Chapter 2

Vector Differentiation

2.1 Ordinary Derivative

1. The first-order ordinary derivative of the vector which is a function of a scalar s is defined by

$$\frac{d\boldsymbol{A}(s)}{ds} \equiv \lim_{\Delta s \to 0} \frac{\Delta \boldsymbol{A}(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\boldsymbol{A}(s+\Delta s) - \boldsymbol{A}(s)}{\Delta s}. \tag{2.1}$$

2.2 Partial Derivative

1. The first-order partial derivative of the vector which is a function of t,x,y,z is defined by

$$\frac{\partial \boldsymbol{A}}{\partial t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{A}(t+\Delta t, x, y, z) - \boldsymbol{A}(t, x, y, z)}{\Delta t}, \tag{2.2a}$$

$$\frac{\partial \boldsymbol{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\boldsymbol{A}(t, x+\Delta x, y, z) - \boldsymbol{A}(t, x, y, z)}{\Delta x}, \tag{2.2b}$$

$$\frac{\partial \boldsymbol{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\boldsymbol{A}(t, x, y+\Delta y, z) - \boldsymbol{A}(t, x, y, z)}{\Delta y}, \tag{2.2c}$$

$$\frac{\partial \boldsymbol{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\boldsymbol{A}(t, x, y, z+\Delta z) - \boldsymbol{A}(t, x, y, z)}{\Delta z}. \tag{2.2d}$$

2.3 Gradient, Divergence and Curl Operators

2.3.1 Vector differential operator

1. The vector differential operator ∇ is defined by

$$\nabla \equiv \sum_i \hat{\boldsymbol{e}}_i \frac{\partial}{\partial x_i}. \tag{2.3}$$

2.3.2 Gradient

1. The gradient of a scalar function ϕ is defined by

$$\nabla\phi \equiv \sum_i \hat{\boldsymbol{e}}_i \frac{\partial \phi}{\partial x_i}. \tag{2.4}$$

The gradient of a scalar function transforms like a vector.

2.3.3 Divergence

1. The divergence of a vector $\boldsymbol{A}$ is defined by

$$\nabla \cdot \boldsymbol{A} = \left(\hat{\boldsymbol{e}}_i \frac{\partial}{\partial x_i} \right) \cdot (\hat{\boldsymbol{e}}_j A_j) = \frac{\partial A_i}{\partial x_i}. \tag{2.5}$$

The divergence of a vector transform like a scalar.

2.3.4 Curl

1. The curl of a vector $\boldsymbol{A}$ is defined by

$$(\nabla \times \boldsymbol{A})_i \equiv \left(\hat{\boldsymbol{e}}_i \frac{\partial}{\partial x_i} \right) \times (\hat{\boldsymbol{e}}_j A_j) = \epsilon_{ijk} \hat{\boldsymbol{e}}_i \frac{\partial A_k}{\partial x_j}. \tag{2.6}$$

The curl of a vector transforms like a vector.

2.3.5 Laplacian

1. The Laplacian of a scalar function ϕ is defined by

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_i} \right). \tag{2.7}$$

The Laplacian transforms like a scalar.

2.3.6 Solenoidal

1. We call a vector $\boldsymbol{A}$ solenoidal if

$$\nabla \cdot \boldsymbol{A} = 0. \tag{2.8}$$

2.3.7 Irrotational

1. We call a vector $\boldsymbol{A}$ irrotational if

$$\nabla \times \boldsymbol{A} = 0. \tag{2.9}$$

2.4 Exercise

2.4.1 $\nabla \times \nabla \phi = 0$

1. Let us prove that $\nabla \times \nabla \phi = 0$.

$$\begin{aligned}
(\nabla \times \nabla \phi) &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi \\
&= \frac{1}{2} \left[\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} + \epsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \right] \phi \\
&= \frac{1}{2} \left[\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} - \epsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \right] \phi \\
&= 0
\end{aligned} \tag{2.10}$$

2.4.2 $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

1. Let us prove that $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$.

$$\begin{aligned}
\nabla \cdot (\nabla \times \boldsymbol{A}) &= \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} A_k \\
&= \frac{1}{2} \left[\epsilon_{ijk} \frac{\partial^2}{\partial x_i \partial x_j} + \epsilon_{jik} \frac{\partial^2}{\partial x_j \partial x_k} \right] A_k \\
&= \frac{1}{2} \epsilon_{ijk} \left[\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial x_k} \right] A_k \\
&= 0
\end{aligned} \tag{2.11}$$

2.4.3 $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

1. Let us prove that $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$.

$$\begin{aligned}
[\nabla \times (\nabla \times \boldsymbol{A})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \boldsymbol{A})_k \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \frac{\partial}{\partial x_l} A_m \\
&= \epsilon_{kij} \epsilon_{klm} \frac{\partial^2}{\partial x_j \partial x_l} A_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial^2}{\partial x_j \partial x_l} A_m \\
&= \frac{\partial^2 A_j}{\partial x_j \partial x_i} - \frac{\partial^2 A_i}{\partial x_j \partial x_j} \\
&= \frac{\partial}{\partial x_i} \left(\frac{\partial A_j}{\partial x_j} \right) - \frac{\partial^2 A_i}{\partial^2 x_j} \\
&= \frac{\partial}{\partial x_i} (\nabla \cdot \boldsymbol{A}) - \nabla^2 A_i \\
&= [\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}]_i.
\end{aligned} \tag{2.12}$$

2.4.4 $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$

1. Let us prove that $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$.

$$\begin{aligned}
\nabla^2(fg) &= \frac{\partial^2}{\partial x_i^2}(fg) \\
&= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}(fg) \\
&= \frac{\partial}{\partial x_i} \left[\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \right] \\
&= \frac{\partial^2 f}{\partial x_i^2} g + \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + f \frac{\partial^2 g}{\partial x_i^2} \\
&= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g.
\end{aligned} \tag{2.13}$$