Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #6

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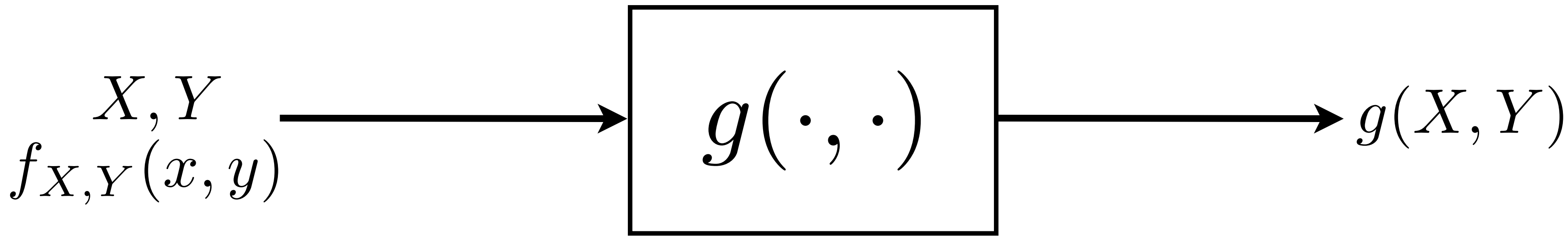
Outline

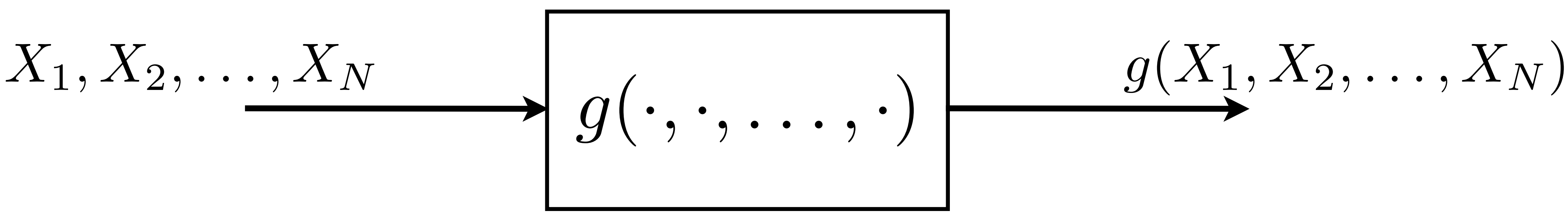
- Transformation of Random Variable
- Multiple random variables

Expected Value of Function of Random Variables



Mean of $g(X,Y)$

$$\bar{g} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dxdy$$



$$\begin{aligned}\bar{g} &= E[g(X_1, X_2, \ldots, X_N)] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_N) f_{X_1,\ldots,X_N}(x_1, \ldots, x_N) dx_1 \ldots dx_N\end{aligned}$$

Example

- Find the mean.

$$X_1, X_2, \ldots, X_N \longrightarrow \boxed{g(X_1, \ldots, X_N)} \longrightarrow \sum_{n=1}^{N} \alpha_i X_i$$

$$\begin{aligned} E[g(X_1, \ldots, X_N)] &= E\left[\sum_{n=1}^{N} \alpha_i X_i\right] \\ &= \sum_{i=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \alpha_i x_i f_{X_1, \ldots, X_N}(x_1, \ldots, x_N)\, dx_1 \ldots dx_N \end{aligned}$$

