Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #11

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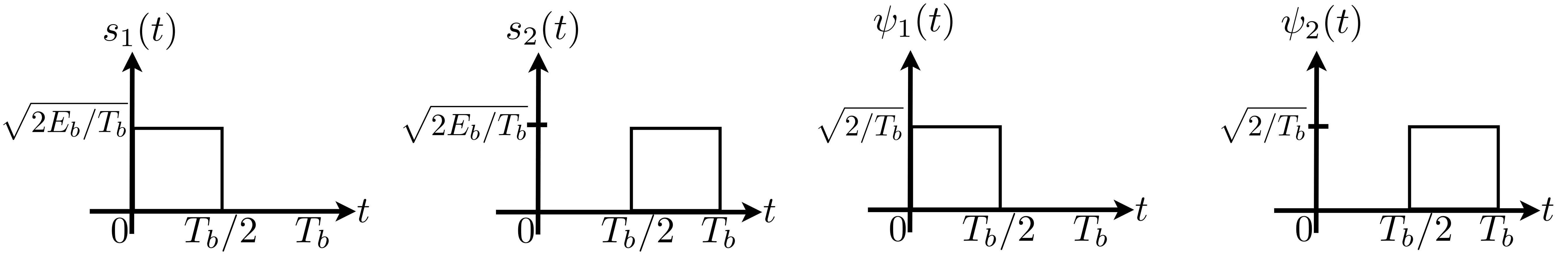
Outline

- Matched filter
- Optimum detection for binary antipodal signals
- M-ary PAM
- M-ary orthogonal signals

Example of binary PPM

- Binary PPM signals

$$s_m(t) = s_{m1}\psi_1(t) + s_{m2}\psi_2(t), \quad j = 1, 2$$



$$s_{11} = \int_0^{T_b} s_1(t)\psi_1(t)\,dt = \sqrt{E_b}$$

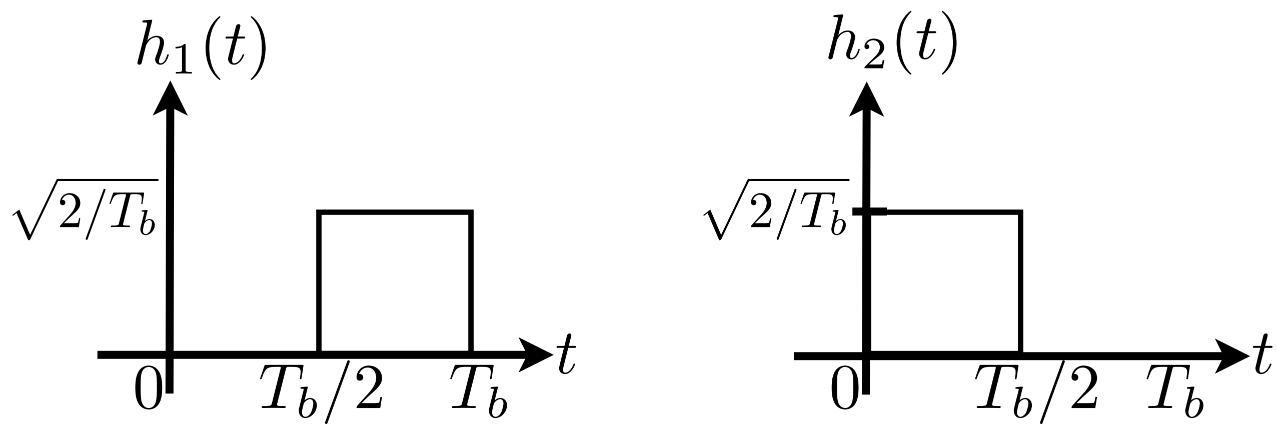
$$s_{12} = \int_0^{T_b} s_1(t)\psi_2(t)\,dt = 0$$

$$s_{21} = \int_0^{T_b} s_2(t)\psi_1(t)\,dt = 0$$

$$s_{22} = \int_0^{T_b} s_2(t)\psi_2(t)\,dt = \sqrt{E_b}$$

- Matched filter

$$h_1(t) = \psi_1(T_b - t), \quad h_2(t) = \psi_2(T_b - t)$$



- If $s_1(t)$ is transmitted, the sampled output signals are

$$\mathbf{y} = [y_1,\ y_2] = [\sqrt{E_b} + n_1,\ n_2]$$

where $n_k = \int_0^{T_b} n(t)\psi_k(t)\, dt$ with $n_k \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$

