$\boxed{1}$ If $|n\rangle$ is the n^{th} harmonic oscillator eigenstate, evaluate :

- <u>knowns</u>

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 .$$

$$a^+ = \frac{p}{\sqrt{2m\hbar\omega}} + i\sqrt{\frac{m\omega}{2\hbar}}\, x .$$

$$a = \frac{p}{\sqrt{2m\hbar\omega}} - i\sqrt{\frac{m\omega}{2\hbar}}\, x$$

$$a^+a = \frac{1}{\hbar\omega} H - \frac{1}{2} \rightarrow H = \hbar\omega\left(a^+a + \frac{1}{2}\right).$$

$$\begin{cases} a|n\rangle = \sqrt{n}\cdot|n-1\rangle \\ a^+|n\rangle = \sqrt{n+1}\;|n+1\rangle . \end{cases}$$

$$a^+ + a = 2\cdot\frac{p}{\sqrt{2m\hbar\omega}} = \sqrt{\frac{2}{m\hbar\omega}}\; p$$

$$\therefore p = \sqrt{\frac{m\hbar\omega}{2}}\,(a^+ + a)$$

$$a^+ - a = 2i\sqrt{\frac{m\omega}{2\hbar}}\, x = i\sqrt{\frac{2m\omega}{\hbar}}\, x$$

$$\therefore \quad x = -i\sqrt{\frac{\hbar}{2m\omega}}\,(a^+ - a)$$

(a) $\langle n | a^{\dagger s} | n \rangle$, $\langle n | a^s | n \rangle$

$$\langle n | a^{\dagger s} | n \rangle = \langle n | a^s | n \rangle = \delta_{s0} \cdot 1.$$

$\langle m | n \rangle = \delta_{mn}$

($s \neq 0$ 이면.. $|n\rangle$ 이.. 달라지므로 orthogonality 에 의해 0)

(b) $$\langle n | x | n \rangle = -i \sqrt{\frac{\hbar}{2m\omega}} \, \cancelto{0}{\langle n | a^\dagger - a | n \rangle}$$

(from (a))

$$= 0.$$

$$\langle n | x^2 | n \rangle = \left(-i \sqrt{\frac{\hbar}{2m\omega}} \right)^2 \langle n | (a^\dagger - a)^2 | n \rangle$$

(from (a))

$$= -\frac{\hbar}{2m\omega} \langle n | \cancelto{0}{(a^\dagger)^2 + a^2} - a^\dagger a - a a^\dagger | n \rangle$$

$$= \frac{\hbar}{2m\omega} \left[\langle n | a^\dagger a | n \rangle + \langle n | a a^\dagger | n \rangle \right]$$

$$= \frac{\hbar}{2m\omega} \left[\langle n | a^\dagger \cdot \sqrt{n} \, | n-1 \rangle + \langle n | a \cdot \sqrt{n+1} \, | n+1 \rangle \right]$$

$$= \frac{\hbar}{2m\omega} \left[\sqrt{n} \, \langle n | \cdot \sqrt{n} \, | n \rangle + \sqrt{n+1} \, \langle n | \sqrt{n+1} \, | n \rangle \right]$$

$$= \frac{\hbar}{2m\omega} (n + n + 1) = \frac{\hbar}{2m\omega} (2n+1).$$

$$\langle n|x^4|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \langle n|(a^\dagger - a)^4|n\rangle$$

a^+와 a의 갯수가 같을 경우만 남는다!!

$\langle n|$ $|n\rangle$에 ~~끼어~~ 싸여 있기 때문에.

$(a^+-a)(a^+-a)(a^+-a)(a^+-a)$

a^+	a^+	$(-a)$	$(-a)$	a^+a^+aa
a^+	$(-a)$	(a^+)	$(-a)$	a^+aa^+a
a^+	$(-a)$	$(-a)$	(a^+)	a^+aaa^+
$(-a)$	(a^+)	(a^+)	$(-a)$	aa^+a^+a
$(-a)$	(a^+)	$(-a)$	(a^+)	aa^+aa^+
$(-a)$	$(-a)$	(a^+)	(a^+)	aaa^+a^+

6개 항만 contribute!