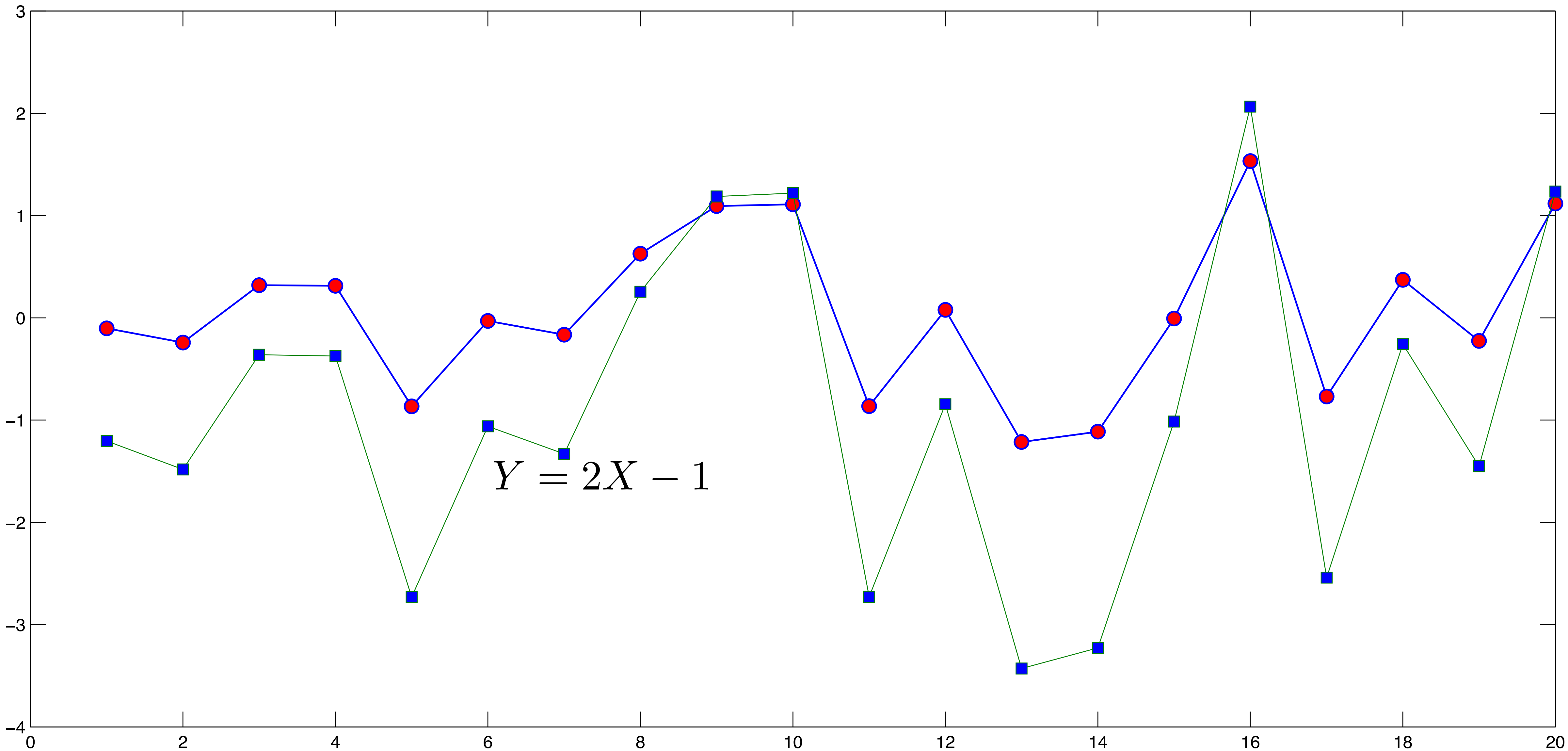
- Note

$$\int_{-\infty}^{\infty} \alpha_i x_i f_{X_i}(x_i)\, dx_i = E[\alpha_i X_i] = \alpha_i E[X_i]$$

