Communication Signals

(Haykin Sec. 2.4 and Ziemer Sec.2.1.4-Sec. 2.4)

KECE321 Communication Systems I

Lecture #3, March 12, 2011

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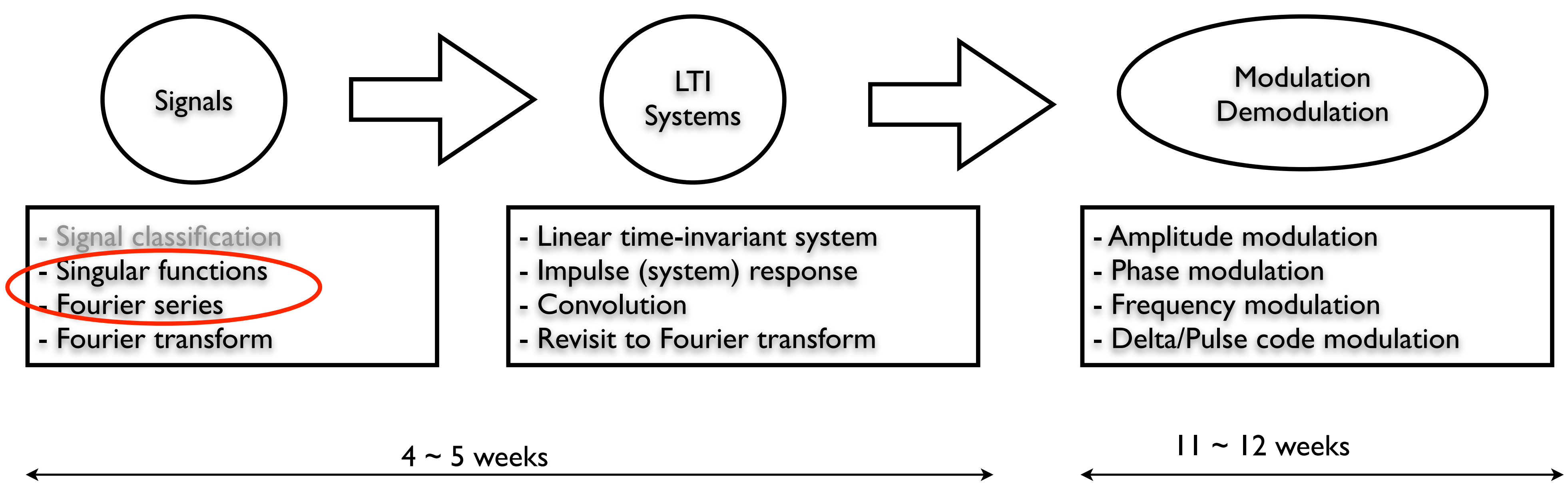
Review

- Signal classification
- Phasor signal and spectra
  - Representation of sinusoidal function in terms of phasor signals
  - Amplitude and phase spectra
    - which gives the dual time-frequency nature of sinusoidal signals

Summary of Today's Lecture

- Singular functions
  - Unit step function
  - Unit impulse function (Dirac delta function)
  - Signum function
- Fourier series
  - Generalized Fourier series
  - Complex Fourier series



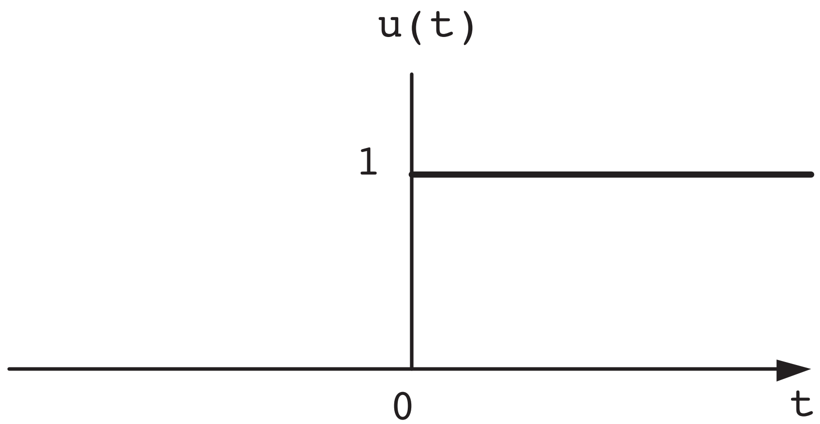
Unit Step Function

- **Definition**

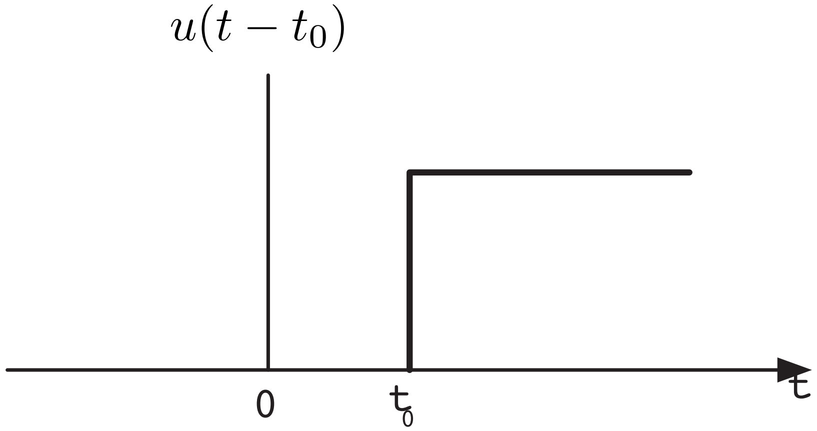
$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

