Communication Signals

(Haykin Sec. 2.1 and Ziemer Sec.2.4-Sec. 2.5)

KECE321 Communication Systems I

Lecture #4, March 14, 2012

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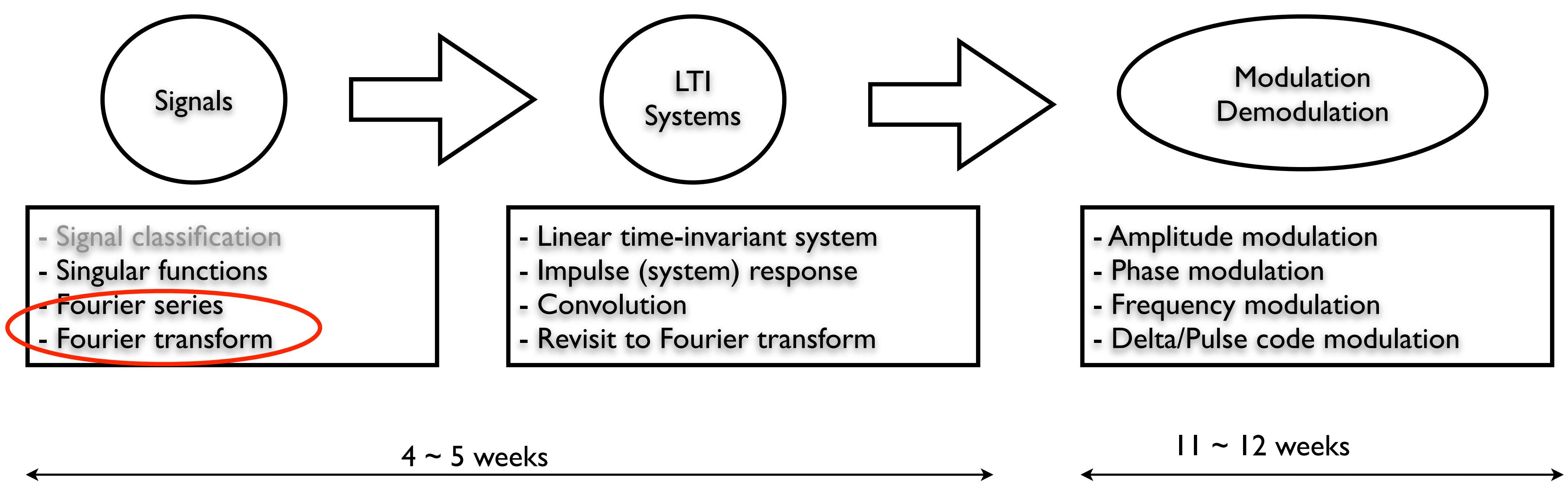
Review

- Singular functions
  - Unit step function
  - Dirac delta function (Unit impulse function)
  - Signum (Sign) function
- Generalized Fourier series
  - Integral-square error

Summary of Today's Lecture

- Fourier series
  - Generalized Fourier series
  - Complex Fourier series
  - Examples
- Fourier transform



Generalized Fourier Series

- Generalized Fourier series:
  - representation of signals as a series of orthogonal functions
- Recall the vector space:
  - Given any vector $\mathbf{A}$ in three-dimensional space can be expressed in terms of three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ that do not all lie in the sample plane

$$\mathbf{A} = A_1\mathbf{x} + A_2\mathbf{y} + A_3\mathbf{z}$$

    - where A_1, A_2, and A_3 are appropriately chosen constants.
    - The vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are said to be *linearly independent* since no one of them can be expressed as a linear combination of the other two. For example, it is impossible to write $\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$, no matter what choice is made for the constants α and β
  - Such a set of linearly independent vectors is said to form a *basis set* for a three-dimensional vector space. Such vectors *span* a three-dimensional vector space in the sense that any vector $\mathbf{A}$ can be expressed as a linear combination of them.

- Similarly, consider the problem of representing a time function, or signal, $x(t)$ on a T-second interval $(t_0, t_0 + T)$, as a similar expansion.

  - We consider a set of time functions $\phi_1(t),\ \phi_2(t), \cdots, \phi_N(t)$, which are specified independently of $x(t)$, and seek a series expansion of the form

$$x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t), \qquad t_0 \le t \le t_0 + T$$

independent of time

the N coefficients X_n are independent of time and the subscript a indicates that $x_a(t)$ is considered an approximation.

  - We assume that the $\phi_n(t)'\mathrm{s}$ are linearly independent; that is, no one of them can be expressed as a weighted sum of the other $N-1$. A set of linearly independent $\phi_n(t)'\mathrm{s}$ will be called a *basis function set*.

- We now wish to examine the error in the approximation of $x(t)$ by $x_a(t)$. As in the case of ordinary vectors, the expansion $x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t)$ is easiest to use if the $\phi_n(t)'\text{s}$ are orthogonal on the interval $(t_0, t_0 + T)$.

  - That is,

$$\int_{t_0}^{t_0+T} \phi_m(t)\phi_n^*(t)\,dt = c_n\delta_{nm} \triangleq \begin{cases} c_n, & n = m \\ 0, & n \neq m \end{cases} \quad \text{(all } m \text{ and } n\text{)}$$

    where, if $c_n = 1$ for all n, the $\phi_n(t)'\text{s}$ are said to be normalized.

    - A normalized orthogonal wet of functions is called on *orthogonal basis set*.

      - δ_{mn} is called the Kronecker delta function, is defined as unity if $m = n$, and zero otherwise.