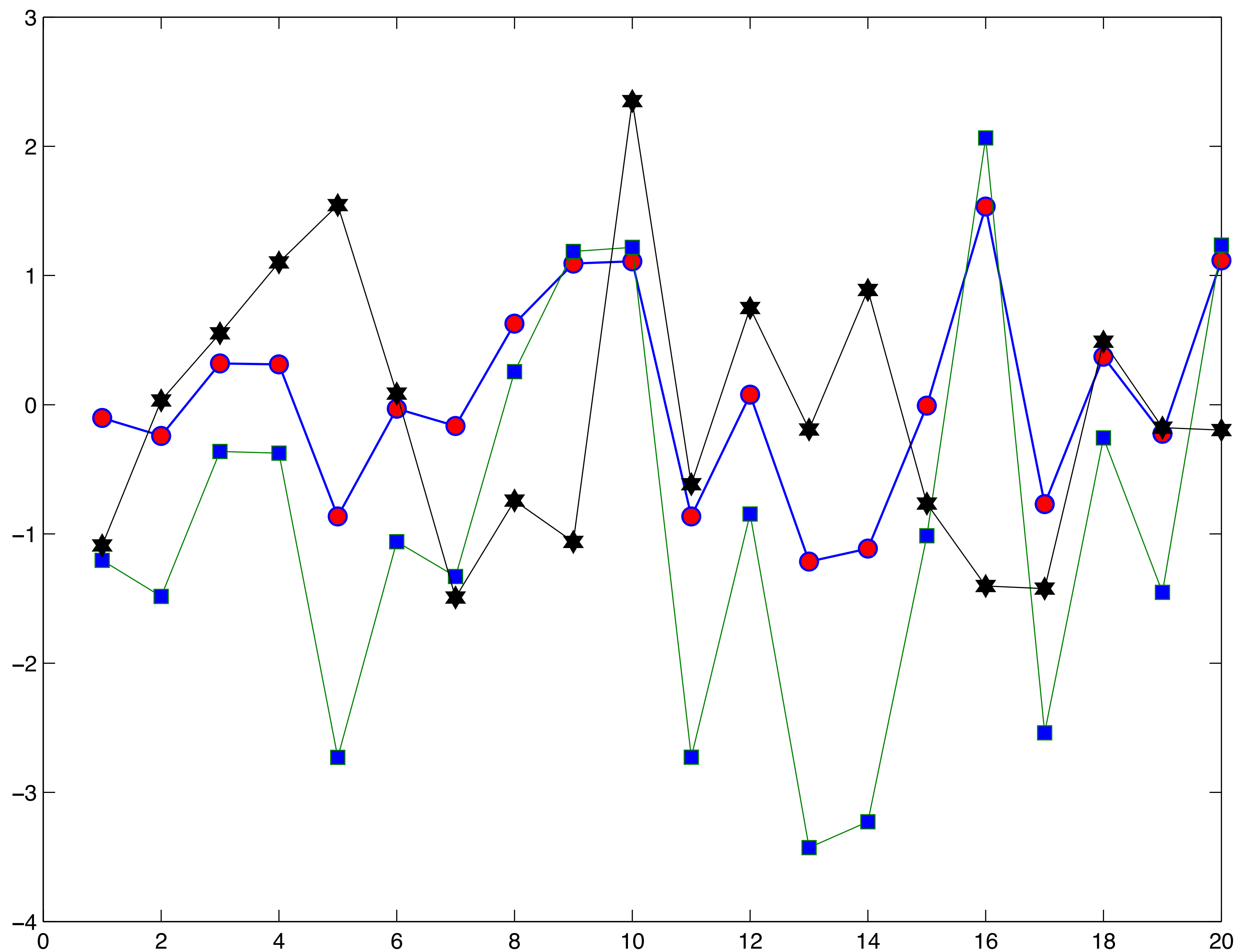
- Hence, we have

$$E\left[\sum_{i=1}^{N} \alpha_i X_i\right] = \sum_{i=1}^{N} \alpha_i E[X_i]$$

Similarity of Two Random Variables



Similarity of Two Random Variables



Correlation, Covariance and Correlation Coefficient

- Correlation of X and Y

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dxdy$$

  - Independent case

$$E(XY) = E(X)E(Y)$$

- Covariance

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

- Correlation coefficient

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

- Example: Two RVs X and Y are independent distributed as

$$X \sim \mathcal{N}(3,4) \qquad Y \sim \mathcal{N}(-1,2)$$

- Determine the covariance of the two random variables $Z = X - Y,$ and $W = 2X + 3Y$

$$\begin{aligned} E(Z) &= E(X) - E(Y) = 3 + 1 = 4, \\ E(W) &= 2E(X) + E(Y) = 6 - 3 = 3, \end{aligned}$$

$$\begin{aligned} E(X^2) &= \mathrm{Var}(X) + (E(X))^2 = 4 + 9 = 13 \\ E(Y^2) &= \mathrm{Var}(Y) + (E(Y))^2 = 2 + 1 = 3, \\ E(XY) &= E(X)E(Y) = -3. \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(W,Z) &= E(WZ) - E(W)E(Z) \\ &= E(2X^2 - 3Y^2 + XY) - E(Z)E(W) \\ &= 2 \times 13 - 3 \times 3 - 3 - 4 \times 3 \\ &= 2. \end{aligned}$$

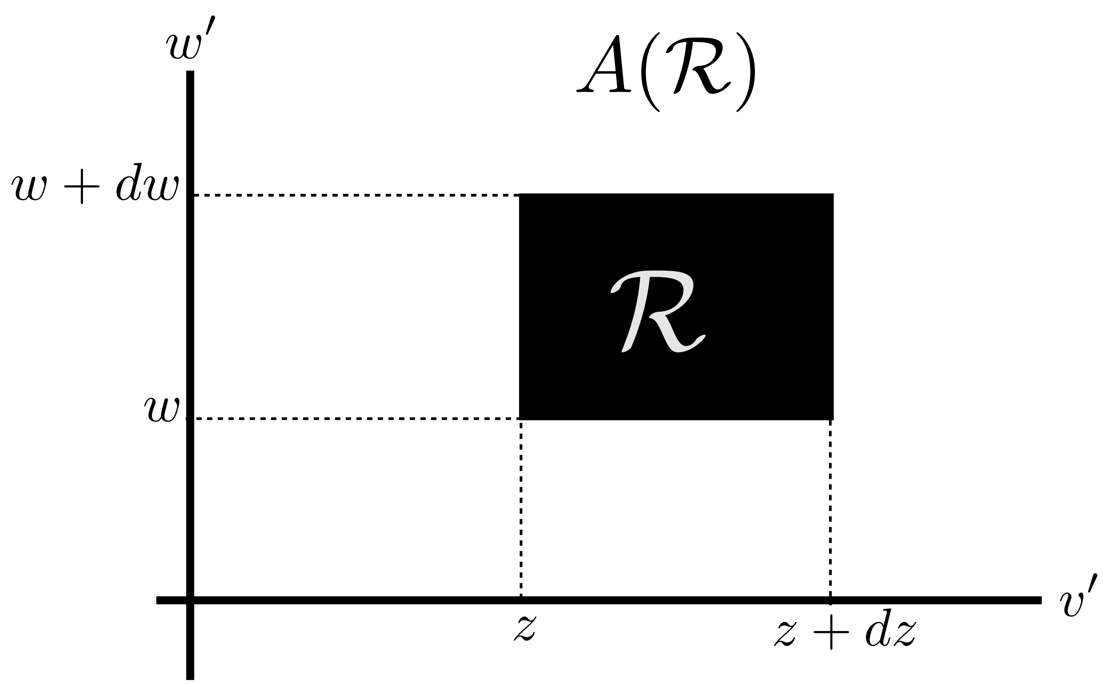
Transformation of Multiple Random Variables

