Mobile Communications (KECE425)

Lecture Note 24

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Summary

- Singular value decomposition (SVD)
- MIMO channel in SVD

Hermitian Matrix

- The matrix A is called Hermitian matrix if

$$A = A^H$$

- If matrix is Hermitian, its eigenvalues are real.

- Example of Hermitian matrix: $F = H^H H$ or $F = H H^H$ in MIMO channels.

Positive and Semi-Positive Matrix

- Definition
  - Define the inner product between two complex vectors

$$< \mathbf{x}, \mathbf{y} >= \sum_{j=1}^{n} x_j y_j^*$$

  - The matrix A is positive definite matrix if $< A\mathbf{x}, \mathbf{x} > > \ 0$ for $\mathbf{x} \neq \mathbf{0}$
  - The matrix A is positive semi-definite if $< A\mathbf{x}, \mathbf{x} > \geq \ \mathbf{0}$ for all $\mathbf{x}$.

- Properties of positive definite (or semi-positive definite) matrix
  - Its eigenvalues are positive (or non-negative) real values.

- Example of semi-positive definite matrix: $F = H^H H$ or $F = HH^H$ in MIMO channel

Singular Value Decomposition

- Handy mathematical technique that has application to many problems.
- Given any $m \times n$ matrix A, algorithm to find matrices U, V, and D such that

$$
\begin{aligned}
A &= UDV^H \\
U &: \text{is } m \times m \text{ and orthonormal, that is, } UU^H = I, \\
D &: \text{is } m \times n \text{ and diagonal} \\
V &: \text{ is } n \times n \text{ and orthonormal, that is, } VV^H = I.
\end{aligned}
$$

- SVD

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

1) $m > n$

$$= \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mm} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}^H$$

2) $n > m$

$$= \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mm} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_m} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}^H$$

3) $n = m$,

$$= \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mm} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_m} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}^H$$

- *Rank* of the matrix A is defined as the number of non-zero values in D.
  - We can show the rank of A is at most $\min(m, n)$, that is, the maximum value of rank A is $\min(m, n)$.

- Consider $D^H D$.

1) $m = n$

$$D^H D \;=\; \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

2) $m > n$

$$D^H D \;=\; \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} & 0 & \cdots & 0 \end{bmatrix}_{n\times m} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}_{n\times n}$$

3) $n > m$

$$D^H D = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_m} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_m} & 0 & \cdots & 0 \end{bmatrix}_{n\times m}$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

at most m non-zero eigenvalues $(m < n)$.

- Consider $A^H A$:

$$A^H A \;=\; (UDV^H)^H(UDV^H) \;=\; VD^H U^H UDV^H = V(D^H D)V^H$$

  - Let $F = A^H A$ and $\Sigma = D^H D$. Note that Σ is $n \times n$ diagonal matrix.
  - Then $F = V\Sigma V^H$.
  - Now consider FV:

$$\begin{aligned} FV \;&=\; V\Sigma \overset{=I}{\boxed{V^H V}} = V\Sigma \\ &=\; \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \end{aligned}$$

$$FV = V\Sigma \overset{= I}{V^H V} = V\Sigma$$

$$= \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \begin{array}{c} \begin{matrix} \quad\mathbf{v}_1 & \qquad\mathbf{v}_2 & & \qquad\qquad\mathbf{v}_n \end{matrix} \\ \begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} & \cdots & \lambda_n v_{1n} \\ \lambda_1 v_{21} & \lambda_2 v_{22} & \cdots & \lambda_n v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_1 v_{n1} & \lambda_2 v_{n2} & \cdots & \lambda_n v_{nn} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} | & | & | & | \\ \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \\ | & | & | & | \end{bmatrix}$$

$$FV = \begin{bmatrix} | & | & & | \\ \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} | & | & & | \\ F\mathbf{v}_1 & F\mathbf{v}_2 & \cdots & F\mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