- $a^+a^+aa|n\rangle = a^+a^+a \cdot \sqrt{n}|n-1\rangle$

$$= a^+a^+ \sqrt{n}\sqrt{n-1}\,|n-2\rangle$$

$$= a^+ \sqrt{n}\sqrt{n-1}\sqrt{n-1}\,|n-1\rangle$$

$$= n(n-1).\,|n\rangle$$

- $a^\dagger a a^\dagger a|n\rangle = \sqrt{n}\cdot\sqrt{n}\cdot\sqrt{n}\cdot\sqrt{n}\,|n\rangle = n^2|n\rangle$
- $a^\dagger a a a^\dagger|n\rangle = \sqrt{n+1}\cdot\sqrt{n+1}\cdot\sqrt{n}\sqrt{n}\,|n\rangle = n(n+1)|n\rangle$
- $a a^\dagger a^\dagger a|n\rangle = \sqrt{n}\sqrt{n}\sqrt{n+1}\sqrt{n+1}\,|n\rangle = n(n+1)|n\rangle$
- $a a^\dagger a a^\dagger|n\rangle = \sqrt{n+1}\sqrt{n+1}\sqrt{n+1}\sqrt{n+1}\,|n\rangle = (n+1)^2|n\rangle$
- $a a a^\dagger a^\dagger|n\rangle = \sqrt{n+1}\sqrt{n+2}\sqrt{n+2}\sqrt{n+1}\,|n\rangle = (n+1)(n+2)|n\rangle$

$$\therefore \langle n|x^4|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^2\Big[n(n-1)+n^2+n(n+1)+n(n+1)+(n+1)^2$$
$$+(n+1)(n+2)\Big]$$

$$=\left(\frac{\hbar}{2m\omega}\right)^2\Big[n^2-n+n^2+n^2+n+n^2+n+n^2+2n+1$$
$$+n^2+3n+2\Big]$$

$$=\left(\frac{\hbar}{2m\omega}\right)^2\Big[6n^2+6n+3\Big] = \left(\frac{\hbar}{2m\omega}\right)^2\cdot 3(2n^2+2n+1)$$

(c) $\langle n|p|n\rangle = 0.$

$$\langle n|p^2|n\rangle = \left(\sqrt{\frac{m\hbar\omega}{2}}\right)^2\langle n|(a^\dagger+a)^2|n\rangle$$

$$= \frac{m\hbar\omega}{2}\langle n|a^\dagger a + a a^\dagger|n\rangle$$

$$= \frac{m\hbar\omega}{2}(2n+1).$$

$$\langle n|p^4|n\rangle = \left(\frac{m\hbar\omega}{2}\right)^2 3(2n^2+2n+1).$$

(d) $\langle m| a^{+s} |n\rangle = \langle m| (a^+)^{s-1} \cdot \sqrt{n+1}\, |n+1\rangle$

$= \sqrt{n+1} \cdot \sqrt{n+2} \cdot \dots \cdot \sqrt{n+s-1}\, \langle m| a^+ |n+s-1\rangle$

$= \sqrt{n+1}\, \sqrt{n+2} \cdots \sqrt{n+s-1}\, \sqrt{n+s}\, \langle m|n+s\rangle$

$= \sqrt{\dfrac{(n+s)!}{n!}} \cdot \delta_{m,n+s}.$

$\langle m| a^s |n\rangle = \langle m| a^{s-1} \cdot \sqrt{n}\, |n-1\rangle$

$= \langle m| a^{s-2} \cdot \sqrt{n}\, \sqrt{n-1}\, |n-2\rangle$ $\quad$ $2 \to 1.$ $\; s-1 \to s-2$

$= \cdots$

$= \sqrt{n}\, \sqrt{n-1} \cdots \cancel{\sqrt{n+s-2}}\, \sqrt{n-s+2}\, \langle m| a |n-s+1\rangle$