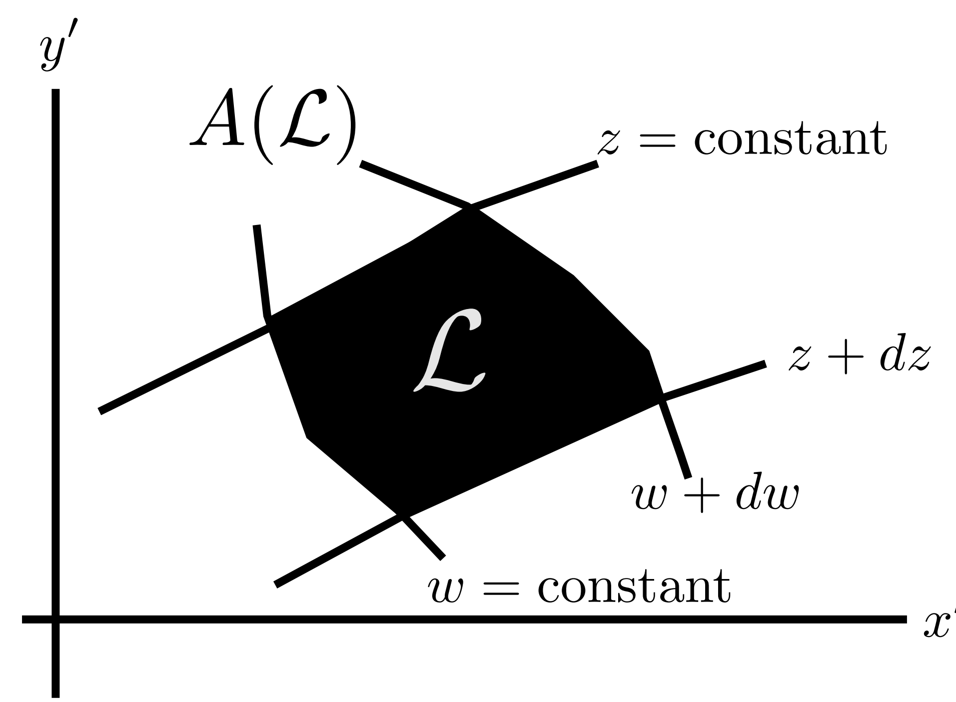
$X,\ Y$ → $f_{X,Y}(x,y)$

$$\begin{aligned} Z &= g(X,Y) \\ W &= h(X,Y) \end{aligned}$$

$Z,\ W$ → $f_{Z,W}(z,w)$

$$x = \phi(z,w)$$
$$y = \psi(z,w)$$





- Joint PDF of Z and W

$$
\begin{aligned}
P[z < Z \le z + dz,\ \ w < W \le w + dw] &= \int_{\mathcal{R}}\int f_{ZW}(\zeta, \eta)\, d\zeta d\eta \\
&= f_{Z,W}(z, w) A(\mathcal{R}) \\
&= \int_{\mathcal{L}}\int f_{X,Y}(\zeta, \eta)\, d\zeta d\eta \\
&= f_{X,Y}(x, y) A(\mathcal{L})
\end{aligned}
$$

- Note that

$$
A(\mathcal{L}) = \text{mag}\begin{vmatrix} \frac{\partial \phi}{\partial z} & \frac{\partial \phi}{\partial w} \\ \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \end{vmatrix} dz\, dw
$$

  - Jacobian

$$
J \triangleq \frac{\partial(\phi, \psi)}{\partial(z, w)}
$$

- Then, we have

$$f_{Z,W}(z,w)A(\mathcal{R}) = f_{X,Y}(x,y)A(\mathcal{L})$$

- where

$$A(\mathcal{R}) = dzdw$$

and

$$A(\mathcal{L}) = |J|dzdw$$

- Hence, we have

$$f_{Z,W}(z,w) = f_{X,Y}(x,y)|J|$$

Example

- Consider the Gaussian joint PDF given as

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

and transformation given as

$$v = g(X,Y) = \sqrt{X^2+Y^2},$$
$$\theta = h(X,Y) = \tan^{-1}\frac{Y}{X}.$$

- Jacobian

$$J(x,y) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

$$|J| = |\det J(x,y)| = \frac{1}{\sqrt{x^2+y^2}} \quad = \frac{1}{v}$$

- The set of equations

$$\begin{cases} \sqrt{x^2+y^2} = v \\ \tan^{-1} = \theta \end{cases} \longrightarrow \begin{cases} x = v\cos\theta \\ y = v\sin\theta \end{cases}$$

