Communication Systems II

[KECE322_01]

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Lecture #15

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Outline

- Optimum decision rule
  - MAP criterion
  - ML criterion
  - Minimum Euclidean distance rule
  - Maximum correlation rule
- Average symbol rate
  - M-ary PAM
  - M-ary orthogonal signals

Optimum Detector

- Posteriori probability

$$P(\text{signal } \mathbf{s}_m \text{ was transmitted } |\mathbf{y})$$

- Optimum decision rule
  - Select the signal corresponding to the maximum set of posteriori probabilities:

$$\text{choose } m \text{ such that } \{P(\mathbf{s}_m|\mathbf{y})\}_{m=1}^{M} \text{ is maximum}$$

  - which is called "*maximum a posteriori (MAP)*" criterion.

- Bays' rule

$$P(\mathbf{s}_m|\mathbf{y}) = \frac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}$$

  - where $P(\mathbf{s}_m)$ is called "a priori probability".

MAP criterion

$$\text{choose } m \text{ such that } \{P(\mathbf{s}_m|\mathbf{y})\}_{m=1}^{M} \text{ is maximum}$$
$$= \text{choose } m \text{ such that } \{\tfrac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}\}_{m=1}^{M} \text{ is maximum}$$

- For equally probable case, that is, $P(\mathbf{s}_m) = \dfrac{1}{M}$, MAP criterion becomes

$$\text{choose } m \text{ such that } \{f(\mathbf{y}|\mathbf{s}_m)\}_{m=1}^{M} \text{ is maximum}$$

  - which is called "maximum likelihood (ML)" criterion.

Definition

- likelihood function: $f(\mathbf{y}|\mathbf{s}_m)$
- Log-likelihood function: $\ln f(\mathbf{y}|\mathbf{s}_m)$

MAP criterion

$$\text{choose } m \text{ such that } \{P(\mathbf{s}_m|\mathbf{y})\}_{m=1}^{M} \text{ is maximum}$$

$$= \text{choose } m \text{ such that } \{\frac{f(\mathbf{y}|\mathbf{s}_m)P(\mathbf{s}_m)}{f(\mathbf{y})}\}_{m=1}^{M} \text{ is maximum}$$

for equally probable case =>

$$= \text{choose } m \text{ such that } f(\mathbf{y}|\mathbf{s}_m)$$

$$= \text{choose } m \text{ such that } \ln f(\mathbf{y}|\mathbf{s}_m)$$

4-PAM case

- Likelihood function

$$f(y|s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1, 2, \ldots, M$$

- Log-Likelihood function

$$\ln f(y|s_m) = -\frac{1}{2}\log(\pi N_0) - \frac{(y-s_m)^2}{N_0}, \quad m = 1, 2, \ldots, M$$

- ML criterion

$$\max_m \left[-\frac{1}{2}\log(\pi N_0) - \frac{(y-s_m)^2}{N_0} \right], \quad m = 1, 2, \ldots, M$$

$$= \max_m \left[-\frac{(y-s_m)^2}{N_0} \right], \quad m = 1, 2, \ldots, M$$

$$= \min_m \left[(y-s_m)^2 \right], \quad m = 1, 2, \ldots, M$$

$$= \min_m ||y-s_m||, \quad m = 1, 2, \ldots, M$$

- Generally, the output of the demodulator over AWGN channel can be written as

$$y_k = s_{mk} + n_k, \quad k = 1, 2, \ldots, N$$

dimension

- Its likelihood function is given as

$$f(y_k|s_{mk}) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_{mk})^2/N_0}, \quad m = 1, 2, \ldots, M$$

- Joint likelihood function

$$f(\mathbf{y}|\mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{k=1}^{N}(y-s_{mk})^2/N_0}, \quad m = 1, 2, \ldots, M$$

- Log-likelihood function

$$\ln f(\mathbf{y}|\mathbf{s}_m) = -\frac{N}{2}\ln(\pi N_0) - \ln \sum_{k=1}^{N} \frac{(y-s_{mk})^2}{N_0}, \quad m = 1, 2, \ldots, M$$

- ML criterion

$$\min_m (y - s_{mk})^2, \quad m = 1, 2, \ldots, M$$

  - which is called *minimum (Euclidean) distance rule*

- Optimum decision rule
  - MAP criterion becomes ML criterion for equally probable case.
  - ML criterion can be reduced to minimum Euclidean distance rule over AWGN channels.
- Calculation of Euclidean distance

$$\begin{aligned} D(\mathbf{y}, \mathbf{s}_m) &= \sum_{n=1}^{N} y_n^2 - 2\sum_{n=1}^{N} y_n s_{mn} + \sum_{n=1}^{N} s_{mn}^2 \\ &= ||\mathbf{y}||^2 - 2\,\mathbf{y}\cdot\mathbf{s}_m + ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M \end{aligned}$$