- **Shifted unit step function**

$$u(t - t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$$



(a)

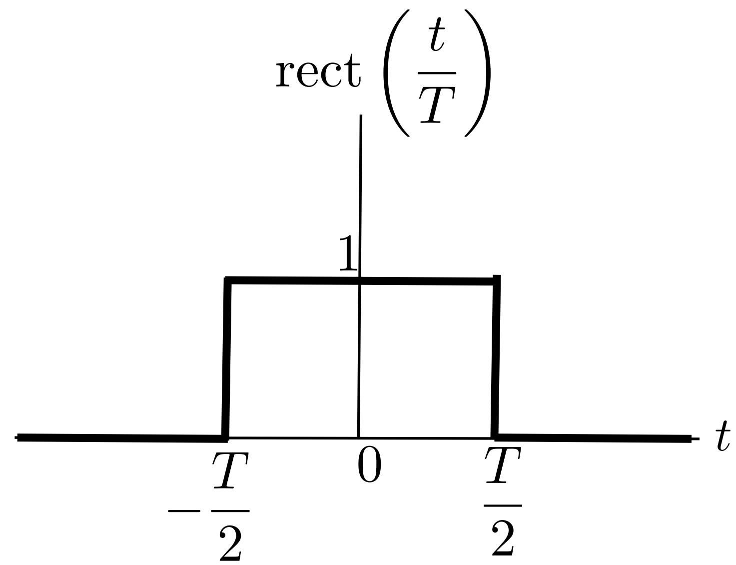


(b)

Unit Impulse Function (Dirac Delta Function)

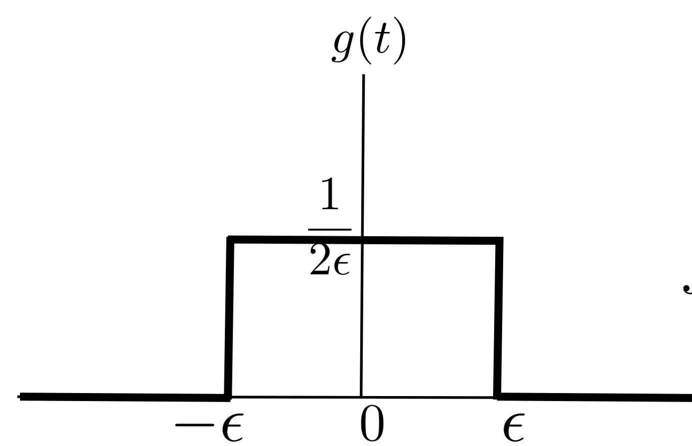
- Rectangular pulse

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{elsewhere} \end{cases}$$



- Consider the rectangular pulse given as

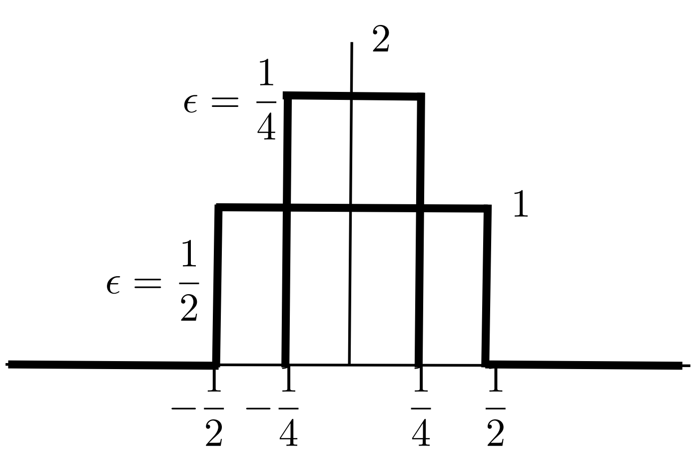
$$g(t) = \frac{1}{2\epsilon}\mathrm{rect}\left(\frac{t}{2\epsilon}\right)$$