- Joint PDF

$$f_{V,\Theta}(v,\theta) = v f_{X,Y}(v\cos\theta, v\sin\theta) = \frac{v}{2\pi\sigma^2} e^{-\frac{v^2}{2\sigma^2}}$$

- Marginal PDF

$$\begin{aligned} f_\Theta(\theta) &= \int_0^\infty f_{V,\Theta}(v,\theta) dv = \frac{1}{2\pi}\int_0^\infty \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}}\, dv = \frac{1}{2\pi}. \\ f_V(v) &= \int_0^{2\pi} f_{V,\Theta}(v,\theta)\, d\theta = \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}}. \end{aligned}$$

$$\longrightarrow f_V(v) = \begin{cases} \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}}, & v \geq 0, \\ 0, & v < 0. \end{cases}$$

Jointly Gaussian Random Variables

- PDF of jointly Gaussian RVs

$$
\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\
&\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-m_X)^2}{\sigma_X^2}+\frac{(y-m_Y)^2}{\sigma_Y^2}-\frac{2\rho(x-m_X)(y-m_Y)}{\sigma_Y}\right]\right\}
\end{aligned}
$$

Conditional CDF and PDF

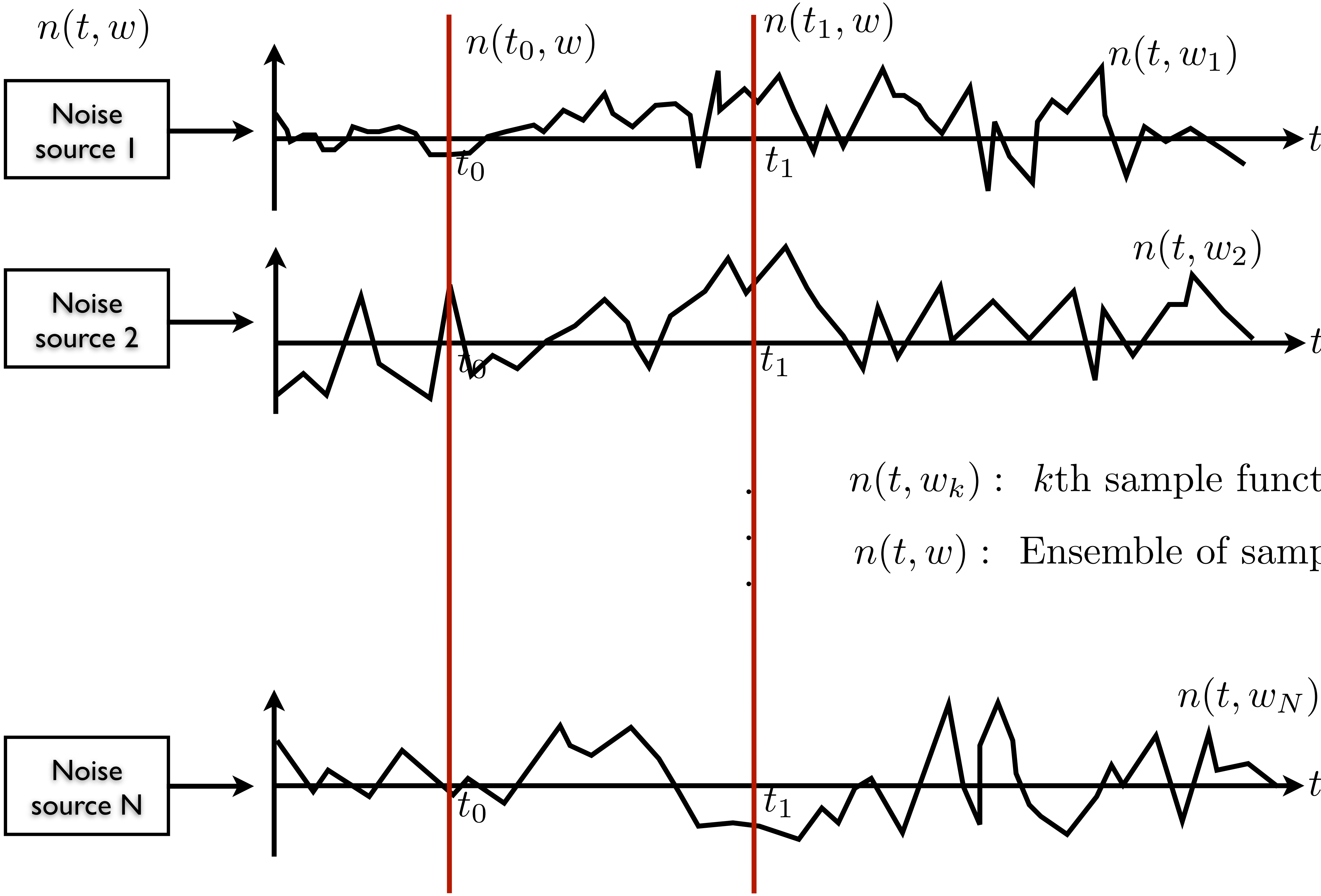
$$F_X(x|Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(t,y)\,dt}{f_Y(y)}$$

$$F_Y(y|X=x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x,t)\,dt}{f_X(x)}\,dt$$

$$f_X(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Random Process: Ensemble of Sample Functions