$= \sqrt{n \cdot (n-1) \cdots (n-s+2)(n-s+1)} \cdot \langle m|n-s\rangle$

$= \sqrt{\dfrac{n!}{(n-s)!}}\; \delta_{m,n-s}$

(e) $\langle m|x|n\rangle = -i\sqrt{\frac{\hbar}{2m\omega}}\langle m|a^{+}-a|n\rangle$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}\Big(\underbrace{\langle m|a^{+}|n\rangle}_{\sqrt{n+1}\,\delta_{m,n+1}} - \underbrace{\langle m|a|n\rangle}_{\sqrt{n}\,\delta_{m,n-1}}\Big)$$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right).$$

$$\langle m|x^2|n\rangle = \left(-\frac{\hbar}{2m\omega}\right)\cdot\langle m|(a^{+})^2 + a^2 - a^{+}a - aa^{+}|n\rangle$$

$$= -\frac{\hbar}{2m\omega}\Big[\langle m|(a^{+})^2|n\rangle \to \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}$$

$$+\langle m|a^2|n\rangle \to \sqrt{n(n-1)}\,\delta_{m,n-2}$$

$$-n\langle m|n\rangle$$

$$-(n+1)\langle m|n\rangle\Big]$$

$$= -\frac{\hbar}{2m\omega}\Big[\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{n(n-1)}\,\delta_{m,n-2}$$

$$-(2n+1)\,\delta_{mn}\Big]$$

(f) $\langle m|p|n\rangle = \sqrt{\frac{m\hbar\omega}{2}} \langle m|a^{+}+a|n\rangle$

$= \sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{n+1}\, \delta_{m,n+1} + \sqrt{n}\, \delta_{m,n-1} \right)$

$\langle m|p^2|n\rangle = \frac{m\hbar\omega}{2} \cdot \Big[\langle m|(a^{+})^2|n\rangle$

$+ \langle m|(a)^2|n\rangle$

$+ \langle m|a^{+}a + aa^{+}|n\rangle \Big]$

$= \frac{m\hbar\omega}{2} \Big[\sqrt{(n+1)(n+2)}\, \delta_{m,n+2} + \sqrt{n(n-1)}\, \delta_{m,n-2}$

$+ (2n+1)\, \delta_{mn} \Big]$

$\boxed{2}$ Coherent states.

As shown in class, only the ground state of the harmonic oscillator has the minimum uncertainty $\Delta x \Delta p = \hbar/2$. However, we can construct the minimum uncertainty wave functions in the following way. That state is called the "coherent state" and it is defined as..

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

that is, it is an eigenstate of an annihilation operator. Since a is not hermitian, its eigenvalue α is in general complex.

(a) Compute $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$ in the state $|\alpha\rangle$, and show that $\Delta x \Delta p = \hbar/2$.

$\Rightarrow$ i) $\langle x\rangle = \langle\alpha|x|\alpha\rangle = -i\sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|a^+ - a|\alpha\rangle$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}\left[\langle\alpha|a^+|\alpha\rangle - \langle\alpha|a|\alpha\rangle\right]$$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}\left[\langle\alpha|a|\alpha\rangle^* - \langle\alpha|a|\alpha\rangle\right]$$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}\left[(\alpha^* - \alpha)\right]\langle\alpha|\alpha\rangle$$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}(\alpha^* - \alpha).$$

$$\langle x^2 \rangle = \langle \alpha | x^2 | \alpha \rangle = -\frac{\hbar}{2m\omega} \langle \alpha | (a^\dagger - a)^2 | \alpha \rangle$$

$$= -\frac{\hbar}{2m\omega} \langle \alpha | (a^\dagger)^2 - a^\dagger a - a a^\dagger + a^2 | \alpha \rangle$$

$$[a, a^\dagger] = a a^\dagger - a^\dagger a = 1.$$

$$= -\frac{\hbar}{2m\omega} \left[\langle \alpha | (a^\dagger)^2 - 2a^\dagger a - 1 + a^2 | \alpha \rangle \right]$$

$$= -\frac{\hbar}{2m\omega} \left[\langle \alpha | (a^\dagger)^2 | \alpha \rangle - 2 \langle \alpha | a^\dagger a | \alpha \rangle + \langle \alpha | a^2 | \alpha \rangle - \langle \alpha | \alpha \rangle \right]$$

$$= -\frac{\hbar}{2m\omega} \left[(\alpha^*)^2 - 2\alpha^* \cdot \alpha + \alpha^2 - 1 \right]$$

$$= -\frac{\hbar}{2m\omega} \left[(\alpha^* - \alpha)^2 - 1 \right].$$

$$\left[\begin{array}{l} a|\alpha\rangle = \alpha|\alpha\rangle \\ \langle \alpha | a^\dagger = \langle \alpha | \alpha^* \end{array} \right]$$