Area=1

$$\int_{-\infty}^{\infty} g(t)\, dt = 1$$

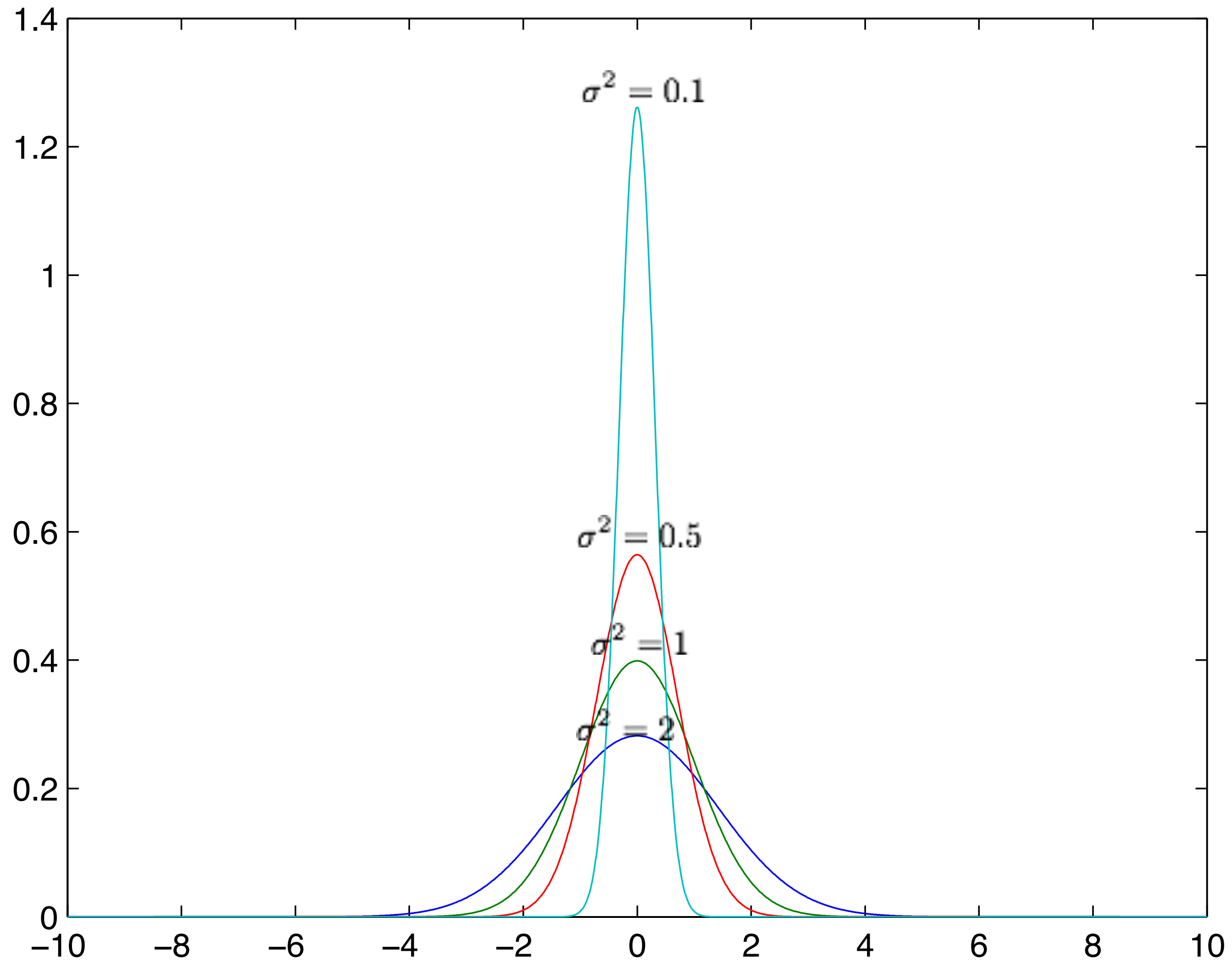
- Now consider $\lim_{\epsilon \to 0} g(t)$ in which case the area is still 1.



- Also consider the Gaussian pulse given as

$$g(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

  - We can prove that $g(t)$ has a unit area, that is, $\int_{t=-\infty}^{\infty} g(t)\, dt = 1$
  - Now if we take $\sigma^2 \rightarrow 0$, $g(t)$ is in narrower gaussian pulse shape



- We define *Dirac delta function* as a function which has the property of $\lim_{\epsilon \to 0} g(t)$ (or $\lim_{\sigma^2 \to 0} g(t)$ in the Gaussian pulse) and denote it as $\delta(t)$.

- Definition of Dirac delta (or unit impulse) function

$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0)$$

or

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$$

where $x(t)$ is any continuous function at time $t = 0$

where $x(t)$ is any continuous function at time $t = t_0$

- By considering the special case $x(t) = 1$ and $x(t) = 0$ for $t < t_1$ and $t > t_2$, the following two properties are obtained:

$$\int_{t_1}^{t_2} \delta(t - t_0)\, dt = 1, \qquad t_1 < t < t_2$$

and

$$\delta(t - t_0) = 0, \qquad t \neq t_0$$

- Some properties of the delta function

1. $\delta(at) = \frac{1}{|a|}\delta(t)$

2. $\delta(-t) = \delta(t)$

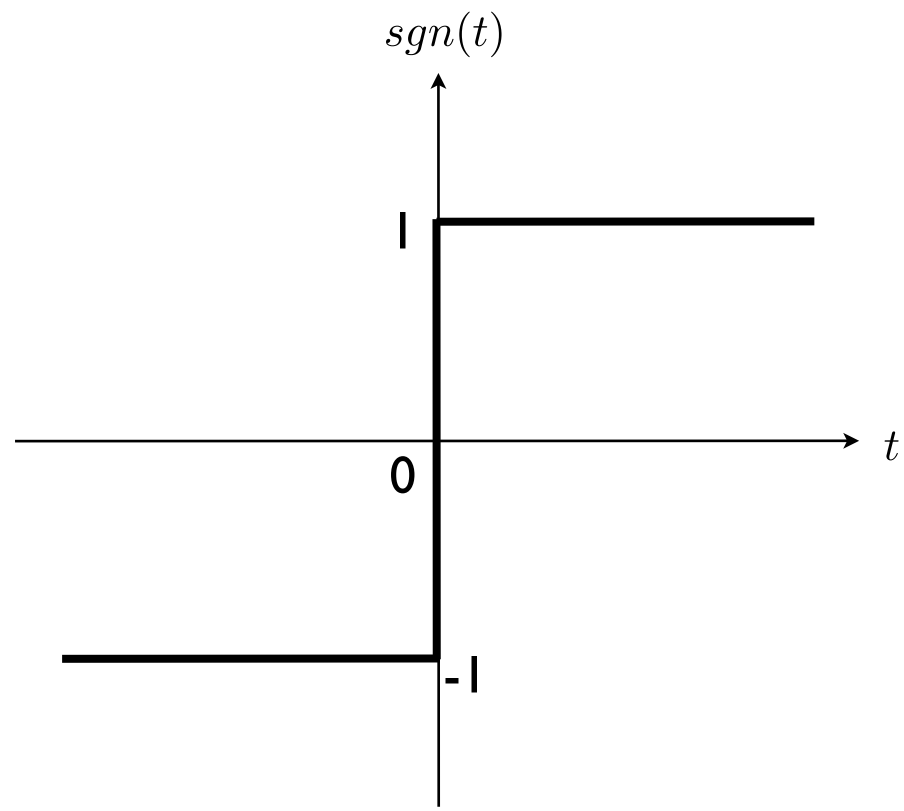
3.
$$\int_{t_1}^{t_2} x(t)\delta(t-t_0)\,dt = \begin{cases} x(t_0), & t_1 < t < t_0 \\ 0, & \text{otherwise} \\ \text{undefined} & \text{for } t_0 = t_1 \text{ or } t_2 \end{cases}$$

4. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$, $x(t)$ continuous at $t = t_0$