$n(t, w_k)$: kth sample function

$n(t, w)$: Ensemble of sample functions

Random Process

- Random process
  - a function of time
  - an ensemble of sample functions

Notation

- Random process

$n(t)$ instead of $n(t, w)$ in the previous noise source example

In general, we denote the capital letter as a function of time such as $X(t)$

- Random variable

Random variable is an ensemble of values of the random process at a certain time such as $X(t_0) \triangleq \{X(t_0, w_k)\}_{k=1}^{K}$ or simply $X(t_0) = X_0$.

Generally, $X(t_j) \triangleq \{X(t_j, w_k)\}_{k=1}^{K} = X_j$.

Statistics of Random Process

- Density function at t_0

  is the density function of the random variable $X(t_0)$ (or simply X_0).

- Mean and variance at t_0

  are the mean and variance of the random variable $X(t_0)$ (or simply X_0).

- Density function, mean and variance at time t_0 can be in general different from those at time t_1.

Statistical Averages

The mean, or expectation, of the random process $X(t)$ is a deterministic function of time denoted by $m_X(t)$ that at each time instant t_0 equals the mean of the random variable $X(t_0)$.

That is, $m_X(t) = E[X(t)]$ for all t.

- Note for the calculation of the mean value at a certain time instant t_0 :

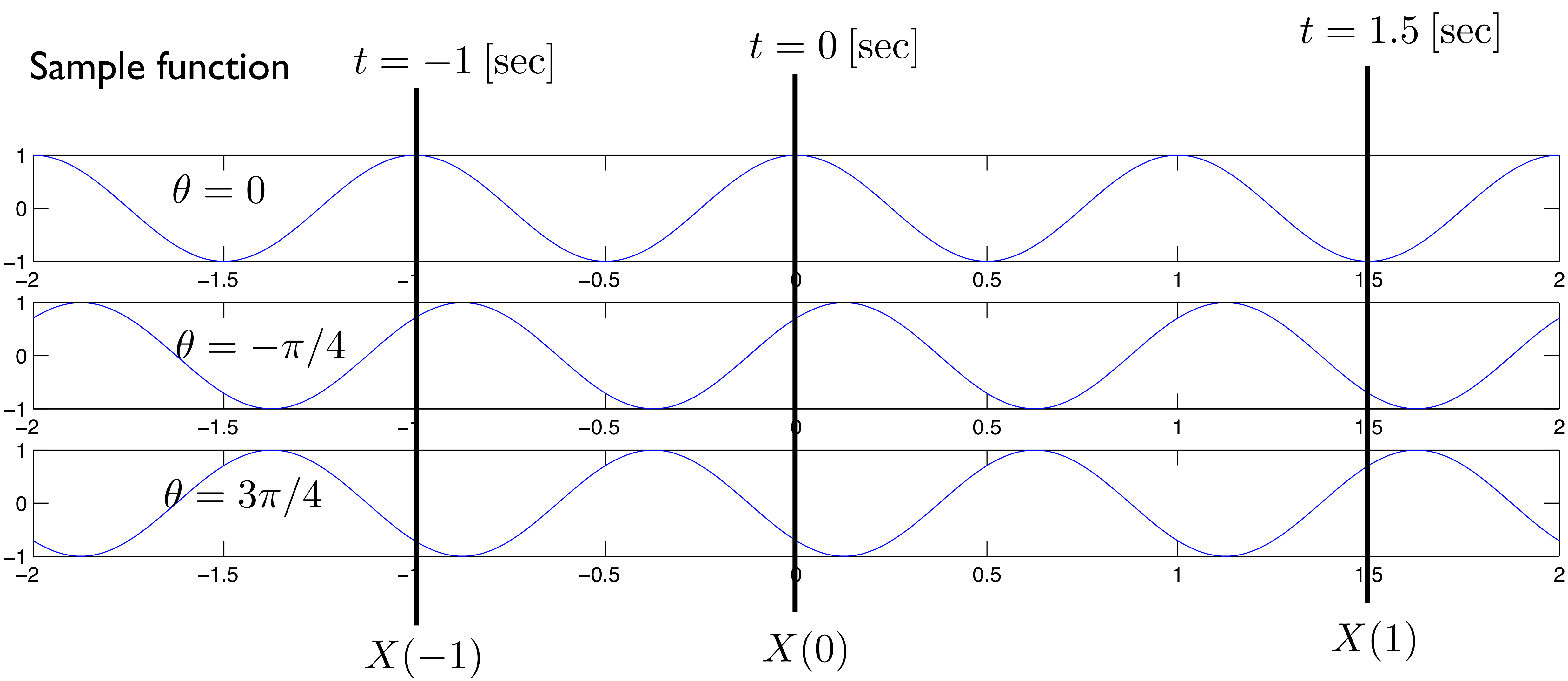
$$E[X(t_0)] = m_X(t_0) = \int_{-\infty}^{\infty} x f_{X(t_0)}(x)\, dx.$$