$$\langle p \rangle = \langle \alpha | p | \alpha \rangle = \sqrt{\frac{m\hbar\omega}{2}} \langle \alpha | a^\dagger + a | \alpha \rangle$$

$$= \sqrt{\frac{m\hbar\omega}{2}} \left[\langle \alpha | a^\dagger | \alpha \rangle + \langle \alpha | a | \alpha \rangle \right]$$

$$= \sqrt{\frac{m\hbar\omega}{2}} \left[\alpha^* + \alpha \right].$$

$$\langle p^2 \rangle = \frac{m\hbar\omega}{2} \langle \alpha | (a^\dagger + a)^2 | \alpha \rangle$$

$$= \frac{m\hbar\omega}{2} \langle \alpha | (a^\dagger)^2 + \cancel{a^\dagger a}\ a^\dagger a + \underbrace{a a^\dagger}_{= a^\dagger a + 1} + a^2 | \alpha \rangle$$

$$= \frac{m\hbar\omega}{2} \left((\alpha^*)^2 + 2\alpha^*\alpha + \alpha^2 + 1 \right)$$

$$= \frac{m\hbar\omega}{2} \left[(\alpha^* + \alpha)^2 + 1 \right].$$

$\therefore$

$$\cancel{\langle x \rangle = i \sqrt{\frac{\hbar}{2m\omega}} (\alpha - \alpha^*)}$$

$$\cancel{\langle x^2 \rangle = \frac{\hbar}{2m\omega} \left[1 - (\alpha - \alpha^*)^2 \right]}$$

$$\langle x \rangle = -i \sqrt{\frac{\hbar}{2m\omega}} (\alpha^* - \alpha)$$

$$\langle x^2 \rangle = -\frac{\hbar}{2m\omega} \left[(\alpha^* - \alpha)^2 - 1 \right]$$

$$\langle p \rangle = \sqrt{\frac{m\hbar\omega}{2}} (\alpha^* + \alpha)$$

$$\langle p^2 \rangle = \frac{m\hbar\omega}{2} \left[(\alpha^* + \alpha)^2 + 1 \right]$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{-\frac{\hbar}{2m\omega}\left[\cancel{(\alpha^* - \alpha)^2} - 1\right] + \frac{\hbar}{2m\omega}\cancel{(\alpha^* - \alpha)^2}}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{\frac{m\hbar\omega}{2}\left[\cancel{(\alpha^* + \alpha)^2} + 1\right] - \frac{m\hbar\omega}{2}\cancel{(\alpha^* + \alpha)^2}}$$

$$= \sqrt{\frac{m\hbar\omega}{2}}$$

$$\therefore \Delta x \Delta p = \sqrt{\frac{\hbar}{2\cancel{m}\cancel{\omega}}} \cdot \sqrt{\frac{\cancel{m}\hbar\cancel{\omega}}{2}} = \frac{\hbar}{2}.$$

(b) Show that the state $|\alpha\rangle$ can be written in the form

$$|\alpha\rangle = C\, e^{\alpha a^\dagger} |0\rangle.$$

$$\Rightarrow \quad e^{\alpha a^\dagger} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \cdot (a^\dagger)^n$$

$$(a^\dagger)^n |0\rangle = \sqrt{n!}\, |n\rangle$$

$\therefore$ $\cancel{|\alpha}$

$$e^{\alpha a^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \cdot (a^\dagger)^n |0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$a\, e^{\alpha a^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \cdot a |n\rangle$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \cdot \sqrt{n} \cdot |n-1\rangle$$

$$= \alpha \cdot \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \cdot |n-1\rangle$$

$$= \alpha \cdot e^{\alpha a^\dagger} |0\rangle.$$

$$\therefore \quad |\alpha\rangle = C \cdot e^{\alpha a^\dagger} |0\rangle$$

(Since.. $a|\alpha\rangle = \alpha \cdot C \cdot e^{\alpha a^\dagger} |0\rangle = \alpha |\alpha\rangle$).

(c) Prove that if $f(a^\dagger)$ is any polynomial in $a^\dagger$, then

$$a f(a^\dagger)|0\rangle = \frac{df(a^\dagger)}{da^\dagger}|0\rangle.$$

Using this fact, compute C.