$$\downarrow$$

$$F\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

- SVD for a matrix A is

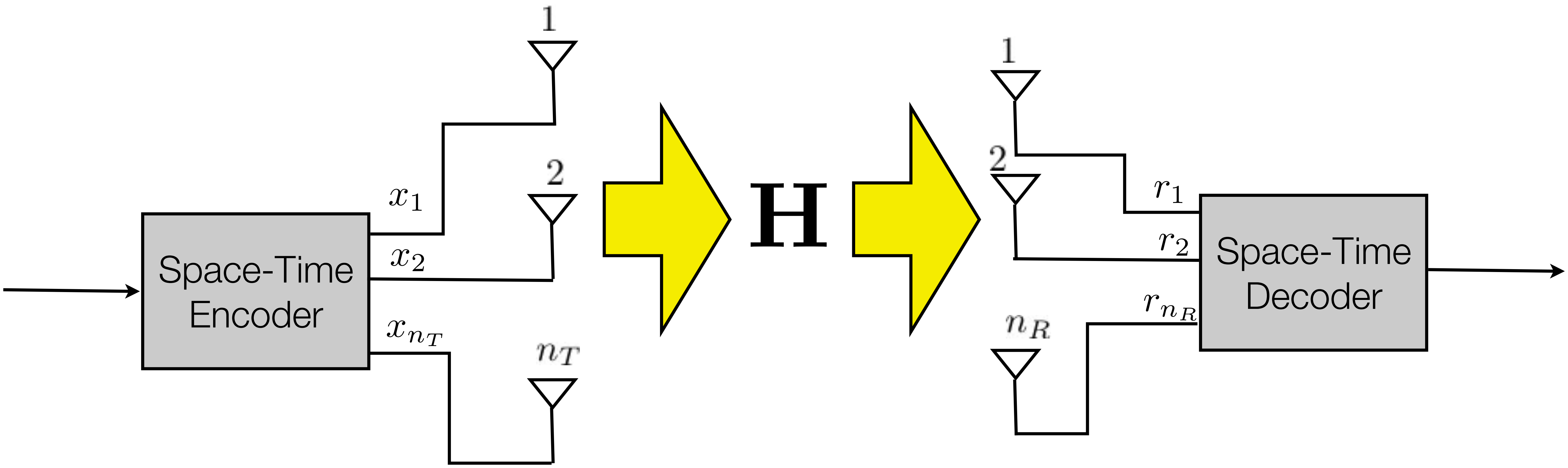
$$A = UDV^H$$

  - Each column vector of V is the eigenvector of $A^H A$.

- D is diagonal matrix with the square root of eigenvalues of $A^H A$ in its diagonal elements.

- We can also show that each column vector of U is the eigenvector of AA^H.

MIMO for Spatial Multiplexing



- MIMO spatial multiplexing means the transmission of multiple data streams from the transmit antennas instead of signal data stream such as in diversity systems.

- Received signal over MIMO channels

$$\begin{aligned} r_1 &= h_{11}x_1 + h_{12}x_2 + \cdots + h_{1n_T}x_{n_T} + n_1 \\ r_2 &= h_{21}x_1 + h_{22}x_2 + \cdots + h_{2n_T}x_{n_T} + n_2 \\ &\vdots \\ r_{n_R} &= h_{n_R1}x_1 + h_{n_R2}x_2 + \cdots + h_{n_Rn_T}x_{n_T} + n_{n_R} \end{aligned}$$

or in vector form with the channel matrix H

$$\mathbf{r} = H\mathbf{x} + \mathbf{n}$$

where

$$\begin{aligned} \mathbf{r} &= [r_1\, r_2 \ldots r_{n_R}]^T, \\ \mathbf{x} &= [x_1\, x_2 \ldots x_{n_T}]^T, \\ \mathbf{n} &= [n_1\, n_2 \ldots n_{n_R}]^T \end{aligned}$$

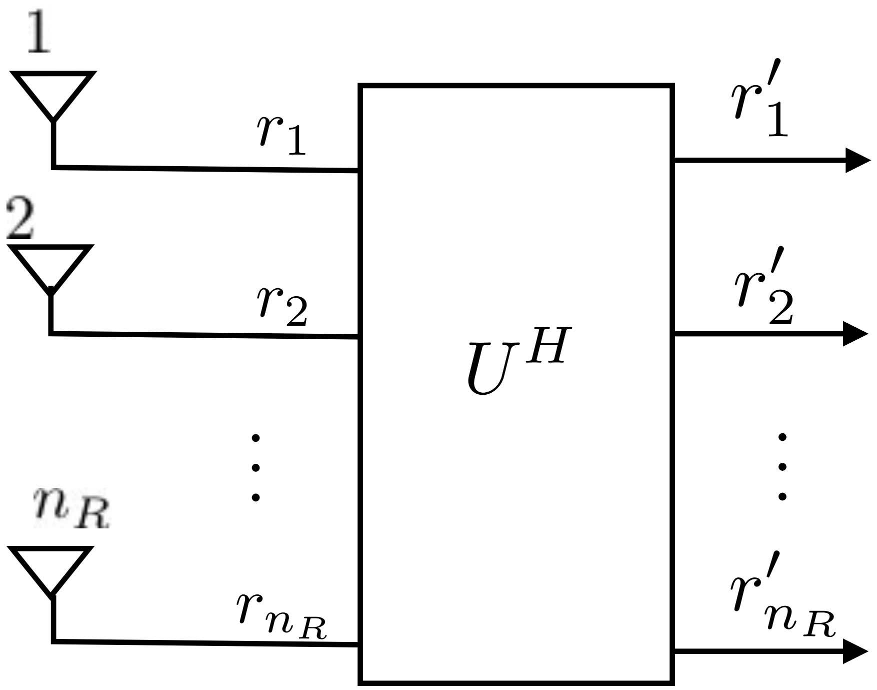
- MIMO channel matrix, H can be also factored by SVD such as:

$$H = UDV^H$$

- Then we can rewrite the received signal in vector form as

$$\mathbf{r} \quad = \quad UDV^H\mathbf{x} + \mathbf{n}$$

- At the receiver consider the following linear signal processing:



$$\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$$

where

$$\begin{aligned} \mathbf{r}' &= U^H\mathbf{r}, \\ \mathbf{x}' &= V^H\mathbf{x}, \\ \mathbf{n}' &= U^H\mathbf{n} \end{aligned}$$

where we assume that the receiver knows the channels, that is, H, by estimation, perfectly.

$$\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$$

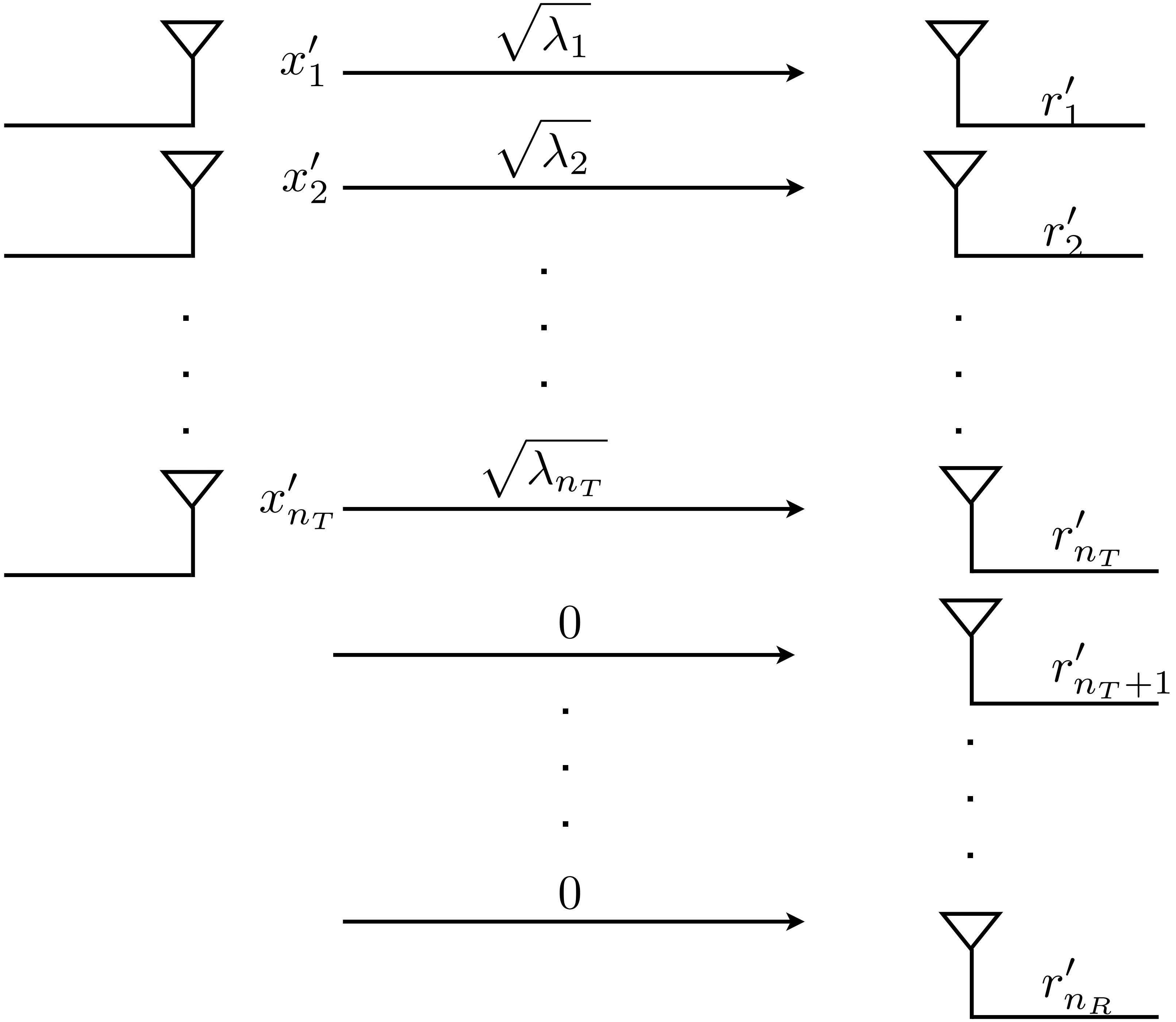
1) When $n_R > n_T$, recall D has the form of

