Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #10

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Outline

- Matched filter
- Optimum detection for binary antipodal signals

Correlation-Type Demodulator for Binary Orthogonal Signals

- Signal waveform

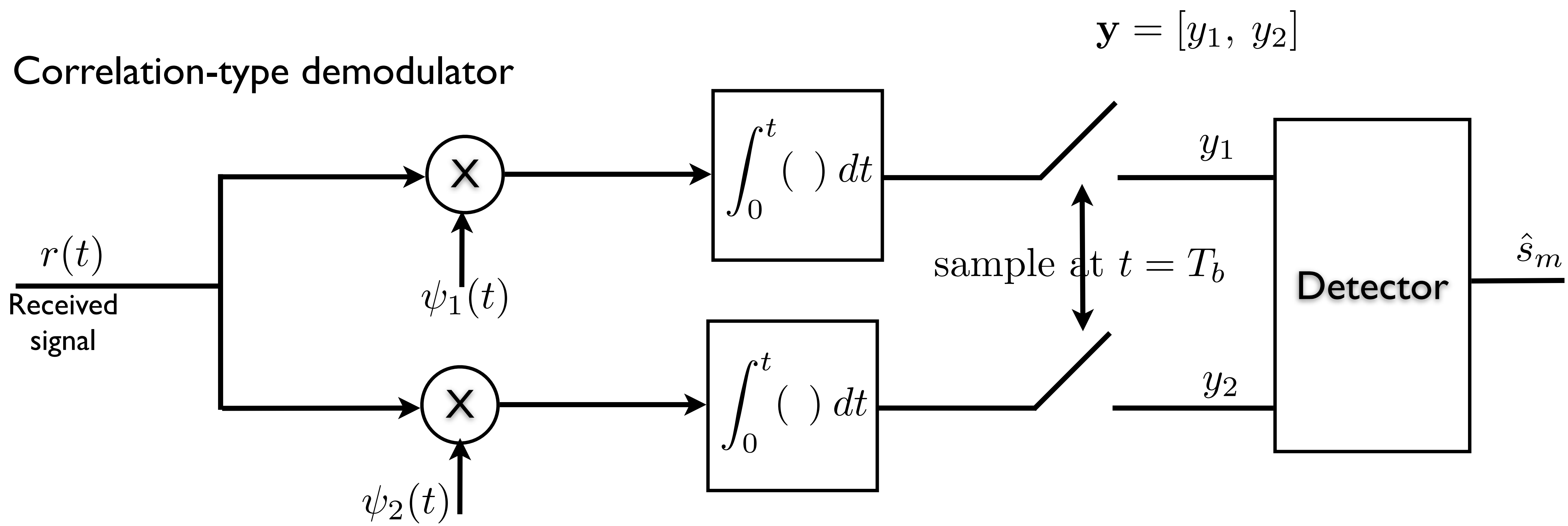
$$r(t) = s_m(t) + n(t), \quad 0 \le t \le T_b, \quad m = 1, 2.$$

where $s_1(t) = \sqrt{\mathcal{E}_b}\psi_1(t)$, and $s_2(t) = \sqrt{\mathcal{E}_b}\psi_2(t)$

  - Note that in vector form, the transmit signals are

$$\mathbf{s}_1 = [\sqrt{\mathcal{E}_b},\ 0], \text{ and } \mathbf{s}_2 = [0,\ \sqrt{\mathcal{E}_b}]$$

- Correlation-type demodulator



12년 9월 26일 수요일

- Correlator output waveforms

$$y_m(t) = \int_0^t r(\tau)\phi_m(\tau)\, d\tau, \;\; m = 1, 2.$$

- Sampled signal at $t = T_b$

$$y_m = y_m(T_b) = \int_0^{T_b} r(\tau)\phi_m(\tau)\, d\tau, \;\; m = 1, 2.$$

- For $s_1(t) = s_{11}\phi_1(t)$, so that $r(t) = s_{11}\psi_1(t) + n(t)$.