Signum (or Sign) Function

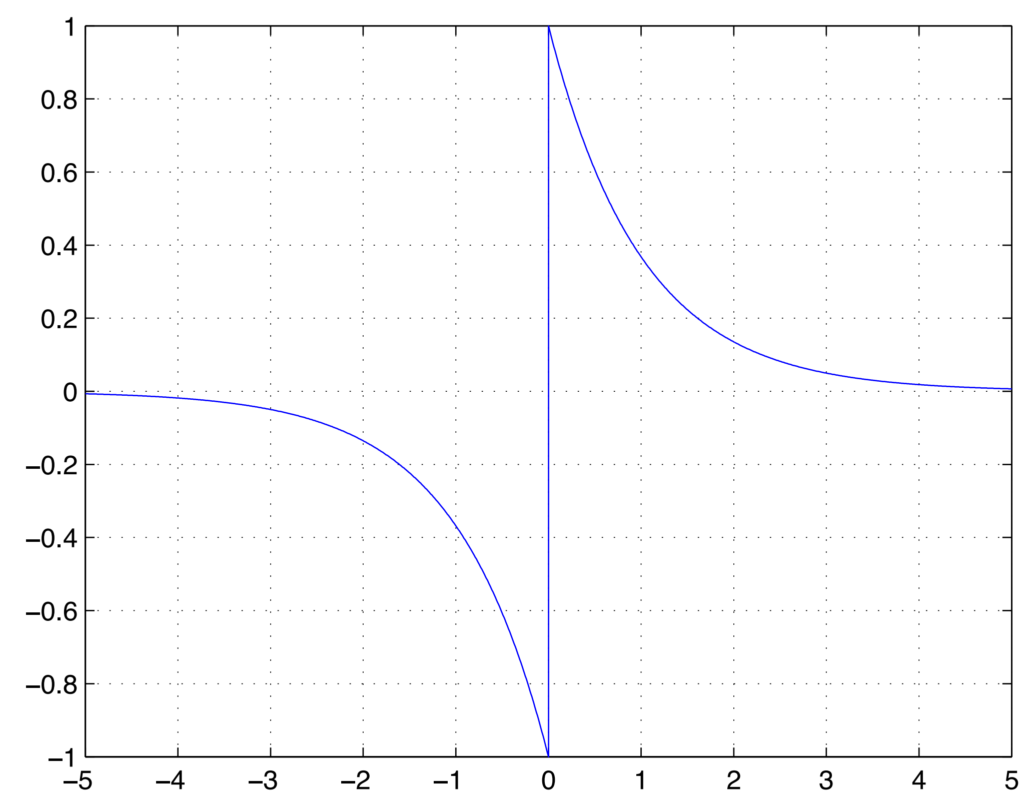
- Definition

$$sgn(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



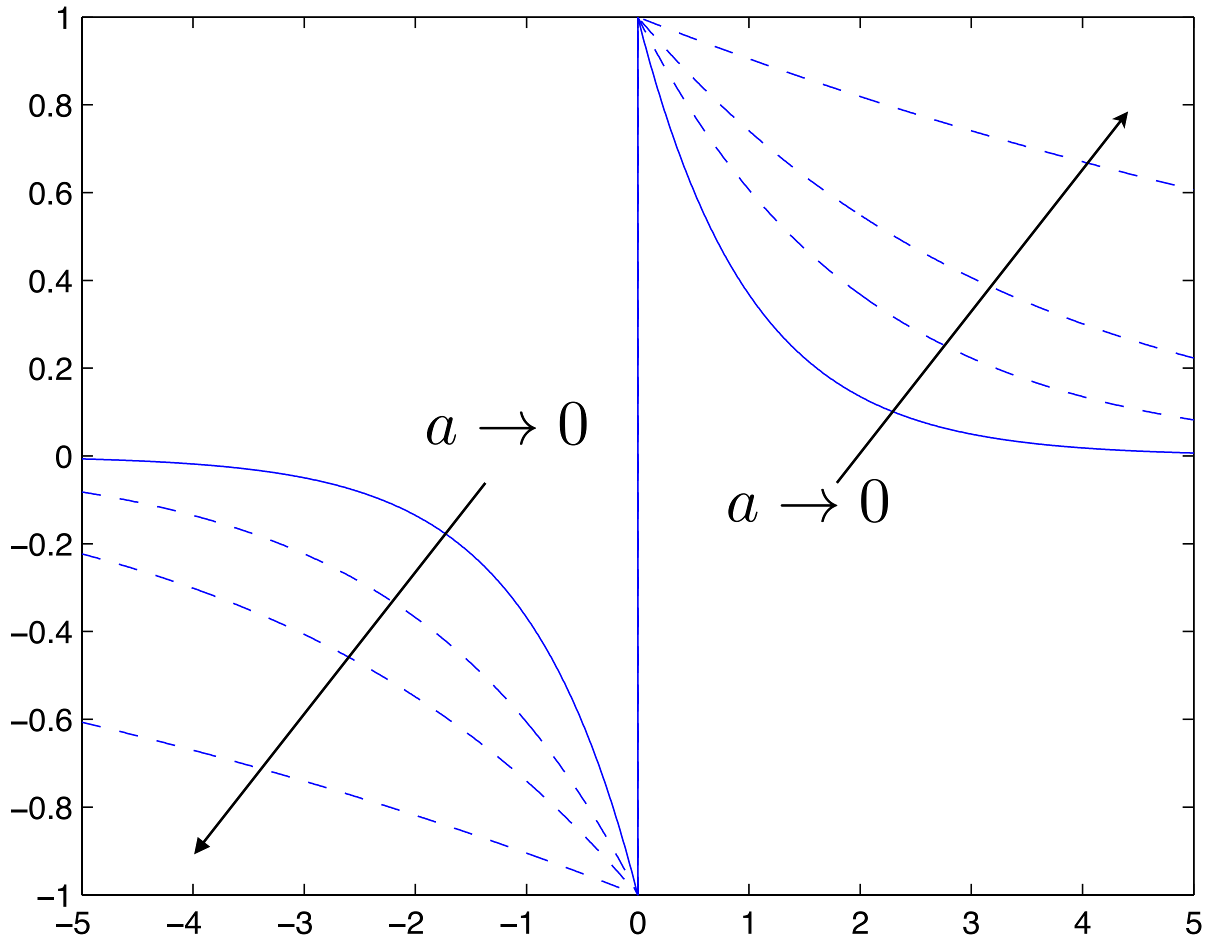
- Odd-symmetric double exponential pulse

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$



- We can derive the signum function from the odd-symmetric double-exponential function such as

$$\lim_{a \to 0} g(t) = sgn(t)$$



J. Fourier

- Joseph Fourier
  - was born in Auxerre, France on March 21, 1768 and died in Paris on May 4, 1830.
  - was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems to heat transfer and vibration.
  - The Fourier transform and Fourier's Law are also named in his honor.
  - Fourier is also generally credited with the discovery of the greenhouse effect.
  - Detailed biography can be found at http://en.wikipedia.org/wiki/Joseph_Fourier.

Fourier’s Insight

- Fourier’s insight was that (under certain circumstances), one can write a series expansion for a 2π- *periodic function* in terms of sines and cosines.
- Then it was proved that any periodic signal can be converged to the sum of orthogonal sines and cosines (or exponential) functions.

Generalized Fourier Series

- Generalized Fourier series:
  - representation of signals as a series of orthogonal functions
- Recall the vector space:
  - Given any vector $\mathbf{A}$ in three-dimensional space can be expressed in terms of three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ that do not all lie in the sample plane

$$\mathbf{A} = A_1\mathbf{x} + A_2\mathbf{y} + A_3\mathbf{z}$$

    - where A_1, A_2, and A_3 are appropriately chosen constants.
    - The vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are said to be *linearly independent* since no one of them can be expressed as a linear combination of the other two. For example, it is impossible to write $\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$, no matter what choice is made for the constants α and β
  - Such a set of linearly independent vectors is said to form a *basis set* for a three-dimensional vector space. Such vectors *span* a three-dimensional vector space in the sense that any vector $\mathbf{A}$ can be expressed as a linear combination of them.

- Similarly, consider the problem of representing a time function, or signal, $x(t)$ on a T-second interval $(t_0, t_0 + T)$, as a similar expansion.

  - We consider a set of time functions $\phi_1(t),\ \phi_2(t), \cdots, \phi_N(t)$, which are specified independently of $x(t)$, and seek a series expansion of the form

$$x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t), \qquad t_0 \le t \le t_0 + T$$

independent of time

the N coefficients X_n are independent of time and the subscript a indicates that $x_a(t)$ is considered an approximation.