  - Minimum distance rule choose $\mathbf{s}_m$ to give the minimum distance metric which is equivalent to choose minimum value of the metric given as

$$D'(\mathbf{y}, \mathbf{s}_m) = -2\,\mathbf{y}\cdot\mathbf{s}_m + ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M$$

  or choose the maximum distance metric given as

$$C(\mathbf{y}, \mathbf{s}_m) = 2\,\mathbf{y}\cdot\mathbf{s}_m - ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M$$

- Correlation metric

$$C(\mathbf{y}, \mathbf{s}_m) = 2\,\mathbf{y} \cdot \mathbf{s}_m - ||\mathbf{s}_m||^2, \quad m = 1, 2, \ldots, M$$

- We choose $\mathbf{s}_m$ which gives maximum correlation metric.
- If all the signals have equal energy, that is, $||\mathbf{s}_m||^2 = \mathcal{E}_s$, for all *m*
  - we can just neglect the term $||\mathbf{s}_m||^2$.

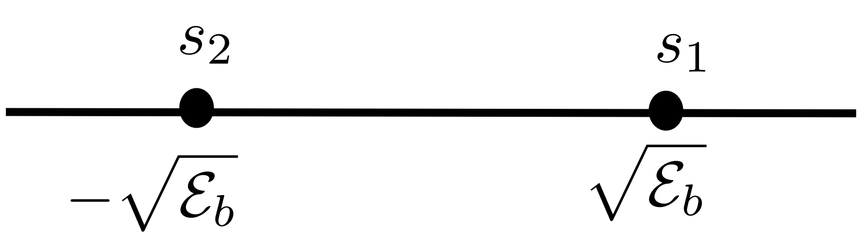
Summary of Optimum Decision Rule

- Optimum decision rule is MAP criterion.
- MAP criterion is equivalent to ML criterion for equally probable case.
- ML criterion is equivalent to minimum Euclidean distance rule over AWGN channels.
- Minimum Euclidean distance rule is equivalent to maximum correlation rule.

Probability of Error for M-ary Pulse Amplitude Modulation

- Bit error rate of binary PAM signals

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$



$$d_{12} = 2\sqrt{\mathcal{E}_b} \implies \mathcal{E}_b = \frac{d_{12}^2}{4}$$

  - We can rewrite the BER of binary PAM signals as

$$P_2 = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

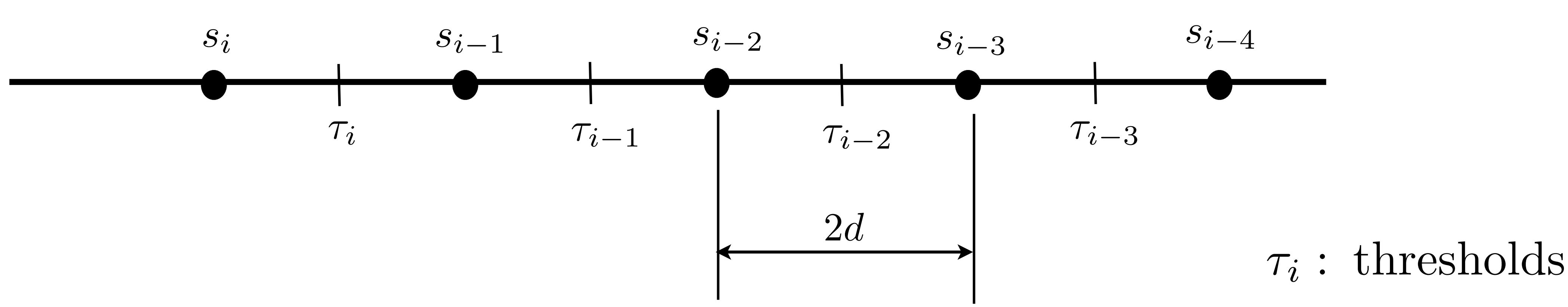
- In the case of M-ary PAM, the input to the detector is

$$y = s_m + n,$$

  - Optimum decision rule for equally probable case chooses the maximum correlation metrics

$$C(y, s_m) = 2ys_m - s_m^2 = 2(y - s_m/2)s_m$$

  - Equivalently, the optimum threshold may compare y with a set of M-1 thresholds, which are placed at the midpoints of successive amplitude levels. Thus, a decision is made in favor of the amplitude level that is closest to y.