Example

Random process:

$$X(t) = A\cos(2\pi f_0 t + \Theta) \quad \text{where} \quad \Theta \sim \mathcal{U}[0, 2\pi] \qquad f_\Theta(\theta) = \frac{1}{2\pi}, \ \ 0 \le \theta \le 2\pi$$

Sample function



Mean value

$$m_X(t_0) = E[X(0)] = \int_0^{2\pi} X(0) f_\Theta(\theta)\, d\theta = \int_0^{2\pi} A\cos(2\pi f_0 t + \theta)\frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi}\left[A\sin(2\pi f_0 t + \theta)\Big|_{\theta=0}^{2\pi}\right] = \frac{1}{2\pi}\left[A\sin(2\pi f_0) - A\sin(2\pi f_0)\right] = 0$$

The mean value is not a function of time but is always zero regardless of time.

Autocorrelation Function of the Random Process

- Definition

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2)\, dx_1 dx_2$$

- Autocorrelation function $R_X(t_1, t_2)$ is a deterministic function of two variables t_1 and t_2.

Example

Random process:

$$X(t) = A\cos(2\pi f_0 t + \theta) \text{ where } \Theta \sim \mathcal{U}[0, 2\pi] \qquad f_\Theta(\theta) = \frac{1}{2\pi},\ 0 \le \theta \le 2\pi$$

Find the autocorrelation function.

$$R_X(t_1, t_2) = E[A\cos(2\pi f_0 t_1 + \Theta) A\cos(2\pi f_0 t_2 + \Theta)]$$

$$= \frac{A^2}{2}\{E[\cos(2\pi f_0(t_1 - t_2))] + \underbrace{E[\cos(2\pi f_0(t_1 + t_2) + 2\Theta)]}_{0}\}$$

$$= \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2))$$

Wide-Sense Stationary Processes

- Definition

A process $X(t)$ is *wide-sense stationary (WSS)* if the following conditions are satisfied:

1. $m_X(t) = E[X(t)]$ is independent of t.
2. $R_X(t_1, t_2)$ depends only on the time difference $\tau = t_2 - t_1$ and not on t_1 and t_2 individually.

- Notation for the WSS process

$$m_X(t) \to m_X$$

$$R_X(t_1, t_2) \to R_X(\tau)$$

Example

$$Y(t) = A\cos(2\pi f_0 t + \theta)$$

where $\Theta \sim \mathcal{U}[0, \pi]$ $\qquad f_\Theta(\theta) = \frac{1}{\pi},\ 0 \le \theta \le \pi$

Is $Y(t)$ stationary process?

$$m_Y(t) = E[A\cos(2\pi f_0 t + \Theta)]$$

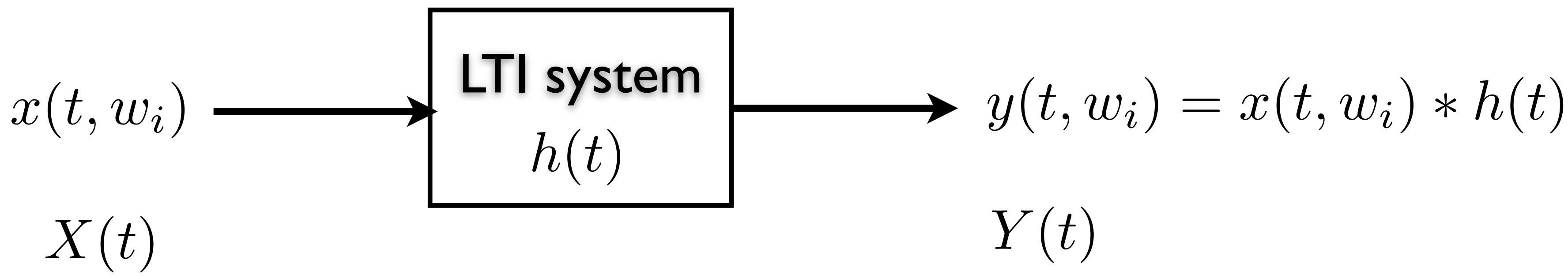
$$= \int_0^\pi \frac{1}{\pi} A\cos(2\pi f_0 t + \theta)\, d\theta$$

$$= \frac{A}{\pi}\sin(2\pi f_0 t + \theta)\Big|_{\theta=0}^{\pi} = -\frac{2A}{\pi}\sin(2\pi f_0 t)$$

$Y(t)$ is not stationary process.

