\mathcal{X}} ||\mathbf{r} - \mathbf{H}\mathbf{x}||^2 \leq d^2\right\}$$

Search only noiseless receive points (defined as the lattice $\mathbf{Hx}$) that lie within a hypersphere of radius d around the receive signal $\mathbf{r}$.

MIMO Sphere Decoder

- ML performance with reduced complexity.
- Search only noiseless received points (defined as the lattice $\mathbf{Hx}$) that lies within a hypersphere of radius d around the received signal $\mathbf{r}$.
- Initial radius d is selected according to the noise variance per antenna.
- Complexity of algorithm depends on the noise level and the channel conditions.
- B. Hassibi and H. Vikalo
  - "On the sphere decoding algorithm I. Expected complexity", IEEE Trans. Signal Processing, 2005

Sphere Decoder Algorithm

$$\mathbf{r} - \mathbf{H}\mathbf{x} = \mathbf{r} - \mathbf{Q}\mathbf{R}\mathbf{x} = \mathbf{Q}(\mathbf{Q}^{\mathbf{H}}\mathbf{r} - \mathbf{R}\mathbf{x})$$

$$\begin{aligned} \hat{\mathbf{x}} &= \arg\left\{\min_{x\in\mathcal{X}} ||\mathbf{r} - \mathbf{H}\mathbf{x}||^2 \leq d^2\right\} \\ &= \arg\left\{\min_{x\in\mathcal{X}} ||\mathbf{Q}^H\mathbf{r} - \mathbf{R}\mathbf{x}||^2 \leq d^2\right\} \end{aligned}$$

$n_T \times n_T$ upper trianguler matrix

$$\mathbf{H} = \mathbf{Q}\begin{bmatrix} \mathbf{R} \\ 0_{(n_R - n_T)\times n_T} \end{bmatrix}$$

$\mathbf{Q} = [\mathbf{Q}_1 \ \mathbf{Q}_2]$ is an $n_R \times n_R$ orthogonal matrix.

$$\left\|\mathbf{r} - [\mathbf{Q}_1\ \mathbf{Q}_2]\begin{bmatrix}\mathbf{R}\\0\end{bmatrix}\mathbf{x}\right\|^2 = \left\|\begin{bmatrix}\mathbf{Q}_1^H\\\mathbf{Q}_2^H\end{bmatrix}\mathbf{r} - \begin{bmatrix}\mathbf{R}\\0\end{bmatrix}\mathbf{x}\right\|^2$$

$$= ||\mathbf{Q_1}^H\mathbf{r} - \mathbf{R}\mathbf{x}||^2 + ||\mathbf{Q}_2^H\mathbf{r}||^2$$

In other words, the Sphere decoding becomes

$$d^2 - ||\mathbf{Q}_2^H\mathbf{r}||^2 \geq ||\mathbf{Q}_1^H\mathbf{r} - \mathbf{R}\mathbf{x}||^2$$

Define $\mathbf{y} = \mathbf{Q}_1^H\mathbf{r}$ and $R^2 = d^2 - ||\mathbf{Q}_2^H\mathbf{r}||^2$

$$||\mathbf{y} - \mathbf{R}\mathbf{x}||^2 \leq R^2$$

In matrix form:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{n_T-2} \\ y_{n_T-1} \\ y_{n_T} \end{bmatrix} \approx$$

$$\begin{bmatrix} r_{1,1} & \cdots & r_{1,n_T-2} & r_{1,n_T-1} & r_{1,n_T} \\ & \ddots & \vdots & \vdots & \vdots \\ & & r_{n_T-2,n_T} & r_{n_T-2,n_T-1} & r_{n_T-2,n_T} \\ & & & r_{n_T-1,n_T-1} & r_{n_T-1,n_T} \\ & & & & r_{n_T,n_T} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n_T-2} \\ x_{n_T-1} \\ x_{n_T} \end{bmatrix}$$

The constraint can be expanded as

$$R^2 \geq \sum_{i=1}^{n_T} \left(y_i - \sum_{j=i}^{n_T} r_{i,j} x_j \right)^2$$

Detection of x_{n_T}

$$\begin{aligned} R^2 &\geq (y_{n_T} - r_{n_T,n_T} x_m)^2 \\ &+ (y_{n_T-1} - r_{n_T-1,n_T} x_m - r_{n_T-1,n_T-1} x_{m-1})^2 + \cdots \end{aligned}$$

A necessary condition:

$$R^2 \geq (y_{n_T} - r_{n_T,n_T} x_m)^2$$

Detection of x_{n_T-1}

Define:

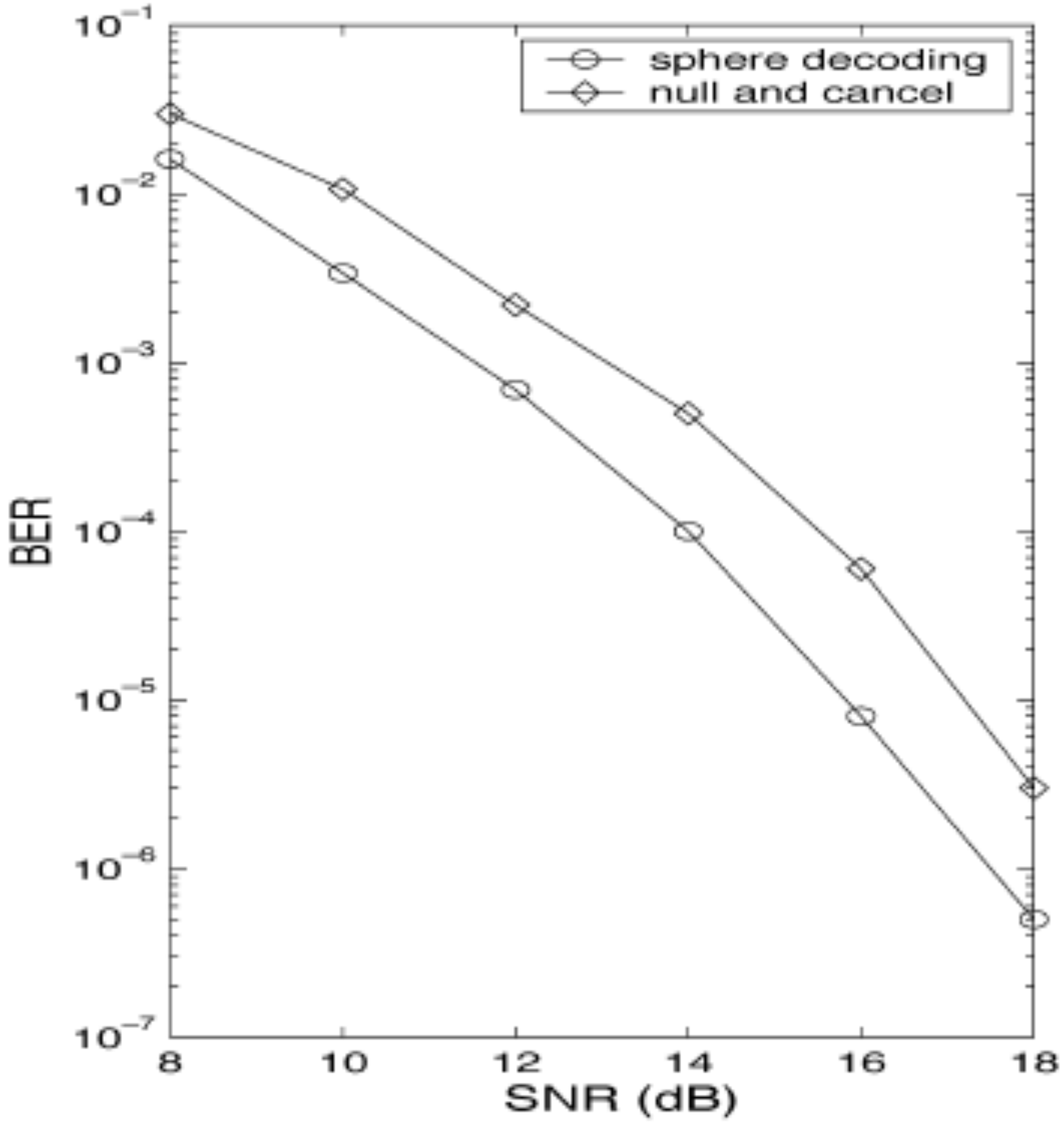
$$R_{n_T-1}^2 = R^2 - (y_{n_T} - r_{n_T,n_T} x_{n_T})^2$$

$$y_{n_T-1|n_T} = y_{n_T-1} - r_{n_T-1,n_T} x_m$$

The range of x_{n_T-1} can be determined as

$$\left\lceil \frac{-R_{n_T-1} + y_{n_T-1|n_T}}{r_{n_T-1,n_T-1}} \right\rceil \leq x_{n_T-1} \leq \left\lfloor \frac{-R_{n_T-1} + y_{n_T-1|n_T}}{r_{n_T-1,n_T-1}} \right\rfloor$$

For the rest, this process is repeated.



4-QAM

$n_T = 20$

[H. Vikalo and B. Hassibi, "On the Sphere-Decoding Algorithm II. Generalization, Second-Order Statistics, and Application to Communications", IEEE Trans. Signal Processing, Vol. 53, No. 8, Aug. 2005]

Antenna Ordering

- Antenna can also be ordered before sphere decoding
- ZF ordering: order antennas by SNR
- MMSE ordering: order antennas by SINR