- Average symbol error rate

$$
\begin{aligned}
P_M &= \frac{1}{M}\left[\Pr(y - s_1 > d) + \Pr(|y - s_2| > d + \cdots + \Pr(|y - s_{M-1}| > d) + \Pr(y - s_M < -d)\right] \\
&= \frac{M-1}{M}\Pr\left(|y - s_m| > d\right) \\
&= \frac{M-1}{M}\frac{2}{\sqrt{\pi N_0}}\int_d^\infty e^{-x^2/N_0}\,dx \\
&= \frac{M-1}{M}\frac{2}{\sqrt{2\pi}}\int_{\sqrt{2d^2/N_0}}^\infty e^{-x^2/2}\,dx \\
&= \frac{2(M-1)}{M}Q\left(\sqrt{2d^2/N_0}\right)
\end{aligned}
$$

- Recall

$$\mathcal{E}_{av} = \frac{d^2(M^2-1)}{3},$$

- We can rewrite the average symbol error rate as

$$P_M = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}}\right).$$

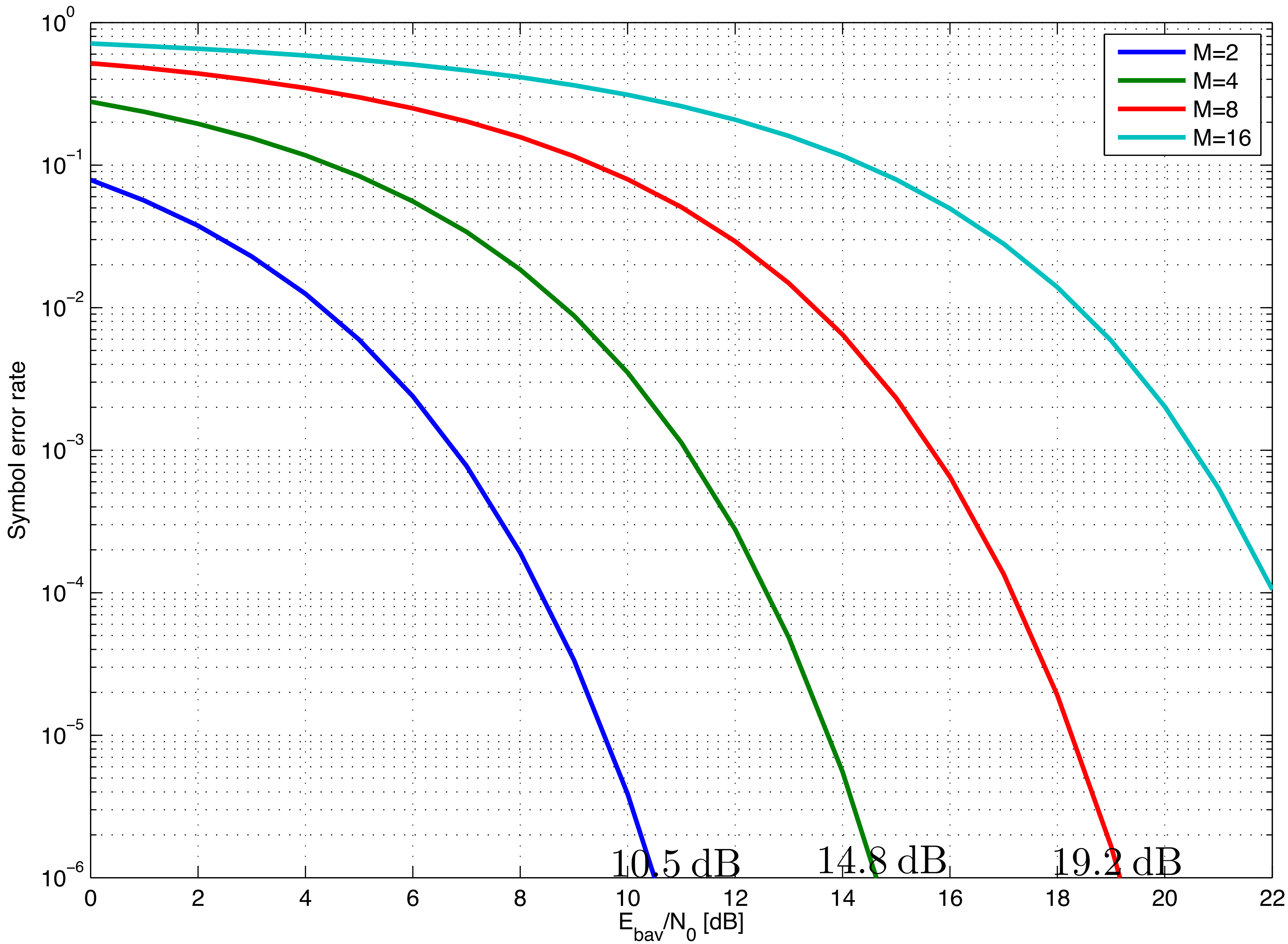
- Note that each signal carries $k = \log_2 M$ bits of information, the average energy per bit is

$$\mathcal{E}_{bav} = \frac{\mathcal{E}_{av}}{k} \implies \mathcal{E}_{av} = k\mathcal{E}_{bav} = (\log_2 M)\mathcal{E}_{bav}$$