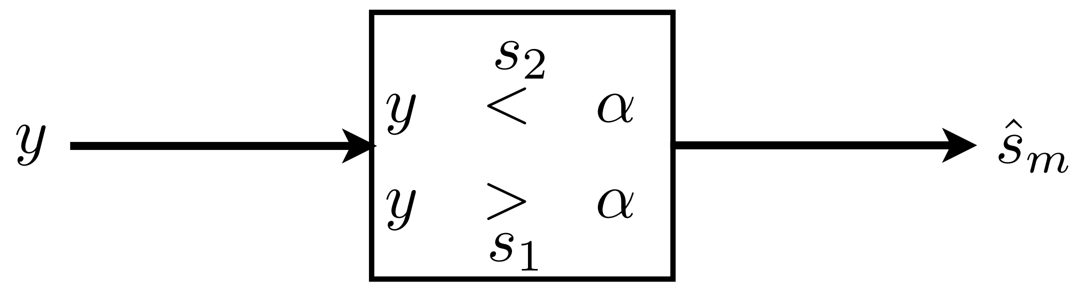
- Output SNR for the first matched filter

$$\left(\frac{S}{N}\right)_o = \frac{(\sqrt{\mathcal{E}_b})^2}{N_0/2} = \frac{2\mathcal{E}_b}{N_0}$$

Performance of the Optimum Receiver: Binary Antipodal Signals

- Output of the demodulator in any signal bit interval

$$y = s_m + n, \quad m = 1, 2$$



- Decision rule
  - If $y > \alpha$, declare $s_1(t)$ was transmitted.
  - If $y < \alpha$, declare $s_2(t)$ was transmitted.

- Average probability of error

$$P_2(\alpha) = P(s_1)\int_{-\infty}^{\alpha} f(y|s_1)\, dy + P(s_2)\int_{\alpha}^{\infty} f(y|s_2)\, dy$$

- Not we want to find the optimum threshold value α, say α^* which minimizes the average probability of error.

- Optimum threshold can by finding the solution of $\left.\frac{dP_2(\alpha)}{d\alpha} = 0\right|_{\alpha=\alpha^*}$

  That is,

$$P(s_1)f(\alpha|s_1) - P(s_2)f(\alpha|s_2) = 0$$

  or equivalently,

$$\frac{f(\alpha|s_1)}{f(\alpha|s_2)} = \frac{P(s_2)}{P(s_1)}$$

  Since $f(\alpha|s_m)$ is Gaussian PDF with mean $\sqrt{\mathcal{E}_b}$ for s_1 and $-\sqrt{\mathcal{E}_b}$ for s_2, we have

$$e^{-(\alpha-\sqrt{\mathcal{E}_b})^2/N_0} e^{-(\alpha+\sqrt{\mathcal{E}_b})^2/N_0} = \frac{P(s_2)}{P(s_1)}$$

- Clearly, the optimum value of the threshold is

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{P(s_2)}{P(s_1)}$$

- For the case of $P(s_1) = P(s_2)$, the optimum threshold is zero. In this case, the average probability of error is

$$\begin{aligned} P_2 &= \frac{1}{2}\int_{-\infty}^{0} f(y|s_1)\,dy + \frac{1}{2}\int_{0}^{\infty} f(y|s_2)\,dy = \int_{-\infty}^{0} f(y|s_1)\,dy \\ &= \frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{0} e^{-(y-\sqrt{\mathcal{E}_b})^2/N_0}\,dy \end{aligned}$$

Change of the variable as $x = (y - \sqrt{\mathcal{E}_b})/\sqrt{N_0/2}$

$$P_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{2\mathcal{E}_b/N_0}} e^{-x^2/2}\,dx = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