  - We assume that the $\phi_n(t)'\mathrm{s}$ are linearly independent; that is, no one of them can be expressed as a weighted sum of the other $N-1$. A set of linearly independent $\phi_n(t)'\mathrm{s}$ will be called a *basis function set*.

- We now wish to examine the error in the approximation of $x(t)$ by $x_a(t)$. As in the case of ordinary vectors, the expansion $x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t)$ is easiest to use if the $\phi_n(t)$'s are orthogonal on the interval $(t_0, t_0 + T)$.

  - That is,

$$\int_{t_0}^{t_0+T} \phi_m(t)\phi_n^*(t)\,dt = c_n \delta_{nm} \triangleq \begin{cases} c_n, & n = m \\ 0, & n \neq m \end{cases} \quad \text{(all } m \text{ and } n\text{)}$$

where, if $c_n = 1$ for all n, the $\phi_n(t)$'s are said to be normalized.

    - A normalized orthogonal wet of functions is called on *orthogonal basis set*.

      - δ_{mn} is called the Kronecker delta function, is defined as unity if $m = n$, and zero otherwise.

- The error in the approximation will be measured in the *integral-square sense* (ISE)

$$\boxed{\text{Error} = \epsilon_N = \int_T |x(t) - x_a(t)|^2\,dt}$$

where $\int_T (\ )\,dt$ denotes the integration over t from t_0 to $t_0 + T$.

- The ISE is an applicable measure of error only when $x(t)$ is an energy signal or a power signal. If $x(t)$ is an energy signal of infinite duration, the limit as $T \to \infty$ is taken.

- We now find the set of coefficients X_n that minimizes the ISE. Substituting $x_a(t)$ into ISE, expressing the magnitude square of the integrand as the integrand times its complex conjugate and expanding, we obtain

$$\epsilon_N = \int_T |x(t)|^2\,dt - \sum_{n=1}^{N}\left[X_n^* \int_T x(t)\phi_n^*(t)\,dt + X_n \int_T x^*(t)\phi_n(t)\,dt\right] + \sum_{n=1}^{N} c_n |X_n|^2$$

  - To find the X_n's that minimizes ϵ_N we add and subtract the quantity

$$\sum_{n=1}^{N} \frac{1}{c_n}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2$$

which yields

$$\epsilon_N = \int_T |x(t)|^2\,dt - \sum_{n=1}^{N} \frac{1}{c_n}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2 + \sum_{n=1}^{N} c_n \left|X_n - \frac{1}{c_n}\int_T x(t)\phi_n^*(t)\,dt\right|^2$$

independent of X_n's

- The first two terms on the right-hand side of ϵ_N are independent of the coefficients X_n. Since the last sum on the right-hand side is nonnegative, we will minimize ϵ_N if we choose each X_n such that the corresponding term in the sum is zero. Thus, since $c_n > 0,$ the choice of

$$X_n = \frac{1}{c_n} \int_T x(t)\phi_n^*(t)\, dt$$

for X_n minimizes the ISE.

- The resulting minimum-error coefficients will be referred to as the *Fourier coefficients*.

- Minimum value for ϵ_n

$$\begin{aligned}(\epsilon_n)_{\min} &= \int_T |x(t)|^2\, dt - \sum_{n=1}^{N} \frac{1}{c_n} \left| \int_T x(t)\phi_n^*(t)\, dt \right|^2 \\ &= \int_T |x(t)|^2\, dt - \sum_{n=1}^{N} c_n |X_n|^2\end{aligned}$$

- If we can find an infinite set of orthonormal functions such that $\lim_{N\to\infty} (\epsilon_N)_{\min} = 0$ for any signal that is integrable square,

$$\int_T |x(t)|^2 \, dt < \infty$$

we say that the $\phi_n(t)$'s are complete. In the sense that the ISE is zero, we may then write

$$x(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \quad \text{(ISE=0)}$$

Assuming a complete orthogonal set of functions, we obtain the relation

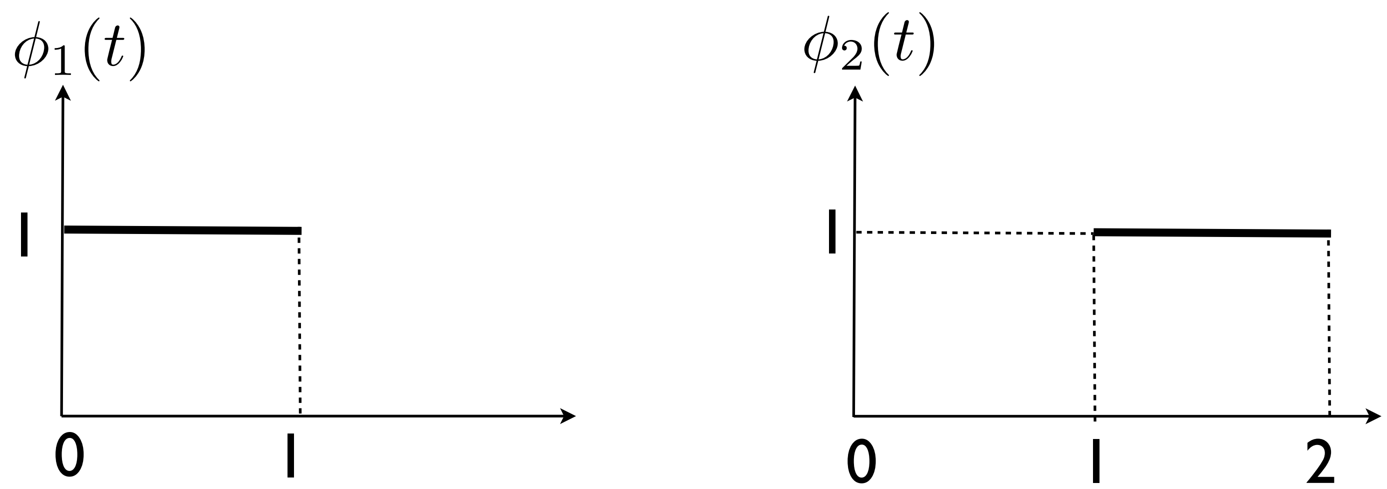
$$\int_T |x(t)|^2 \, dt = \sum_{n=1}^{N} c_n |X_n|^2$$