- Hence, we have

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\mathcal{E}_{bav}}{(M^2-1)N_0}}\right).$$

Average SER curve



Probability of Error for M-ary Orthogonal Signals

- Each of M-ary orthogonal signals has equal energy.

- For equal energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between the received vector $\mathbf{y}$ and each of the M possible transmitted signal vectors $\{\mathbf{s}_m\}$, i.e.,

$$C(y, \mathbf{s}_m) = \mathbf{y} \cdot \mathbf{s}_m = \sum_{k=1}^{M} y_k s_{mk}, \quad m = 1, 2, \ldots, M.$$

- To evaluate the probability of error, let us assume that the signal $\mathbf{s}_1$ is transmitted. Then the vector at the input to the detector is

$$\mathbf{y} = \left(\sqrt{\mathrm{E}_s} + n_1, n_2, n_3, \ldots, n_M \right),$$

  - where $n_1, n_2, n_3, \ldots, n_M$ are zero mean, mutually statistically independent Gaussian random variables with equal variance $N_0/2$.

- Cross-correlation metric

$$\begin{aligned} C(\mathbf{y},\mathbf{s}_1) &= \sqrt{\mathcal{E}_s}(\sqrt{\mathcal{E}_s}+n_1); \\ C(\mathbf{y},\mathbf{s}_2) &= \sqrt{\mathcal{E}_s}n_2; \\ &\vdots \\ C(\mathbf{y},\mathbf{s}_M) &= \sqrt{\mathcal{E}_s}n_M. \end{aligned}$$

  - Note that we can eliminate the scale factor $\sqrt{\mathcal{E}_s}$ for the comparisons.

- PDF of the first correlator output with the elimination of $\sqrt{\mathcal{E}_s}$.

$$f(y_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1-\sqrt{\mathcal{E}_s})^2}{N_0}}$$

  - PDF's of the other M-1 correlator outputs

$$f(y_m) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{y_m^2}{N_0}}, \quad m = 2, 3, \ldots, M$$

- Correct decision probability when $s_1(t)$ is transmitted

$$P_{c|s_1} = \int_{-\infty}^{\infty} \Pr[n_2 < y_1,\, n_3 < y_1,\, \ldots,\, n_M < y_1 \mid s_1] f_{y_1}(y_1)\, dy_1$$

  - For equally probable case,

$$P_c = \int_{-\infty}^{\infty} \Pr(n_2 < y_1,\, n_3 < y_1,\, \ldots,\, n_M < y_1 \mid y_1) f_{y_1}(y_1)\, dy_1$$

  - Note that

$$\begin{aligned} \Pr(n_M < y_1 | y_1) &= \int_{-\infty}^{y_1} f(y_m)\, dy_m, \quad m = 2, 3, \ldots, M \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{\frac{2y_1^2}{N_0}}} e^{-\frac{y_m^2}{2}}\, dy_m \\ &= 1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right). \end{aligned}$$

- Probability of correct decision

$$P_c = \int_{-\infty}^{\infty} \left[1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right)\right]^{M-1} f_{y_1}(y_1)\, dy_1$$

- Probability of symbol error (Symbol error rate)

$$P_M = 1 - \int_{-\infty}^{\infty} \left[1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right)\right]^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-(y-\sqrt{\mathcal{E}_s})^2/N_0}\, dy_1$$

Change of variable $x = \frac{2y_1^2}{N_0}$

Then we have

$$P_M = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - Q\left(x\right)\right]^{M-1} e^{-(x-\sqrt{2\mathcal{E}_s/N_0})}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{1 - \left[1 - Q\left(x\right)\right]^{M-1}\right\} e^{-(x-\sqrt{2\mathcal{E}_s/N_0})}\, dx$$

- Average bit error rarte

$$P_b = \frac{1}{k}\sum_{n=1}^{k} n\binom{n}{k}\frac{P_M}{2^k-1} = \frac{2^{k-1}}{2^k-1}P_M \approx \frac{P_M}{2}, \quad k >> 1.$$