Technique of BER/SER Calculation

Assume a certain signal was transmitted, say $s_1(t)$

Calculate the conditional error probability, $P_2(e|s_1)$

Check if all the conditional probabilities are equal, that is, $P_2(e|s_1) = P_2(e|s_2)$, Then the average probability of error is

$$P_2 = P_2(e|s_1)P(s_1) + P_2(e|s_2)P(s_2)$$

For equally probable case, that is, $P(s_1) = P(s_2) = 1/2$

$$P_2 = \frac{1}{2}(P_2(e|s_1) + P_2(e|s_2)) = P_2(e|s_1)$$

Performance of Binary Orthogonal Signals

- Dimensionality of binary orthogonal signals
  - Two-dimensional transmit signals can be written as

$$s_m(t) = s_{m1}\phi_1(t) + s_{m2}\phi_2(t)$$

$$\mathbf{s}_m = [s_{m1} \;\; s_{m2}], \quad m = 1,\, 2$$

  - The output of the demodulator is also two-dimensional

$$\mathbf{y} = [y_1 \;\; y_2]$$

- If $s_1(t)$ is transmitted, the demodulator outputs are

$$\begin{aligned} y_1 &= \mathcal{E}_b + n_1 \\ y_2 &= n_2 \end{aligned}$$

  - where n_1 and n_2 are statistically independent and identically distributed (I.I.D.) Gaussian random variable with zero mean and variance $\sigma_n^2 = N_0/2$.

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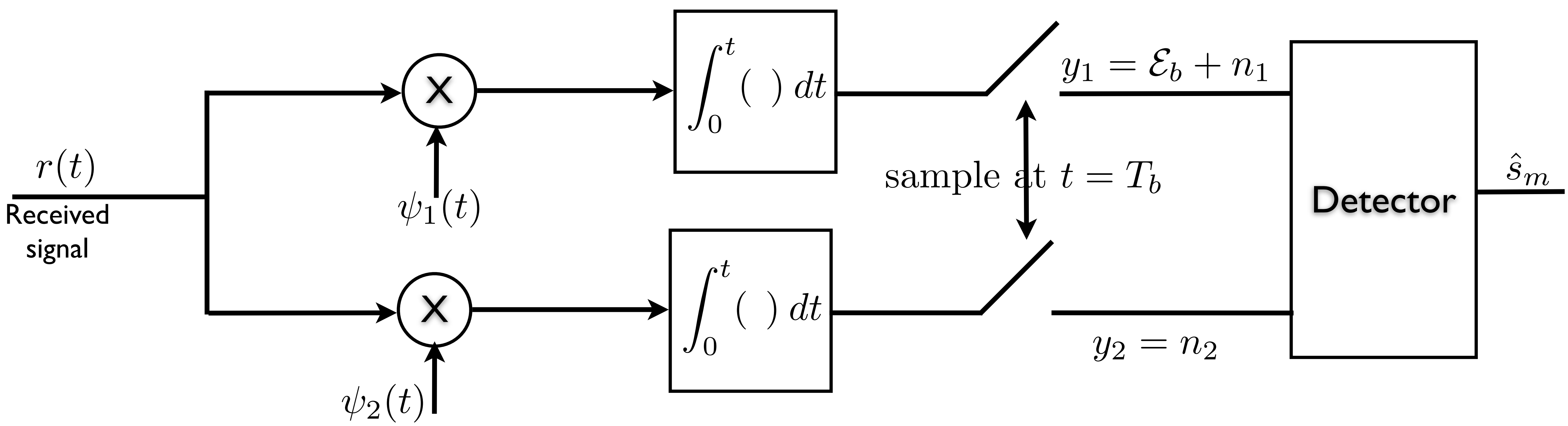
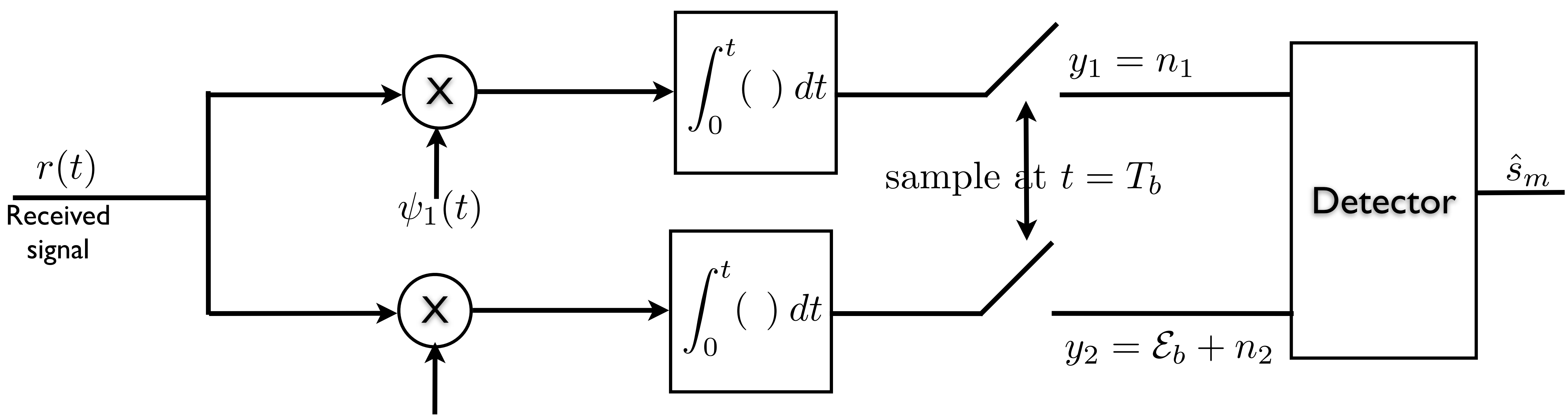
- If $s_1(t)$ is transmitted,