This equation is known as *Parseval's theorem*.

In general, equation $\lim_{N\to\infty} (\epsilon_N)_{\min} = 0$ requires that $x(t)$ be equal to $x_a(t)$ as $N \to \infty$.

- *Example,* The signal $x(t)$ is to be approximated by a two-term generalized Fourier series

$$x(t) = \begin{cases} \sin(\pi t), & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$



- The Fourier coefficients are calculated as

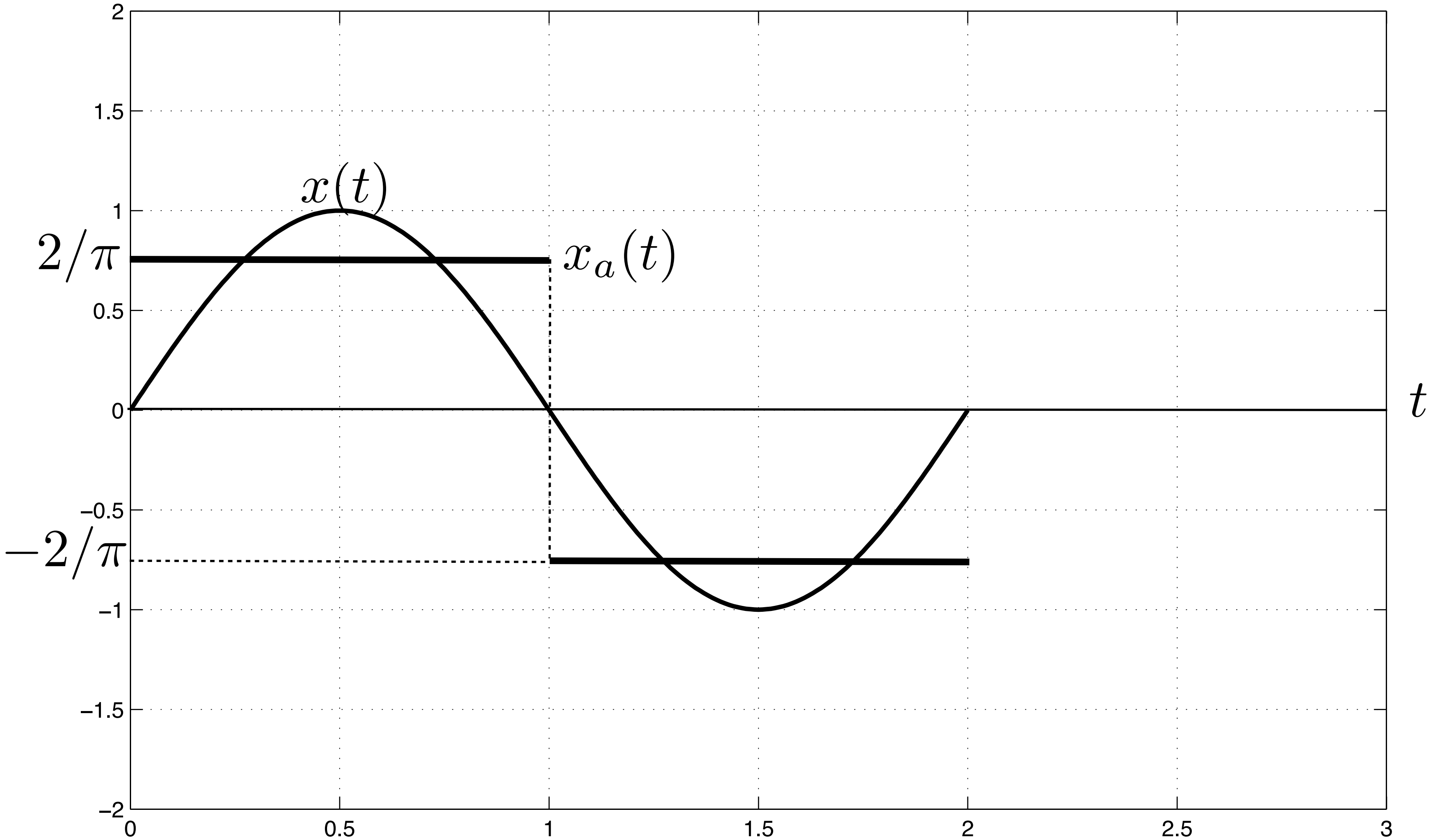
$$X_1 = \int_0^2 \phi_1(t)\sin(\pi t)\,dt = \int_0^1 \sin(\pi t)\,dt = \frac{2}{\pi}$$