$\Rightarrow$ $(a^\dagger)^n$ 에 대해 고려해 보자.

$$\left[\ f(a^\dagger) = \sum_{n=0}^{\infty} C_n \cdot (a^\dagger)^n \quad \text{으로 표현 된다.}\ \right]$$

$$a(a^\dagger)^n = a a^\dagger \cdot (a^\dagger)^{n-1} = (a^\dagger a + 1)(a^\dagger)^{n-1}$$

$$= (a^\dagger)^{n-1} + a^\dagger a (a^\dagger)^{n-1}$$

$$= (a^\dagger)^{n-1} + a^\dagger \left[a a^\dagger \cdot (a^\dagger)^{n-2} \right]$$

$$= (a^\dagger)^{n-1} + a^\dagger \cdot (a^\dagger a + 1)(a^\dagger)^{n-2}$$

$$= 2 \cdot (a^\dagger)^{n-1} + (a^\dagger)^2 \cdot a (a^\dagger)^{n-2}$$

$$= \cdots$$

$$= \cancel{(n-1)}\ (n-1)(a^\dagger)^{n-1} + (a^\dagger)^{n-1} \cdot a \cdot a^\dagger$$

$$= (n-1)(a^\dagger)^{n-1} + (a^\dagger)^{n-1}(a^\dagger a + 1)$$

$$= n \cdot (a^\dagger)^{n-1} + (a^\dagger)^n \cdot a$$

$$\therefore\ a\,(a^\dagger)^n\,|0\rangle = [\,n\,(a^\dagger)^{n-1} + (a^\dagger)^n\,\overbrace{a\,]\,|0\rangle}^{0}$$

$$= n\,(a^\dagger)^{n-1}\,|0\rangle$$

$$\therefore\ a\,f(a^\dagger)\,|0\rangle = \sum_{n=0}^{\infty} C_n \cdot a \cdot (a^\dagger)^n\,|0\rangle$$

$$= \sum_{n=0}^{\infty} C_n \cdot n\,(a^\dagger)^{n-1}\,|0\rangle$$

$$= \underbrace{\sum_{n=1}^{\infty} C_n \cdot n\,(a^\dagger)^{n-1}}_{\frac{d f(a^\dagger)}{d a^\dagger}}\,|0\rangle$$

$$= \frac{d f(a^\dagger)}{d a^\dagger}\,|0\rangle.$$

$\therefore\ f(a^\dagger) = C e^{\alpha a^\dagger}$ 라 하면..

$$\underbrace{a f(a^\dagger)\,|0\rangle}_{|\alpha\rangle} = \alpha \cdot \underbrace{f(a^\dagger)\,|0\rangle}_{|\alpha\rangle} = \frac{d f(a^\dagger)}{d a^\dagger}\,|0\rangle.$$

$$\frac{d f(a^\dagger)}{d a^\dagger} = C e^{\alpha a^\dagger} \cdot \alpha$$

$$a\, f(a^\dagger) \cdot |0\rangle = \frac{df(a^\dagger)}{da^\dagger} |0\rangle$$ 임을 보였다.

여기서 $\langle\alpha|\alpha\rangle=1$ 이라는 조건으로 C 결정하자.

$$|\alpha\rangle = C \cdot e^{\alpha a^\dagger} |0\rangle$$

$$\langle\alpha| = C^* \langle 0| e^{\alpha^* a}$$

$$\langle\alpha|\alpha\rangle = |C|^2 \langle 0| e^{\alpha^* a} \underbrace{e^{\alpha a^\dagger}} |0\rangle$$

$f(a^\dagger)$ 라 하자.

$$\boxed{a^n f(a^\dagger) |0\rangle = \frac{d^n f(a^\dagger)}{da^{\dagger n}} |0\rangle}$$

~~[illegible]~~ 이때 $e^{\alpha^* a} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha^*)^n a^n$ 이므로..