- If $s_2(t)$ is transmitted,



- Decision rule to minimize the average probability of error
  - Compare y_1 with y_2
    - If $y_1 > y_2$ (equivalently $y_1 - y_2 > 0$) , declare $s_1(t)$ was transmitted.
    - Otherwise, declare $s_2(t)$ was transmitted.

- Probability of error
  - Assuming $s_1(t)$ is transmitted, the error occurs when $y_1 - y_2 < 0$.
  - Let

$$z = y_1 - y_2 = \sqrt{\mathcal{E}_b} + n_1 - n_2$$

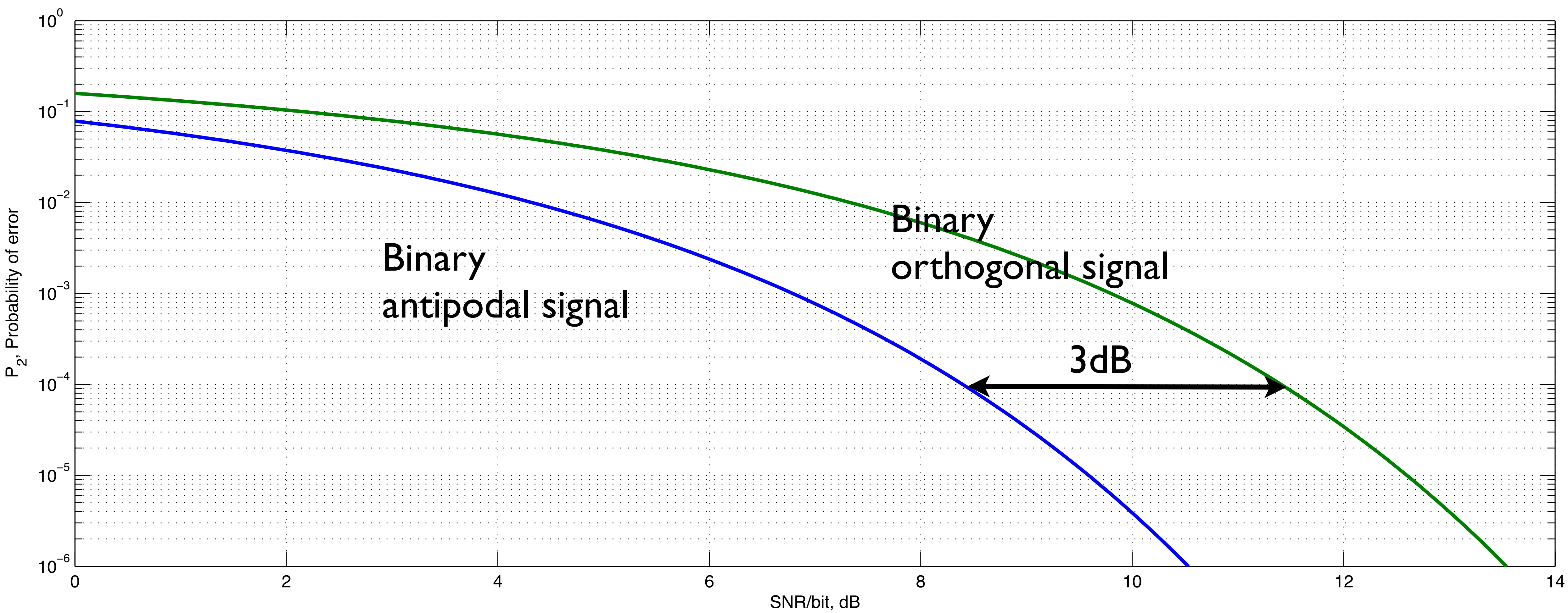
    - Then we can shown

$$z \sim \mathcal{N}(\sqrt{\mathcal{E}_b}, N_0) \qquad \Longrightarrow \qquad f(z) = \frac{1}{\sqrt{2\pi N_0}} e^{-(z-\sqrt{\mathcal{E}_b})^2/2N_0}$$

- Average probability of error

$$P_2 = P(z<0) = \int_{-\infty}^{0} f(z)\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{\mathcal{E}_b/N_0}} e^{-x^2/2}\,dx = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$



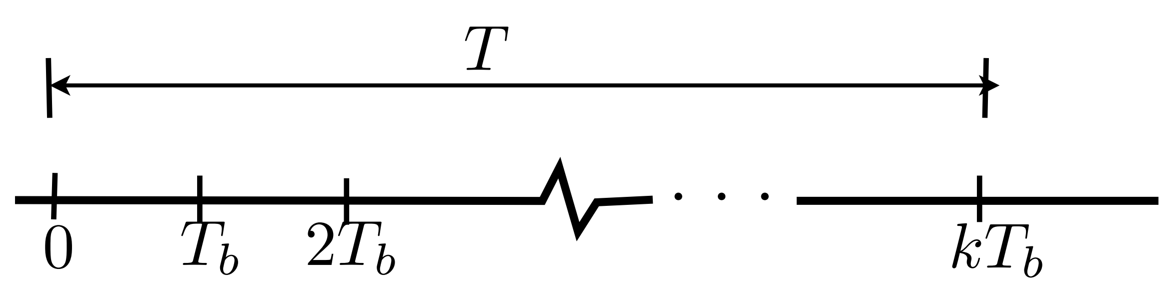
M-ary Pulse Modulation

- M-ary modulation
  - The binary sequence is subdivided into blocks of k bits, called symbols, and each block (or symbol) is represented by one of $M = 2^k$ signal waveforms, each of duration of T.