$$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_{n_T}} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

∗ Then we have

$$\begin{aligned} r_1' &= \sqrt{\lambda_1}x_1' + n_1' \\ r_2' &= \sqrt{\lambda_2}x_2' + n_2' \\ &\vdots \\ r_{n_T} &= \sqrt{\lambda_{n_T}}x_{n_T}' + n_{n_T}' \\ r_{n_T+1} &= n_{n_T+1} \\ &\vdots \\ r_{n_R} &= n_{n_R} \end{aligned}$$

- In this case, the MIMO channel can be modeled as n_T parallel channels with the channel coefficient $\sqrt{\lambda_k}$ for $k = 1, 2, ..., n_T$.



$$\mathbf{r}' = D\mathbf{x}' + \mathbf{n}'$$

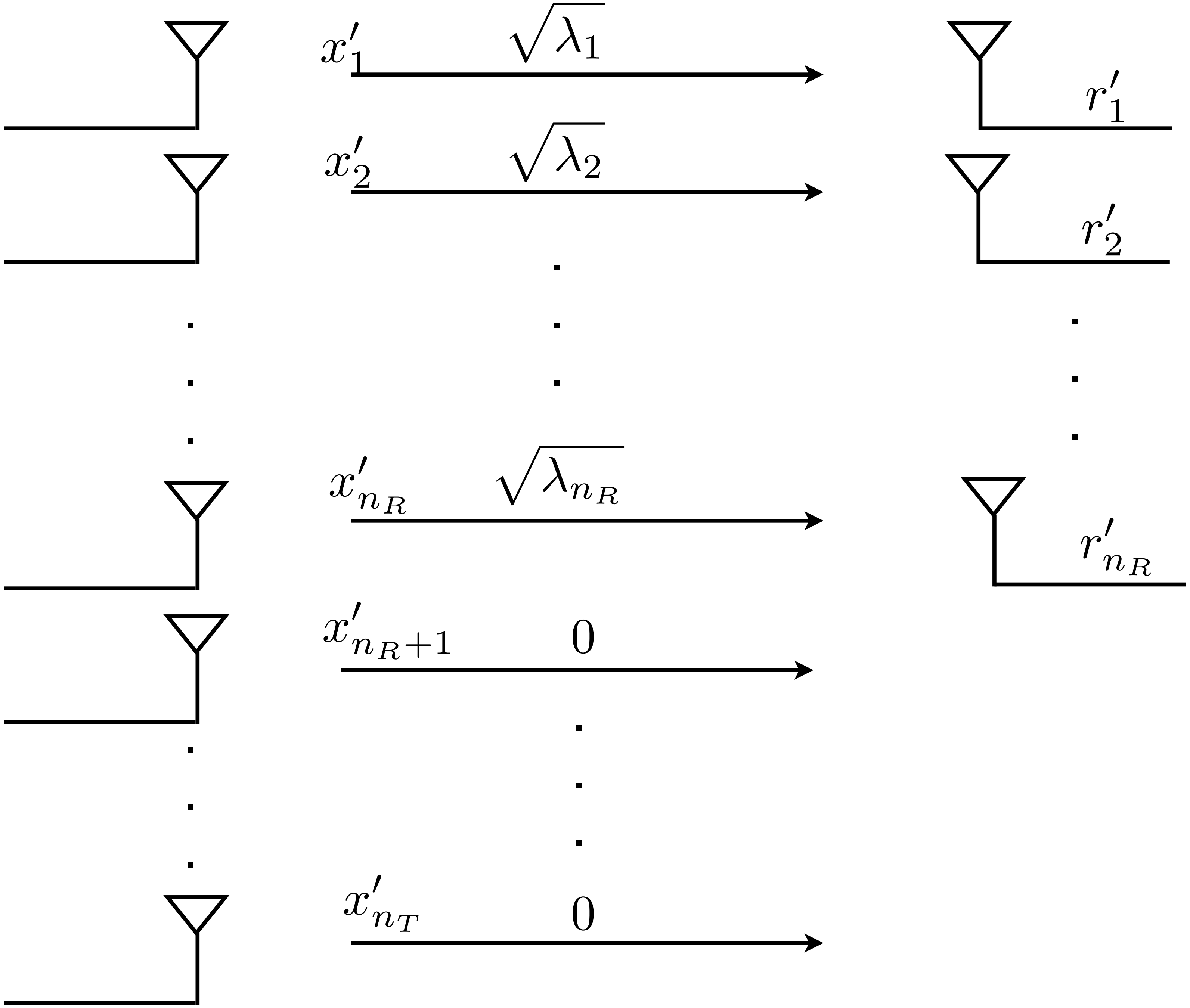
2) When $n_T > n_R$, recall D has the form of