$$y_1 = \int_0^{T_b} [s_{11}\psi_1(\tau) + n(\tau)]\psi_1(\tau)\, d\tau = s_{11} + n_1 = \sqrt{E_b} + n_1$$

$$y_2 = \int_0^{T_b} [s_{11}\psi_1(t) + n(t)]\psi_2(t)\, dt = n_2$$

where

$$n_1 = \int_0^{T_b} n(\tau)\psi_1(\tau)d\tau$$

$$n_2 = \int_0^{T_b} n(\tau)\psi_2(\tau)d\tau$$

- Sampled output in vector form if $s_1(t)$ is transmitted:

$$\mathbf{y} = [y_1,\ y_2] = [\sqrt{\mathcal{E}_b} + n_1,\ n_2]$$

- Sampled output in vector form if $s_2(t)$ is transmitted:

$$\mathbf{y} = [y_1,\ y_2] = [n_1, \sqrt{\mathcal{E}_b} + n_2]$$

Statistical characteristic of the observed signal vector $\mathbf{y}$

- n_1 and n_2 are zero-mean Gaussian random variable with variance $\sigma^2 = N_0/2$.

$$n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$$

- Correlation between n_1 and n_2

$$\begin{aligned} E[n_1 n_2] &= \int_0^{T_b}\int_0^{T_b} E[n(t)n(\tau)\psi_1(t)\psi_2(\tau)\, dt\, d\tau \\ &= \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-\tau)\psi_1(t)\psi_2(\tau)\, dt\, d\tau \\ &= \frac{N_0}{2}\int_0^{T_b} \psi_1(t)\psi_2(\tau)\, dt\, d\tau = 0. \end{aligned}$$

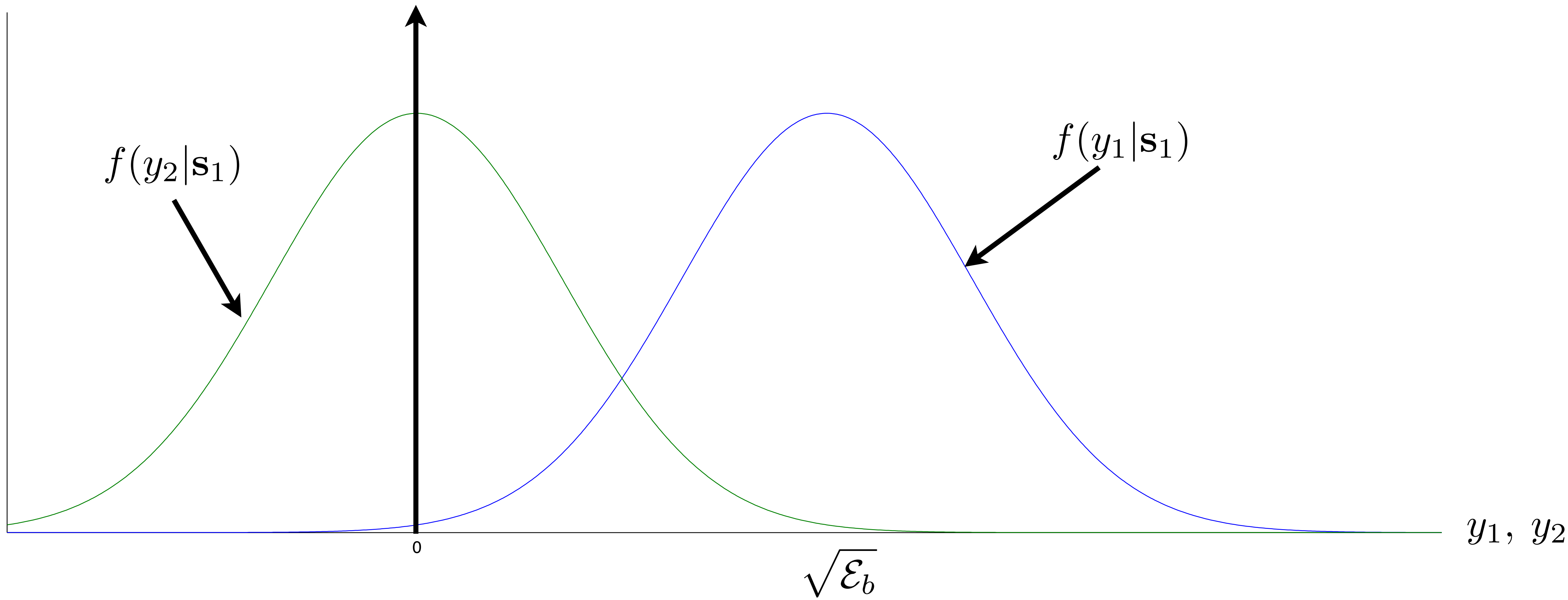
- Conditional joint PDF

$$\begin{aligned} f(y_1, y_2|\mathbf{s}_1) &= \left(\frac{1}{\sqrt{\pi N_0}}\right)^2 \exp\left[-\frac{(y_1-\sqrt{\mathcal{E}_b})^2 + y_2^2}{N_0}\right] \\ f(y_1, y_2|\mathbf{s}_2) &= \left(\frac{1}{\sqrt{\pi N_0}}\right)^2 \exp\left[-\frac{y_1^2 + (y_2-\sqrt{\mathcal{E}_b})^2}{N_0}\right] \end{aligned}$$