- Symbol (or signaling) rate
  - The number of signals (or symbols) transmitted per second

$$R_s = \frac{1}{T} \quad \text{symbols/sec}$$

- Bit rate

$$R_b = kR_s = \frac{k}{T} \quad \text{bits/sec}$$



- Bit interval

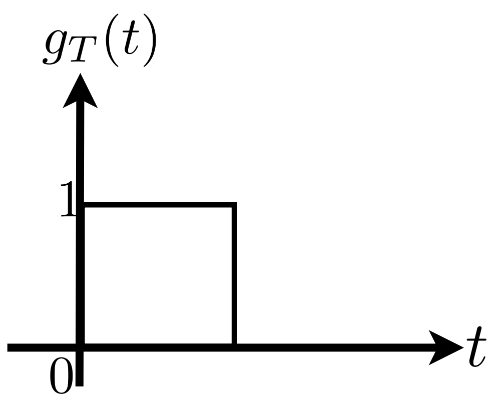
$$T_b = \frac{1}{R_b} = \frac{T}{k}$$

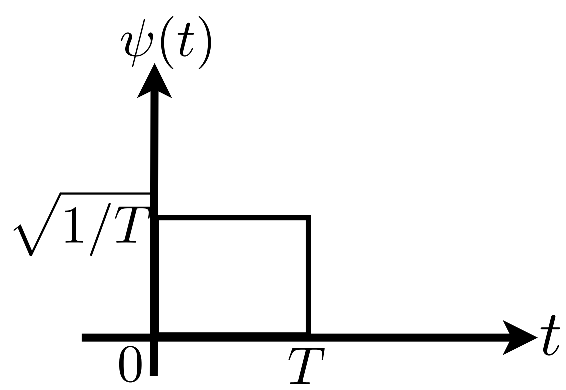
M-ary Pulse Amplitude Modulation (PAM)

- M-ary signal waveforms

$$s_m(t) = A_m g_T(t), \qquad 0 \le t \le T, \quad m = 1, 2, \ldots, M$$

$$= s_m \psi(t) \qquad \text{where } s_m = A_m \sqrt{T}$$





- Energy of each symbol

$$\mathcal{E}_m = \int_0^T s_m^2(t)\, dt = s_m^2 \int_0^T \psi^2(t)\, dt = s_m^2 = A_m^2 T$$

- Average energy

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m = \frac{T}{M} \sum_{m=1}^{M} A_m^2$$

- Signal amplitude

$$A_m = (2m - 1 - M)A, \quad m = 1, 2, \dots, M$$

  - Signal amplitudes are symmetric about the origin and equally spaced by which there is no DC component and the average transmitted energy can be minimized.