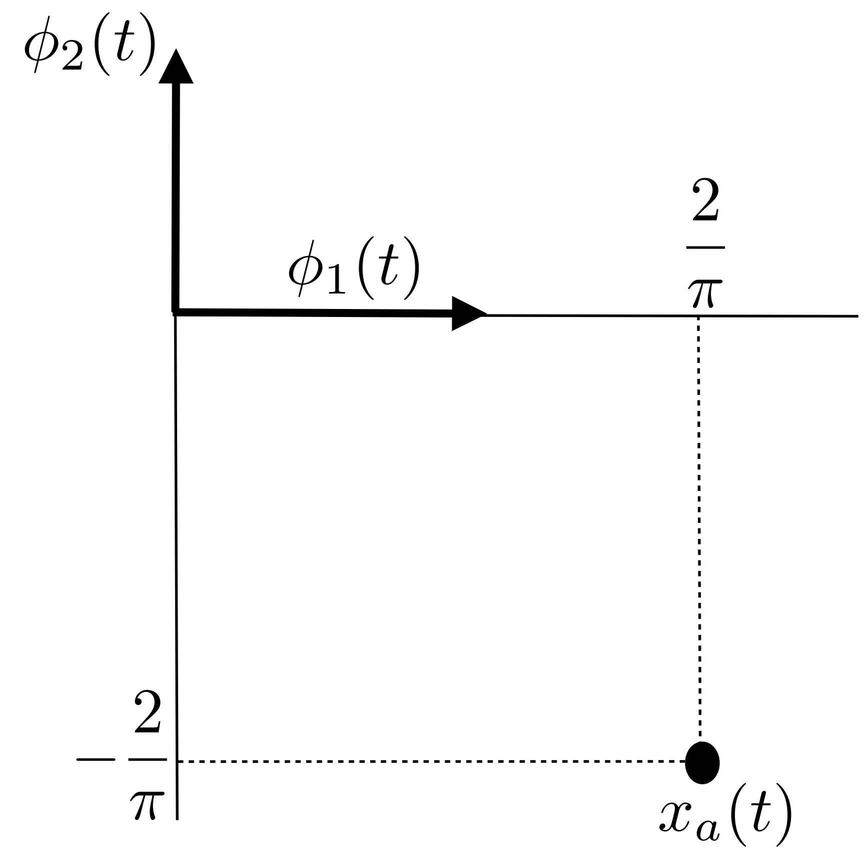
$$X_2 = \int_0^2 \phi_2(t)\sin(\pi t)\,dt = \int_1^2 \sin(\pi t)\,dt = -\frac{2}{\pi}$$

- Thus the generalized two-term Fourier series approximation for this signal is

$$x_a(t) = \frac{2}{\pi}\phi_1(t) - \frac{2}{\pi}\phi_2(t) = \frac{\pi}{2}\left[\text{rect}\left(t-\frac{1}{2}\right) - \text{rect}\left(t-\frac{3}{2}\right)\right]$$

Space interpretation





Minimum ISE

$$(\epsilon_N)_{\min} = \int_0^2 \sin^2(\pi t)\,dt - 2\left(\frac{2}{\pi}\right)^2 = 1 - \frac{8}{\pi^2} \approx 0.189$$

Complex Exponential Fourier Series

- Consider a signal $x(t)$ defined over the interval $(t_0, t_0 + T)$ with the definition

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

we define the complex exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \qquad t_0 \leq t \leq t + 0 + T_0$$

where

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} \, dt$$

- It can be shown to represent the signal $x(t)$ exactly in the interval $(t_0, t_0 + T_0)$, except at a point of jump discontinuity where it converges to the arithmetic mean of the left-hand and right-hand limits.

- Outside the interval $(t_0, t_0 + T_0)$, nothing is guaranteed.

- However, we note that the right-hand side of the complex exponential Fourier series is period with period T_0, since it is the sum of periodic rotating phasors with harmonic frequencies.

> If $x(t)$ is periodic with period T_0, the Fourier series is an accurate representation for $x(t)$ for all t (except at points of discontinuity).

- A useful observation about a complete orthonormal-series expansion of a signal is that the series is unique.

  - For example, if we somehow find a Fourier expansion for a signal $x(t)$, we know that no other Fourier expansion for that $x(t)$ exists, since $\{e^{jn\omega_0 t}\}$ forms a complete set.

Example

- Consider the signal $x(t) = \cos(\omega_0 t) + \sin^2(2\omega_0 t)$ where $\omega_0 = 2\pi/T_0$. Find the complex exponential Fourier series.

- Solution: Using trigonometric identities and Euler's theorem, we obtain

$$\begin{aligned} x(t) &= \cos(\omega_0 t) + \frac{1}{2} - \frac{1}{2}\cos(4\omega_0 t) \\ &= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2} - \frac{1}{4}e^{j\omega_0 t} - \frac{1}{4}e^{-j\omega_0 t} \end{aligned}$$

- Hence,

$$\begin{aligned} X_0 &= \frac{1}{2} \\ X_1 &= \frac{1}{2} = X_{-1} \\ X_4 &= \frac{1}{4} = X_{-4} \end{aligned}$$