$\Rightarrow$

$$f(y_1, y_2|\mathbf{s}_m) = f(y_1|\mathbf{s}_m)f(y_2|\mathbf{s}_m)$$

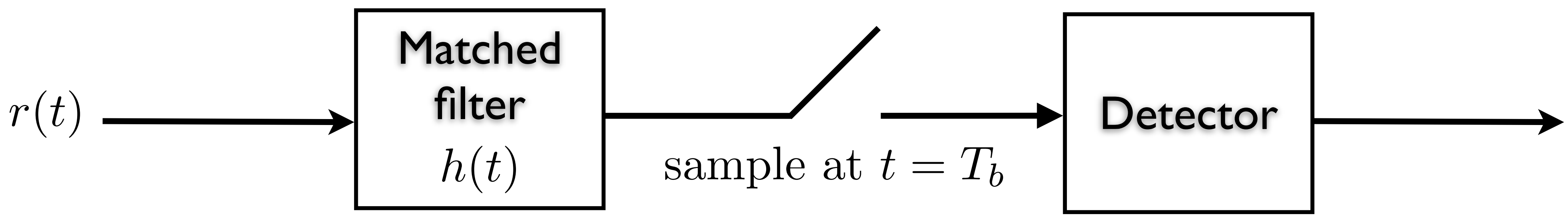
- Conditional PDF when $s_1(t)$ is transmitted.



Matched Filter Type Demodulator

- Binary antipodal signals

$$r(t) = s_m\psi(t) + n(t), \quad 0 \le t \le T_b, \quad m = 1, 2$$



- Impulse response of matched filter

$$h(t) = \psi(T_b - t), \quad 0 \le t \le T_b$$

- Filter output

$$y(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau$$

- Sampling at time $t = T_b$

$$y(T_b) = \int_0^{T_b} r(\tau)h(T_b - \tau)\, d\tau$$

Since $h(T_b - \tau) = \psi(\tau)$

the sampled output signal is

$$\begin{aligned} y(T_b) &= \int_0^{T_b} [s_m \psi(\tau) + n(\tau)]\psi(\tau)\, d\tau \\ &= s_m + n \end{aligned}$$

where

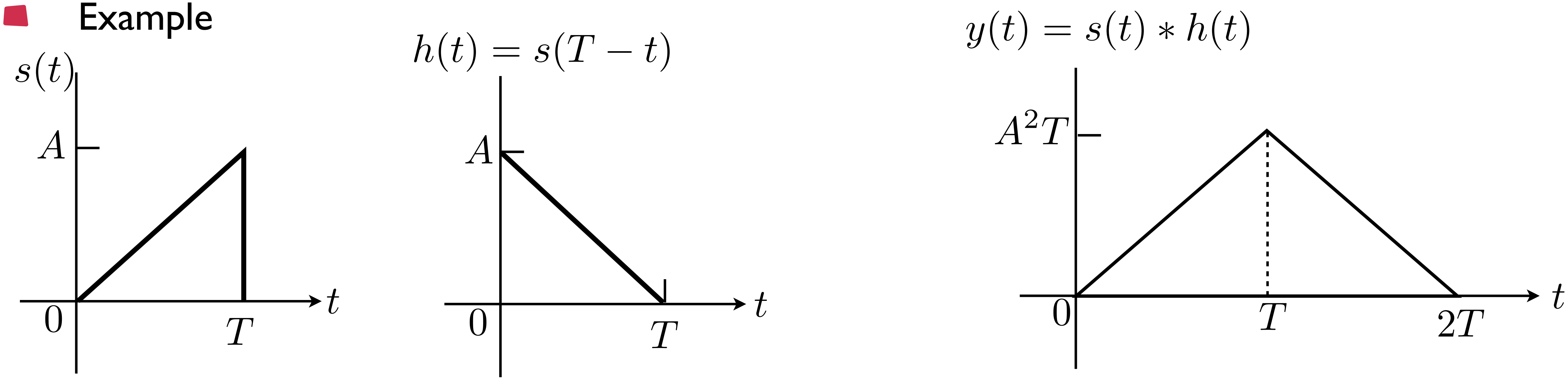
$$n = \int_0^{T_b} n(\tau)\psi(\tau)\, d\tau$$

✦ The sampled output is exactly the same as the output obtained with a cross-correlator.