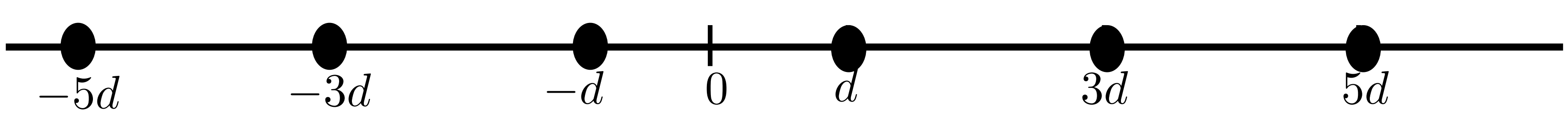
- Average energy

$$\mathcal{E}_{av} = \frac{A^2 T}{M} \sum_{m=1}^{M} (2m - 1 - M)^2 = \frac{A^2 T (M^2 - 1)}{3}$$

- Signal constellation

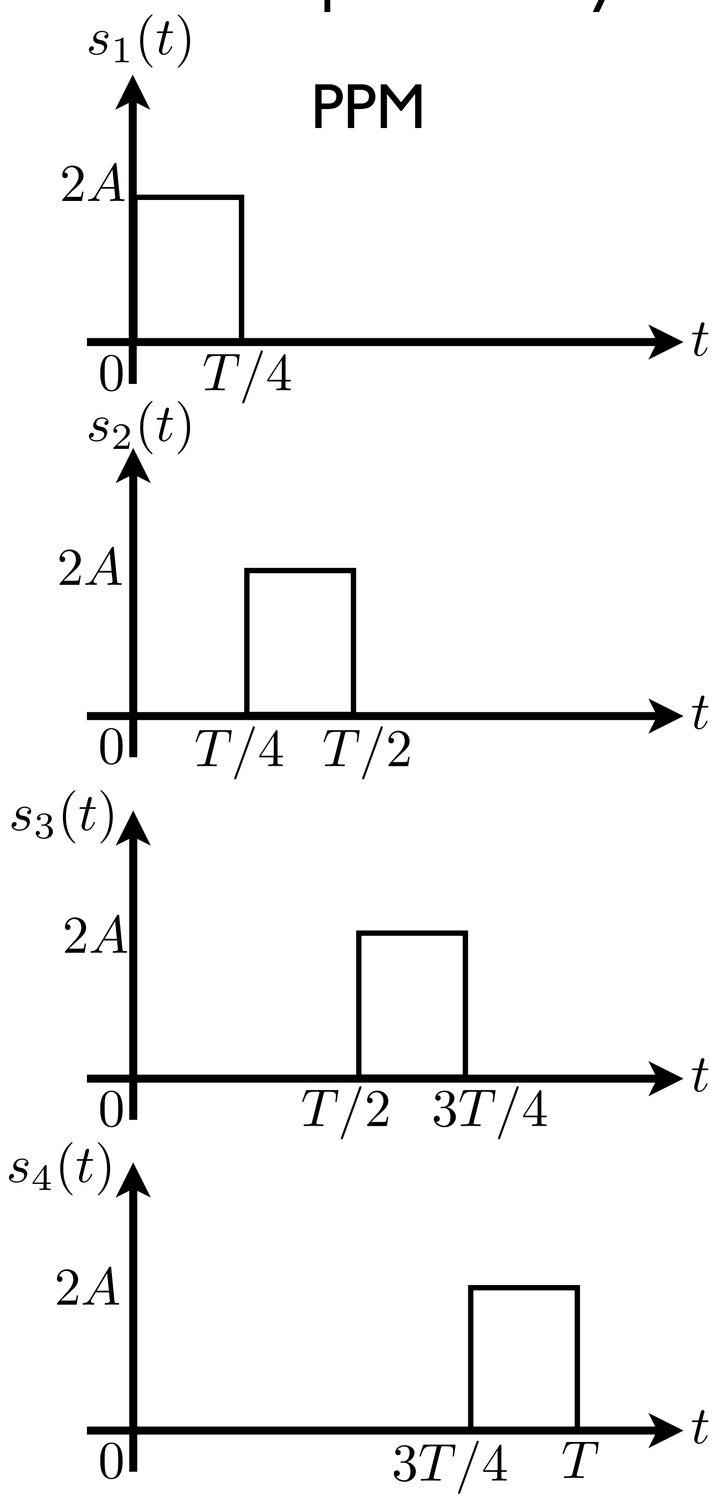
$$\begin{aligned} s_m &= A_m \sqrt{T} = A\sqrt{T}(2m - 1 - M) \\ &= (2m - 1 - M)d, \quad m = 1, 2, \dots, M \end{aligned}$$