$$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_{n_R}} & 0 & \cdots & 0 \end{bmatrix}_{n_R \times n_T}$$

- $\ast$ Then we have

$$\begin{aligned} r_1' &= \sqrt{\lambda_1}x_1' + n_1' \\ &\vdots \\ r_{n_R}' &= \sqrt{\lambda_{n_R}}x_{n_R}' + n_{n_R}' \end{aligned}$$

- In this case, the MIMO channel can be modeled as n_R parallel channels with the channel coefficient $\sqrt{\lambda_k}$ for $k = 1, 2, ..., n_R$.



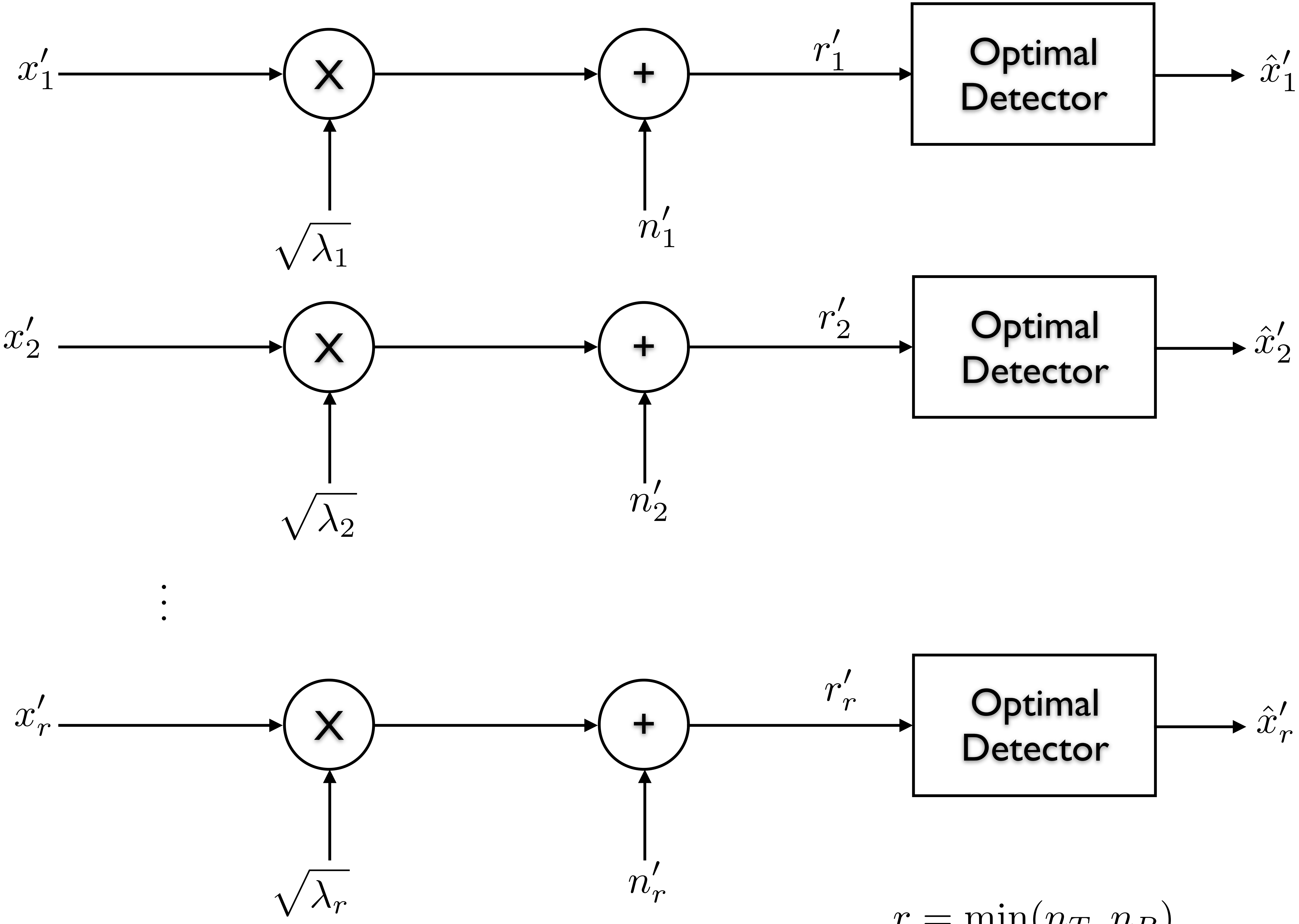
- Rank of a matrix is equal to the number of non-zero eigenvalues.
  - For a matrix of A with the size of $m \times n$, the rank r is given as