Matched Filter

- Definition:
    - A filter whose impulse response $h(t) = s(T-t)$, where $s(t)$ is assumed to be confined to the time interval $0 \leq t \leq T$.

- Example



Binary Orthogonal Signals with Matched Filter

- Binary orthogonal signal waveforms

$$r(t) = s_m(t) + n(t),\ \ 0 \le t \le T_b,\ \ m = 1, 2$$

where

$$< s_1(t), s_2(t) >= \int_0^{T_b} s_1(t) s_2(t)\, dt = 0$$

- Consider matched filters with impulse response given as

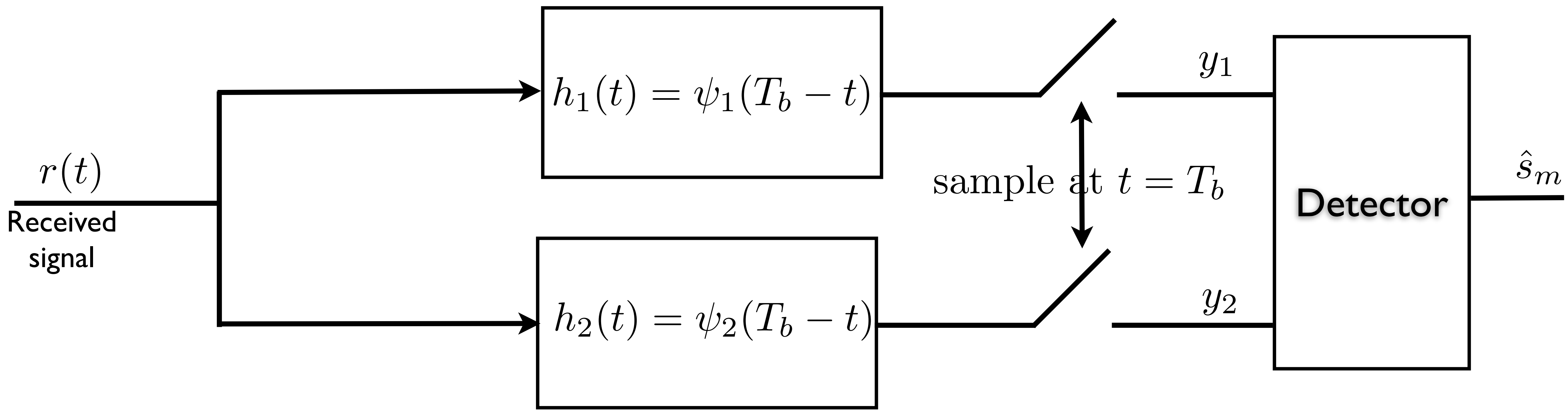
$$\begin{aligned} h_1(t) &= \psi_1(T_b - t), \quad 0 \le t \le T_b \\ h_2(t) &= \psi_2(T_b - t), \quad 0 \le t \le T_b \end{aligned}$$

- Output at the matched filter

$$y_m(t) = \int_0^t r(\tau) h_m(t-\tau)\, d\tau,\ \ m = 1, 2.$$

Sampled output

$$y_m = y_m(T_b) = \int_0^{T_b} r(\tau) h_m(T_b - \tau)\, d\tau$$
$$= \int_0^{T_b} r(\tau) \psi_m(\tau)\, d\tau, \quad m = 1, 2$$