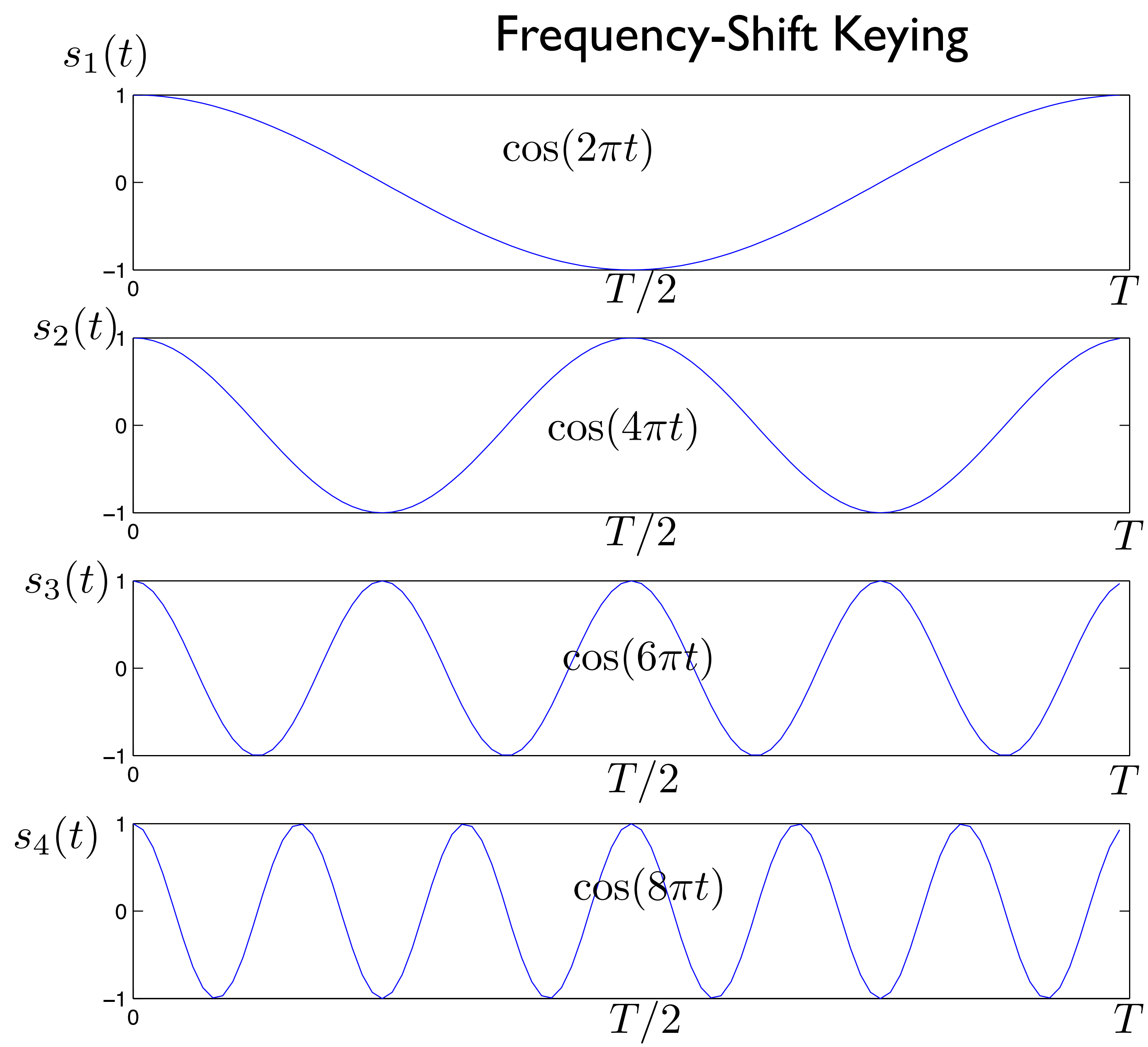
where we define $d = A\sqrt{T}$



M-ary Orthogonal Signals

- Example of 4-ary orthogonal signal waveforms





- Orthogonality condition for $\{s_m(t)\}_{m=1}^{M}$

$$\int_0^T s_i(t)s_j(t)\,dt = 0, \quad i \neq j$$

- Signal waveform expression

$$s_m(t) = \sqrt{\mathcal{E}_s}\psi_m(t), \quad m = 1, 2, \ldots, M$$

  - For PPM,

$$\psi_m(t) = g_T\left(t - \frac{(m-1)T}{M}\right), \quad \frac{(m-1)T}{M} \leq t \leq \frac{mT}{M}$$

  - For frequency shift keying,

$$\psi_m(t) = \sqrt{\frac{2}{T}}\cos(2\pi m t), \quad m = 1, 2, \ldots, M$$

- Dimensionality of M-ary orthogonal signals

  - Dimensionality is M

- Energy