$$r = \min(m, n)$$

- Hence, the MIMO channel with n_T and n_R antennas at the transmitter and the receiver, respectively, the rank r is given as

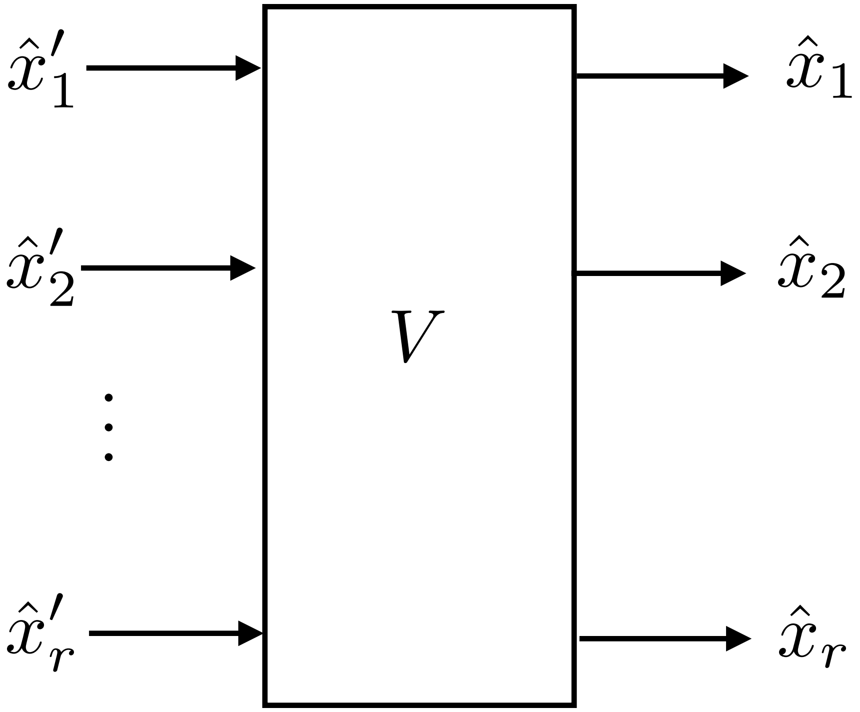
$$r = \min(n_T, n_R)$$

  - So we have r parallel channels.


x'_1
X
+
r'_1
Optimal Detector
$\hat{x}'_1$
$\sqrt{\lambda_1}$
n'_1
x'_2
X
+
r'_2
Optimal Detector
$\hat{x}'_2$
$\sqrt{\lambda_2}$
n'_2
$\vdots$
x'_r
X
+
r'_r
Optimal Detector
$\hat{x}'_r$
$\sqrt{\lambda_r}$
n'_r

$r = \min(n_T, n_R)$

- Recall $\mathbf{x}' = V^H \mathbf{x}$.



  - Also recall $VV^H = I$.

$$\hat{\mathbf{x}} = V\hat{x}' = VV^H\mathbf{x} + V\mathbf{n}' = \mathbf{x} + \mathbf{n}'$$