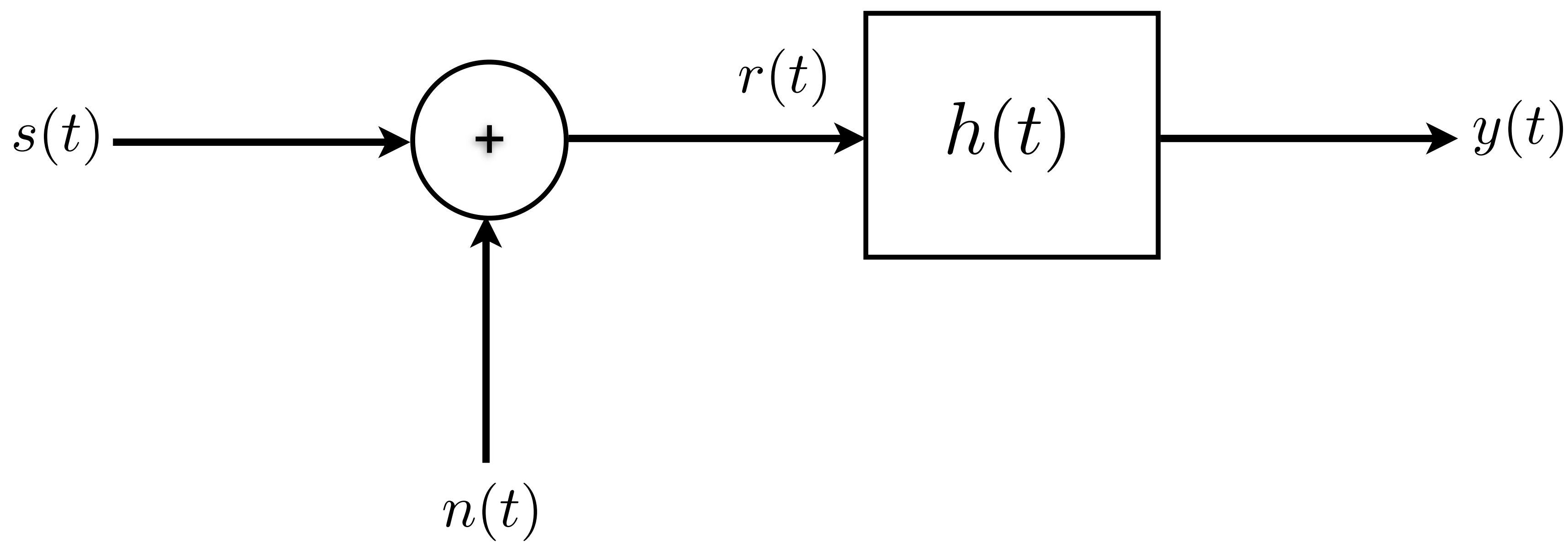
When $s_1(t)$ was transmitted,

$$y_1 = s_{11} + n_1$$
$$y_2 = n_2$$

Properties of Matched Filter

- If a signal $s(t)$ is corrupted by AWGN, the filter with the impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio (SNR).

- Proof



  - Filter output signal

$$y(t) = \int_0^t r(\tau)h(t-\tau)\, d\tau$$

$$= \int_0^t s(\tau)h(t-\tau)\, d\tau + \int_0^t n(\tau)h(t-\tau)\, d\tau$$

- At the sampling instant $t = T$, the signal and noise components are

$$y(T) = \int_0^T s(\tau)h(T-\tau)\,d\tau + \int_0^T n(\tau)h(T-\tau)\,d\tau$$

$$= y_s(T) + y_n(T)$$

signal component   noise component

- Output Signal-to-Noise Ratio (SNR)

$$\left(\frac{S}{N}\right)_0 = \frac{y_s^2(T)}{E[y_n^2(T)]}$$

  - The problem is to select the filter impulse response that maximizes the output SNR.
  - The answer is that the matched filter maximizes the output SNR.

- Variance of the noise term at the output of the filter

$$\begin{aligned} E[y_n^2(T)] &= \int_0^T \int_0^T E[n(\tau)n(t)]h(T-\tau)h(T-t)\,dt\,d\tau \\ &= \frac{N_0}{2}\int_0^T \int_0^T \delta(t-\tau)h(T-\tau)h(T-t)\,dt\,d\tau \\ &= \frac{N_0}{2}\int_0^T h^2(T-t)\,dt \end{aligned}$$

- Output SNR

$$\left(\frac{S}{N}\right)_o = \frac{\left[\int_0^T s(\tau)h(T-\tau)\,d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)\,dt} = \frac{\left[\int_0^T h(\tau)s(T-\tau)\,d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)\,dt}$$

- Cauchy-Schwartz inequality

$$\left[\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt\right]^2 \leq \int_{-\infty}^{\infty} g_1^2(t)\,dt \int_{-\infty}^{\infty} g_2^2(t)\,dt,$$

where equality holds when $g_1(t) = Cg_2(t)$ for any arbitrary constant C.