$$\int_0^T s_m^2(t)\,dt = \mathcal{E}_s \int_0^T \psi_m^2(t)\,dt = \mathcal{E}_s, \quad \text{all } m$$

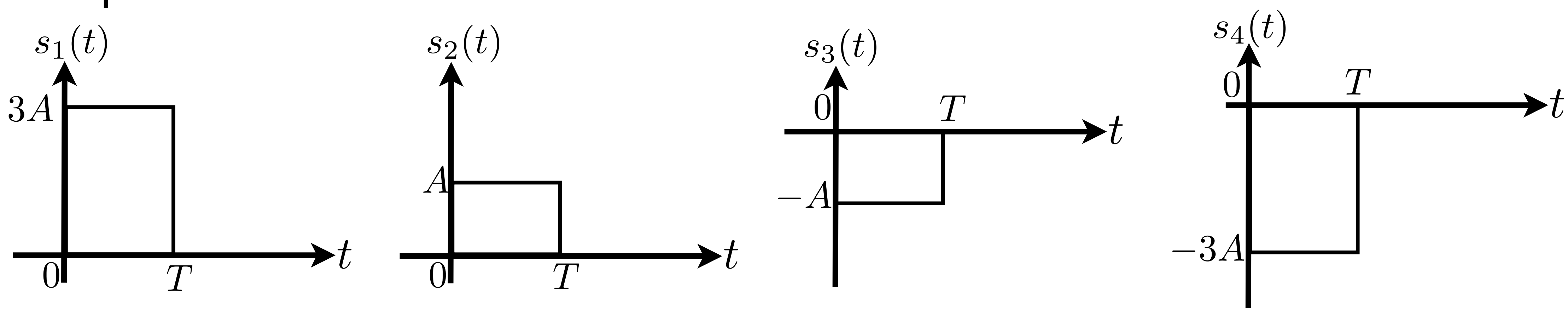
- Geometrical expression

$$\begin{aligned} \mathbf{s}_1 &= (\sqrt{\mathcal{E}_s}, 0, 0, \ldots, 0) \\ \mathbf{s}_2 &= (0, \sqrt{\mathcal{E}_s}, 0, \ldots, 0) \\ &\vdots \\ \mathbf{s}_M &= (0, 0, 0, \ldots, \sqrt{\mathcal{E}_s}) \end{aligned}$$

- Euclidean distance between M signal vectors are mutually equidistant, i.e.,

$$d_{mn} = \sqrt{||\mathbf{s}_m - \mathbf{s}_n||^2} = \sqrt{2\mathcal{E}_s}, \quad \text{for all } m \neq n$$

Example of 4-PAM



- Average energy

$$\mathcal{E}_{av} = 5A^2T = 5d^2 \qquad \text{where } d^2 = A^2T$$