$$e^{\alpha^* a} e^{\alpha a^\dagger} \cdot |0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha^*)^n a^n e^{\alpha a^\dagger} |0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{n!} \frac{d^n}{da^{\dagger n}} \cdot e^{\alpha a^\dagger} |0\rangle$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{n!} \cdot \alpha^n} \cdot e^{\alpha a^\dagger} |0\rangle$$

$$= e^{|\alpha|^2} e^{\alpha a^\dagger} |0\rangle$$

$$\therefore \langle \alpha|\alpha\rangle = |C|^2 \langle 0| e^{|\alpha|^2} \cdot \underbrace{e^{\alpha a^\dagger}}_{\sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n \cdot (a^\dagger)^n} |0\rangle$$

이때.. $n \neq 0$ 이면 $\langle 0|(a^\dagger)^n|0\rangle = 0$

$\therefore$ $n=0$ 항만 남는다.

$$= |C|^2 \cdot e^{|\alpha|} \langle 0|0\rangle = |C|^2 e^{|\alpha|^2} = 1$$

$$\therefore \quad C = e^{-|\alpha|^2/2}$$

$\therefore$ $\cancel{|\alpha\rangle}$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \, e^{\alpha a^\dagger} |0\rangle$$

$\therefore$

$$\cancel{a\, C e^{\alpha a^\dagger} |0\rangle = C e^{\alpha a^\dagger} \alpha |0\rangle}$$

~~C 결정 불가?~~

$\downarrow$

~~Normalization으로 결정..?~~

(d) On the other hand, since the set of the energy eigenstates $\{|n\rangle\}$ forms a complete set, the state $|\alpha\rangle$ can be expanded as..

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle.$$

Show that the coefficient C_n are given by

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0.$$

$\Rightarrow$ $$e^{\alpha a^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

양변 C 곱하면 그래 우리의 $|\alpha\rangle$

$\therefore$ $$C e^{\alpha a^\dagger} |0\rangle = |\alpha\rangle = \sum_{n=0}^{\infty} C \cdot \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

$C \to C_0$ 라고 하면..

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0$$

(e) By normalizing $|\alpha\rangle$, show that $C_0 = \exp(-|\alpha|^2/2)$

$\Rightarrow$ $|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$

$$\langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty} \langle\alpha|n\rangle\langle n|\alpha\rangle$$

$$= \sum_{n=0}^{\infty} C_n^* C_n = \sum_{n=0}^{\infty} |C_n|^2$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |C_0|^2$$

normalized.

$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = 1$$

$$\therefore |C_0|^2 = \frac{1}{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}} = \frac{1}{\exp|\alpha|^2}$$

$$= \exp(-|\alpha|^2)$$

$$\therefore C_0 = \exp(-|\alpha|^2/2).$$

(f) From parts (d) and (e), you can find the probability for the state $|\alpha\rangle$ to contain n quanta. Find it, and it is called the Poisson distribution.

⇒ Harmonic oscillator 의 에너지 (n^{th})

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

↓

단계가 하나 올라갈 때 마다 $\hbar\omega$ 씩 에너지 증가.

즉.. E_n 은 n개의 energy 덩어리 (quanta) 가 있는 상태로 해석!

∴ $|\alpha\rangle$가 n개의 quanta 를 가질 확률

⇓

에너지 측정시 $|n\rangle$ state 가 나올 확률

⇓

$$\underline{\underline{|c_n|^2}}$$

$$|c_n|^2 = \exp(-|\alpha|^2)\cdot\frac{(|\alpha|^2)^n}{n!} \rightarrow \begin{pmatrix} \text{mean} : |\alpha|^2 \\ \text{variance} : |\alpha|^2 \end{pmatrix}$$

$$\left[\ \mathrm{Pois}(\lambda) = \frac{\lambda^k}{k!}\cdot e^{-\lambda} \quad \begin{pmatrix} \text{mean} : \lambda \\ \text{variance} : \lambda. \end{pmatrix}\ \right]$$

(9) Finally, compute the average number of quanta in the coherent state. That is, compute $\langle \alpha | a^\dagger a | \alpha \rangle$

$\rightarrow \quad \langle \alpha | a^\dagger a | \alpha \rangle = \alpha^* \alpha = |\alpha|^2$

3. The Hamiltonian of a particle can be expressed in the form

$$H = \varepsilon_1 a^+ a + \varepsilon_2 (a + a^+), \quad [a, a^+] = 1,$$

where ε_1 and ε_2 are constants.

(a) Find the energies of the eigenstates.

