Chapter 1

Basic Concepts

1.1 Definitions

1.1.1 Vector

1. A vector is a quantity which has both direction and magnitude.

$$\boldsymbol{A}. \tag{1.1}$$

1.1.2 Magnitude of a vector

1. The magnitude of a vector is defined by

$$|\boldsymbol{A}| = A. \tag{1.2}$$

1.1.3 The unit vector

1. The unit vector is defined by

$$\hat{a} = \frac{\boldsymbol{A}}{A}. \tag{1.3}$$

1.1.4 Vector addition

1. Vector addition satisfies commuatative law and associative law.

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}. \tag{1.4a}$$

$$\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) + (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}. \tag{1.4b}$$

1.1.5 Additive identity

1. There is the addictive identity, $\mathbf{0}$, which call a null vector.

$$\boldsymbol{A} + \mathbf{0} = \boldsymbol{A} \tag{1.5}$$

, for any $\boldsymbol{A}$.

1.1.6 Addictive inverse

1. There is the addictive inverser $-\boldsymbol{A}$.

$$\boldsymbol{A} + (-\boldsymbol{A}) = 0 \tag{1.6}$$

, for each $\boldsymbol{A}$.

1.1.7 Scalar multiplication

1. If a scalar a is multiplied to a vector $\boldsymbol{A}$, the product also a vector.

$$a \times \boldsymbol{A} = a\boldsymbol{A}. \tag{1.7}$$

2. The scalar multiplication satisfies distributive law and associative law.

$$(a+b)\boldsymbol{A} = a\boldsymbol{A} + b\boldsymbol{A}, \tag{1.8a}$$

$$a(\boldsymbol{A}+\boldsymbol{B}) = a\boldsymbol{A} + a\boldsymbol{B}, \tag{1.8b}$$

$$a(b\boldsymbol{A}) = (ab)\boldsymbol{A} = ab\boldsymbol{A}. \tag{1.8c}$$

1.1.8 Vector substraction

$$\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B}). \tag{1.9}$$

1.1.9 Representation of vector

1. A vector can be expressed as a linear combination of basis vectors. For example, we can express $\boldsymbol{A}$ of the form

$$\boldsymbol{A} = \sum_{n=1}^{3} A_i \hat{\boldsymbol{e}}_i \tag{1.10}$$

, where $\hat{\boldsymbol{e}}_i$ are unit vectors of the three-dimensional orthogonal coordinate.

1.2 Scalar Product

1.2.1 Scalar Product

1. The scalar product of two vectors is defined by

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta \tag{1.11}$$

, where θ is the angle between two vectors. Scalar product is commutative.

2. In the three-dimensional orthogonal coordinate system, the scalar product of two basis vectors is

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij} \tag{1.12}$$

,where the Kronecker delta δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{1.13}$$

Therefore, in the above coordinate system, the scalar product of two vectors is

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{i,j=1}^{3} (A_i \hat{\boldsymbol{e}}_i) \cdot (B_j \hat{\boldsymbol{e}}_j) = \sum_{i,j=1}^{3} A_i B_j \delta_{ij} = \sum_{i,j=1}^{3} A_i B_i = B_i A_i \tag{1.14}$$

3. We have leaned about the law of cosines.

$$C^2 = A^2 + B^2 - 2AB\cos\theta. \tag{1.15}$$

1.2.2 directional cosines

1. The vector $\boldsymbol{A}$ makes anglea with axes.

$$A_x = A\cos\alpha \tag{1.16a}$$

$$A_y = A\cos\beta \tag{1.16b}$$

$$A_z = A\cos\gamma \tag{1.16c}$$

, where $\cos\alpha,\ \cos\beta,\ \cos\gamma$ is called the directional cosines of $\boldsymbol{A}$.

1.3 Vector Product - Cross Product

1.3.1 Vector product

1. The vector product of two vectors is defined by

$$\boldsymbol{A}\times\boldsymbol{B} = \hat{\boldsymbol{n}}AB\sin\theta \tag{1.17}$$

2. In the three-dimensional orthogonal coordinates, the vector product of two basis vector is

$$\hat{\boldsymbol{e}_i}\times\hat{\boldsymbol{e}_j} = \epsilon_{ijk}\hat{\boldsymbol{e}_k}. \tag{1.18}$$

, where ϵ_{ijk} is called the Levi-Civita symbol.

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3),(2,3,1),(3,1,2), \\ -1, & \text{if } (i,j,k) = (3,2,1),(2,1,3),(1,3,2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.19}$$

3. In the three-dimensional orthogonal coordinates, the vector product of two vector is

$$(A\times\boldsymbol{B})_i = \sum_{j,k=1}^{3}(A_j\hat{\boldsymbol{e}_j})\times(B_k\hat{\boldsymbol{e}_k}) = \sum_{j,k=1}^{3}\epsilon_{ijk}\hat{\boldsymbol{e}}_iA_jB_k \tag{1.20}$$

1.3.2 The law of sines

1. If $\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}=\boldsymbol{0}$, $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ satisfy following relations.

$$\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{B}\times\boldsymbol{C} = \boldsymbol{C}\times\boldsymbol{A}, \tag{1.21a}$$

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}. \tag{1.21b}$$

1.4 Triple Products

1.4.1 Triple scalar product

1. The triple scalar product of three vectors is defined by

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}. \tag{1.22}$$

2. In three-dimensional orthogonal coordinates system, the triple scalar product of three vectors becomes

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C} = \sum_{i,j,k=1}^{3}\epsilon_{ijk}A_iB_jC_k. \tag{1.23}$$

1.4.2 Triple vector product

1. The triple vector product of three vectors is defined by

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \tag{1.24}$$

   The triple vector product is same as

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}). \tag{1.25}$$

   We call this rule BAC-CAB rule.

1.5 Rotational Properties of a Vector

1.5.1 Position Vector

1. The position vector is defined by

$$\boldsymbol{x} = \sum_{i=1}^{3} x_i \hat{\boldsymbol{e}}_i. \tag{1.26}$$

2. Let $\boldsymbol{x}'$ is a vector which has been transformed form $\boldsymbol{x}$ by rotation. Under the rotation, the magnitude of $\boldsymbol{x}$ is cannot changed.

$$x_i'^2 = x_i^2. \tag{1.27}$$

1.5.2 Rotation Transfromation Coefficient

1. The rotation transformation coefficient R_{ij} satisfy

$$x_i' = R_{ij} x_j. \tag{1.28}$$

2. Then, we can verify eq. (1.28).

$$\begin{aligned} x_i^2 &= x_i'^2 &(1.29)\\ &= (R_{ij}x_j)(R_{ik}x_k) &(1.30)\\ &= (R_{ij}R_{ik})(x_i x_k) &(1.31)\\ &= (R_{ij}R_{ik})x_j x_k. &(1.32)\end{aligned}$$

   Therefore,

$$RijR_{ik} = \delta_{jk}. \tag{1.33}$$

1.5.3 Definition of vector

1. If a quantity A transforms like as

$$A_i' = \sum_j R_{ij} A_j, \tag{1.34}$$

   we call $\boldsymbol{A}$ a vector.