Multiple Random Process

- Example in the LTI system



- Independent processes

Two random processes $X(t)$ and $Y(t)$ are independent if, for all $t_1,\ t_2$, the random variables $X(t_1)$ and $Y(t_2)$ are independent. Similarly, $X(t_1)$ and $Y(t_2)$ are uncorrelated if $X(t_1)$ and $Y(t_2)$ are uncorrelated for all $t_1,\ t_2$.

- Cross-Correlation

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

From the definition of cross-correlation, in general, we have

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = R_{YX}(t_2, t_1)$$

- Jointly wide-sense stationary if

1) $X(t)$ and $Y(t)$ are individually stationary

2) $R_{XY}(t_1, t_2)$ depends only on $\tau = t_1 - t_2$

- For jointly stationary process, it follows that

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Example

$X(t),\ Y(t)$: Jointly stationary RPs

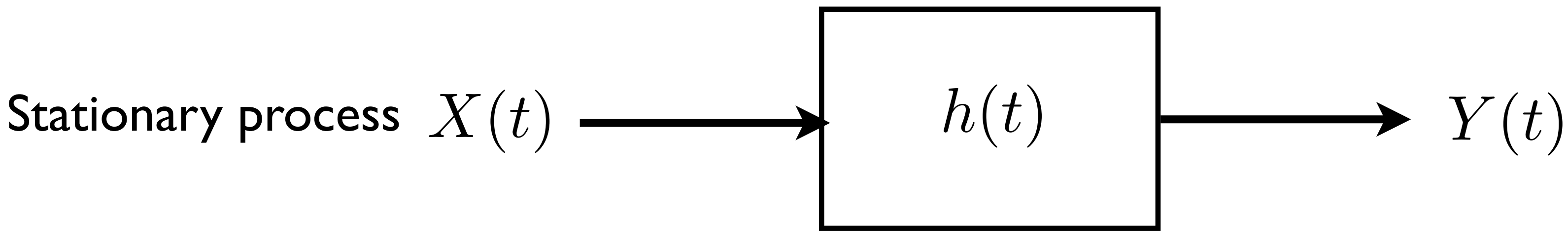
Determine the autocorrelation of the process $Z(t) = X(t) + Y(t)$

$$R_Z(t+\tau, t) = E[Z(t+\tau)Z(t)]$$

$$= E[(X(t+\tau) + Y(t+\tau))(X(t) + Y(t))]$$

$$= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{XY}(-\tau)$$

Random Processes and Linear Systems



$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(\lambda) h(t-\lambda)\, d\lambda$$

If $X(t)$ is stationary process, $Y(t)$ is also stationary!!!

$$E[X(t)] = m_X$$

$$E[Y(t)] = m_Y$$

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$R_Y(\tau) = E[Y(t+\tau)Y(t)]$$

- Mean of $Y(t)$

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,d\tau\right] f_{X(\tau)}(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(\tau)f_{X(\tau)}(x)\,dx\right] h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} E[X(\tau)]h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} m_X h(t-\tau)\,d\tau$$

change of variable

$$\lambda = t - u$$

$$= m_X \int_{-\infty}^{\infty} h(\lambda)\,d\lambda \quad \equiv m_Y$$

- Cross-correlation

$$E[X(t_1)Y(t_2)] = E[X(t_1)Y(t_2)] = E\left[X(t_1)\int_{-\infty}^{\infty} X(s)h(t_2 - s)\,ds\right]$$

$$= \int_{-\infty}^{\infty} E[X(t_1)X(s)]h(t_2 - s)\,ds$$

$$= \int_{-\infty}^{\infty} R_X(t_1 - s)h(t_2 - s)\,ds$$

$$= \int_{-\infty}^{\infty} R_X(t_1 - -t_2 - u)h(-u)\,du$$

$$= \int_{-\infty}^{\infty} R_X(\tau - u)h(-u)\,du$$

$$= R_X(\tau) * h(-\tau) \equiv R_{XY}(\tau)$$

- Auto-correlation

$$E[Y(t_1)Y(t_2)] = E[Y(t_1)Y(t_2)]$$

$$= E\left[\left(\int_{-\infty}^{\infty} X(s)h(t_1 - s)\, ds\right) Y(t_2)\right]$$

$$= \int_{-\infty}^{\infty} R_{XY}(s - t_2)h(t_1 - s)\, ds$$

$$= \int_{-\infty}^{\infty} R_{XY}(u)h(t_1 - t_2 - u)\, du$$

$$= R_{XY}(\tau) * h(\tau)$$

$$= R_X(\tau) * h(-\tau) * h(\tau) \equiv R_Y(\tau)$$