- If we set $g_1(t) = h(t)$ and $g_2(t) = s(T-t)$, it is clear that the output SNR is maximized when $h(t) = Cs(T-t)$, i.e., $h(t)$ is matched to the signal $s(t)$.
  - The scale factor C drops out of the expression for $(S/N)_o$, since it appears in both the numerator and the denominator.

- Output (maximum) SNR obtained with the matched filter is

$$\left(\frac{S}{N}\right)_o = \frac{2}{N_0}\int_0^T s^2(t)\,dt = \frac{2\mathcal{E}_s}{N_0}$$

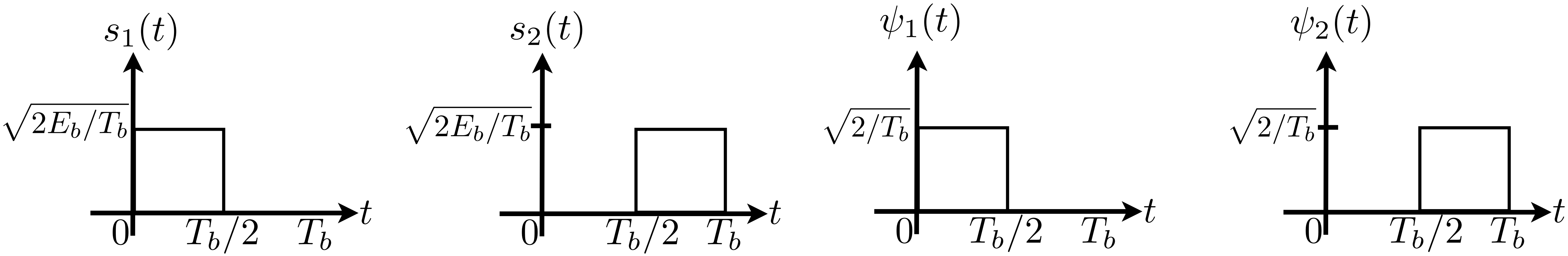
where $\mathcal{E}_s$ is the energy of the signal $s(t)$.

- Note that the output SNR from the matched filter depends on the energy of the waveform $s(t)$ but not on the detailed characteristics of $s(t)$.

Example of binary PPM

- Binary PPM signals

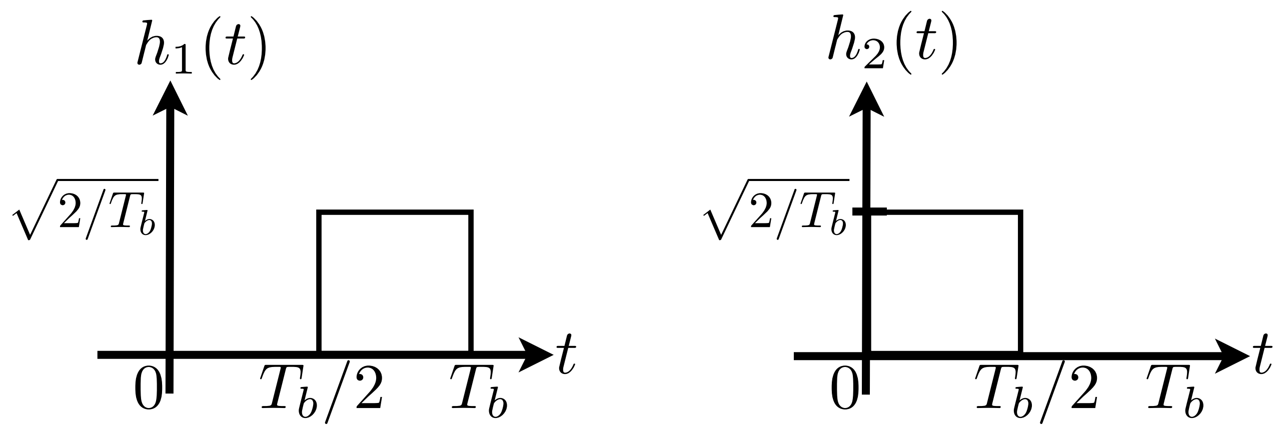
$$s_m(t) = s_{m1}\psi_1(t) + s_{m2}\psi_2(t), \quad j = 1, 2$$



$$\begin{aligned}
s_{11} &= \int_0^{T_b} s_1(t)\psi_1(t)\,dt = \sqrt{E_b} \\
s_{12} &= \int_0^{T_b} s_1(t)\psi_2(t)\,dt = 0 \\
s_{21} &= \int_0^{T_b} s_2(t)\psi_1(t)\,dt = 0 \\
s_{22} &= \int_0^{T_b} s_2(t)\psi_2(t)\,dt = \sqrt{E_b}
\end{aligned}$$