⇒ 만약

$$b = \alpha a + \beta$$
$$b^+ = \alpha a^+ + \beta$$

(α, β are constants.)

라 하자.

H를 b^+ 와 b로 표현해보자!

$$b^+ b = (\alpha a^+ + \beta)(\alpha a + \beta)$$

$$= \alpha^2 a^+ a + \alpha\beta a^+ + \beta\alpha \cdot a + \beta^2$$

$$\therefore b^+ b - \beta^2 = \alpha^2 a^+ a + \alpha\beta (a + a^+)$$

이때, $\alpha^2 = \varepsilon_1$, $\alpha\beta = \varepsilon_2$ 라 하면,

$$b^+ b - \beta^2 = \varepsilon_1 a^+ a + \varepsilon_2 (a + a^+)$$

$$\boxed{\begin{array}{l} \alpha^2 = \varepsilon_1 \quad \alpha\beta = \varepsilon_2 \;\rightarrow\; \alpha^2\beta^2 = \varepsilon_2^2 \\ \beta^2 = \dfrac{\varepsilon_2^2}{\alpha^2} = \dfrac{\varepsilon_2^2}{\varepsilon_1} \end{array}}$$

$$\therefore\ b^+b - \frac{(\varepsilon_2)^2}{\varepsilon_1} = \varepsilon_1 a^+a + \varepsilon_2 (a+a^+).$$

$$\therefore\ H = b^+b - \frac{(\varepsilon_2)^2}{\varepsilon_1}$$ ⇒ 우리가 원래 알고 있었던 Hamiltonian 형태임.

ii) 이제.. b와 b^+의 commutation relation을 구하자.

$$[b, b^+] = [\alpha a + \beta,\ \alpha a^+ + \beta]$$

$$= \alpha^2 [a, a^+] = \alpha^2.$$

iii) 이제.. ground state를 $|0\rangle$ 라고 하고..

$b|0\rangle = 0$ 이라 하면..

$$H|0\rangle = \left(b^+b - \frac{(\varepsilon_2)^2}{\varepsilon_1}\right)|0\rangle = -\frac{(\varepsilon_2)^2}{\varepsilon_1}|0\rangle$$

$\therefore$ ground state energy ⇒ $-\frac{(\varepsilon_2)^2}{\varepsilon_1}$.

iv) first excited state를 $|1\rangle \equiv b^+|0\rangle$ 이라고 하면..

$$H|1\rangle = \left(b^+b - \frac{(\varepsilon_2)^2}{\varepsilon_1}\right) b^+|0\rangle$$

$$= \left(b^+bb^+ - \frac{(\varepsilon_2)^2}{\varepsilon_1} b^+\right)|0\rangle$$

$$= \left[b^+(b^+b + \alpha^2) - \frac{(\varepsilon_2)^2}{\varepsilon_1} b^+\right]|0\rangle$$

$$= \left(\alpha^2 - \frac{(\varepsilon_2)^2}{\varepsilon_1}\right) b^+ |0\rangle = \left[\alpha^2 - \frac{(\varepsilon_2)^2}{\varepsilon_1}\right] |1\rangle .$$

ㄴ) 일반적으로.. $|n\rangle \equiv \frac{(b^+)^n}{\sqrt{n!}} |0\rangle$ 라고 하면..

$$E_n = n\alpha^2 - \frac{(\varepsilon_2)^2}{\varepsilon_1} \qquad (n = 0, 1, 2, \cdots)$$

$$= n \cdot \varepsilon_1 - \frac{(\varepsilon_2)^2}{\varepsilon_1} = \varepsilon_1 \left[n - \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \right].$$

(b) The same except that.. the commutator of a and a^+ is $[a, a^+] = \gamma^2$, where γ is a pure number.

⇒ i) 다 동일하나 $[b, b^+] = \alpha^2 [a, a^+] = \alpha^2 \gamma^2$ 가 된다..

$$[b, b^+] = \varepsilon_1 \gamma^2 .$$

ii) $\therefore E_n = n \cdot \varepsilon_1 \gamma^2 - \frac{(\varepsilon_2)^2}{\varepsilon_1}$

$$= \varepsilon_1 \left[n \cdot \gamma^2 - \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \right].$$

$$(n = 0, 1, 2, \cdots)$$