- Matched filter

$$h_1(t) = \psi_1(T_b - t), \quad h_2(t) = \psi_2(T_b - t)$$



- If $s_1(t)$ is transmitted, the sampled output signals are

$$\mathbf{y} = [y_1, \ y_2] = [\sqrt{E_b} + n_1, \ n_2]$$

where $n_k = \int_0^{T_b} n(t)\psi_k(t)\,dt$ with $n_k \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$

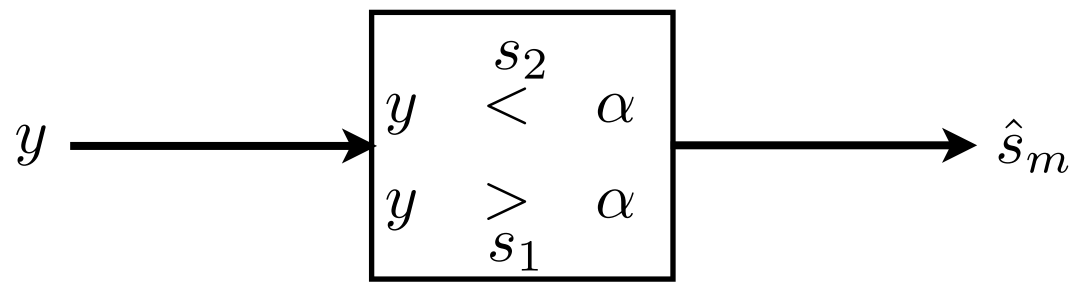
- Output SNR for the first matched filter

$$\left(\frac{S}{N}\right)_o = \frac{(\sqrt{\mathcal{E}_b})^2}{N_0/2} = \frac{2\mathcal{E}_b}{N_0}$$

Performance of the Optimum Receiver: Binary Antipodal Signals

- Output of the demodulator in any signal bit interval

$$y = s_m + n, \quad m = 1, 2$$



- Decision rule
  - If $y > \alpha$, declare $s_1(t)$ was transmitted.
  - If $y < \alpha$, declare $s_2(t)$ was transmitted.

- Average probability of error

$$P_2(\alpha) = P(s_1)\int_{-\infty}^{\alpha} f(y|s_1)\,dy + P(s_2)\int_{\alpha}^{\infty} f(y|s_2)\,dy$$

  - Not we want to find the optimum threshold value α, say α^* which minimizes the average probability of error.

- Optimum threshold can by finding the solution of $\left.\frac{dP_2(\alpha)}{d\alpha} = 0\right|_{\alpha=\alpha^*}$

  That is,

$$P(s_1)f(\alpha|s_1) - P(s_2)f(\alpha|s_2) = 0$$

  or equivalently,

$$\frac{f(\alpha|s_1)}{f(\alpha|s_2)} = \frac{P(s_2)}{P(s_1)}$$

  Since $f(\alpha|s_m)$ is Gaussian PDF with mean $\sqrt{\mathcal{E}_b}$ for s_1 and $-\sqrt{\mathcal{E}_b}$ for s_2, we have

$$e^{-(\alpha-\sqrt{\mathcal{E}_b})^2/N_0} e^{-(\alpha+\sqrt{\mathcal{E}_b})^2/N_0} = \frac{P(s_2)}{P(s_1)}$$

- Clearly, the optimum value of the threshold is

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{P(s_2)}{P(s_1)}$$

- For the case of $P(s_1) = P(s_2)$, the optimum threshold is zero. In this case, the average probability of error is

$$\begin{aligned} P_2 &= \frac{1}{2}\int_{-\infty}^{0} f(y|s_1)\,dy + \frac{1}{2}\int_{0}^{\infty} f(y|s_2)\,dy = \int_{-\infty}^{0} f(y|s_1)\,dy \\ &= \frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{0} e^{-(y-\sqrt{\mathcal{E}_b})^2/N_0}\,dy \end{aligned}$$

Change of the variable as $x = (y - \sqrt{\mathcal{E}_b})/\sqrt{N_0/2}$

$$P_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{2\mathcal{E}_b/N_0}} e^{-x^2/2}\,dx = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$