- The error in the approximation will be measured in the *integral-square sense* (ISE)

$$\boxed{\text{Error} = \epsilon_N = \int_T |x(t) - x_a(t)|^2\,dt}$$

where $\int_T (\ )\,dt$ denotes the integration over t from t_0 to $t_0 + T$.

- The ISE is an applicable measure of error only when $x(t)$ is an energy signal or a power signal. If $x(t)$ is an energy signal of infinite duration, the limit as $T \to \infty$ is taken.

- We now find the set of coefficients X_n that minimizes the ISE. Substituting $x_a(t)$ into ISE, expressing the magnitude square of the integrand as the integrand times its complex conjugate and expanding, we obtain

$$\begin{aligned}\epsilon_N &= \int_T |x(t)|^2\,dt - \sum_{n=1}^{N}\left[X_n^* \int_T x(t)\phi_n^*(t)\,dt + X_n \int_T x^*(t)\phi_n(t)\,dt\right] \\ &\quad + \sum_{n=1}^{N} c_n |X_n|^2\end{aligned}$$

  - To find the X_n's that minimizes ϵ_N we add and subtract the quantity

$$\sum_{n=1}^{N} \frac{1}{c_n}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2$$

which yields

$$\epsilon_N = \int_T |x(t)|^2\,dt - \sum_{n=1}^{N} \frac{1}{c_n}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2 + \sum_{n=1}^{N} c_n \left|X_n - \frac{1}{c_n}\int_T x(t)\phi_n^*(t)\,dt\right|^2$$

independent of X_n's

- The first two terms on the right-hand side of ϵ_N are independent of the coefficients X_n. Since the last sum on the right-hand side is nonnegative, we will minimize ϵ_N if we choose each X_n such that the corresponding term in the sum is zero. Thus, since $c_n > 0,$ the choice of

$$X_n = \frac{1}{c_n} \int_T x(t)\phi_n^*(t)\, dt$$

for X_n minimizes the ISE.

- The resulting minimum-error coefficients will be referred to as the *Fourier coefficients*.

- Minimum value for ϵ_n

$$\begin{aligned}(\epsilon_n)_{\min} &= \int_T |x(t)|^2\, dt - \sum_{n=1}^{N} \frac{1}{c_n} \left| \int_T x(t)\phi_n^*(t)\, dt \right|^2 \\ &= \int_T |x(t)|^2\, dt - \sum_{n=1}^{N} c_n |X_n|^2\end{aligned}$$

- If we can find an infinite set of orthonormal functions such that $\lim_{N\to\infty} (\epsilon_N)_{\min} = 0$ for any signal that is integrable square,

$$\int_T |x(t)|^2 \, dt < \infty$$

we say that the $\phi_n(t)$'s are complete. In the sense that the ISE is zero, we may then write

$$x(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \quad \text{(ISE=0)}$$

Assuming a complete orthogonal set of functions, we obtain the relation

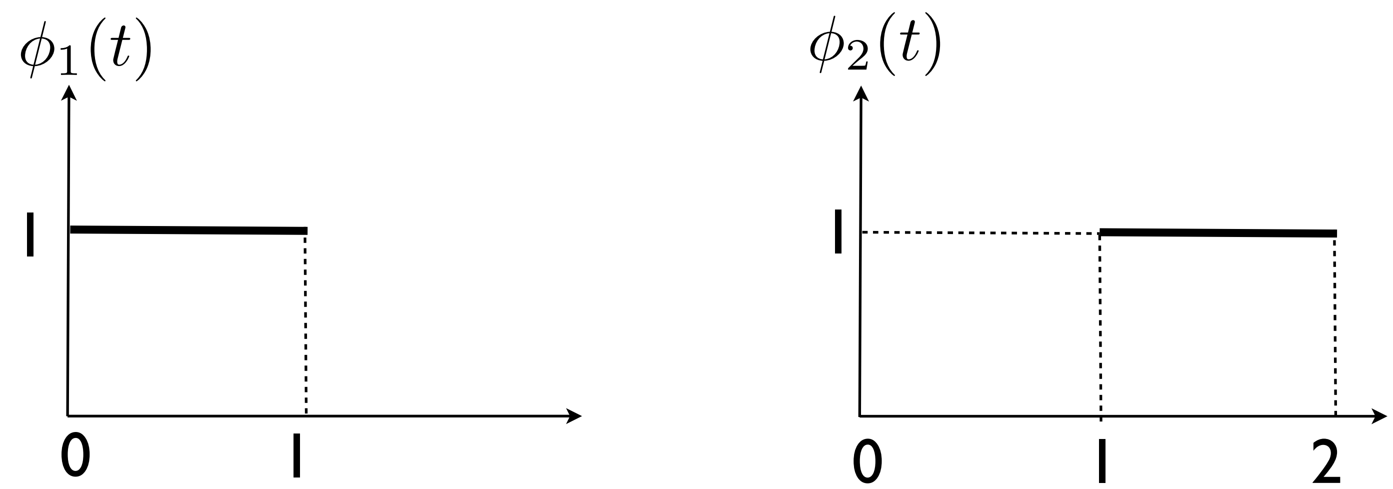
$$\int_T |x(t)|^2 \, dt = \sum_{n=1}^{N} c_n |X_n|^2$$

This equation is known as *Parseval's theorem*.

In general, equation $\lim_{N\to\infty} (\epsilon_N)_{\min} = 0$ requires that $x(t)$ be equal to $x_a(t)$ as $N \to \infty$.

- *Example,* The signal $x(t)$ is to be approximated by a two-term generalized Fourier series

$$x(t) = \begin{cases} \sin(\pi t), & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$



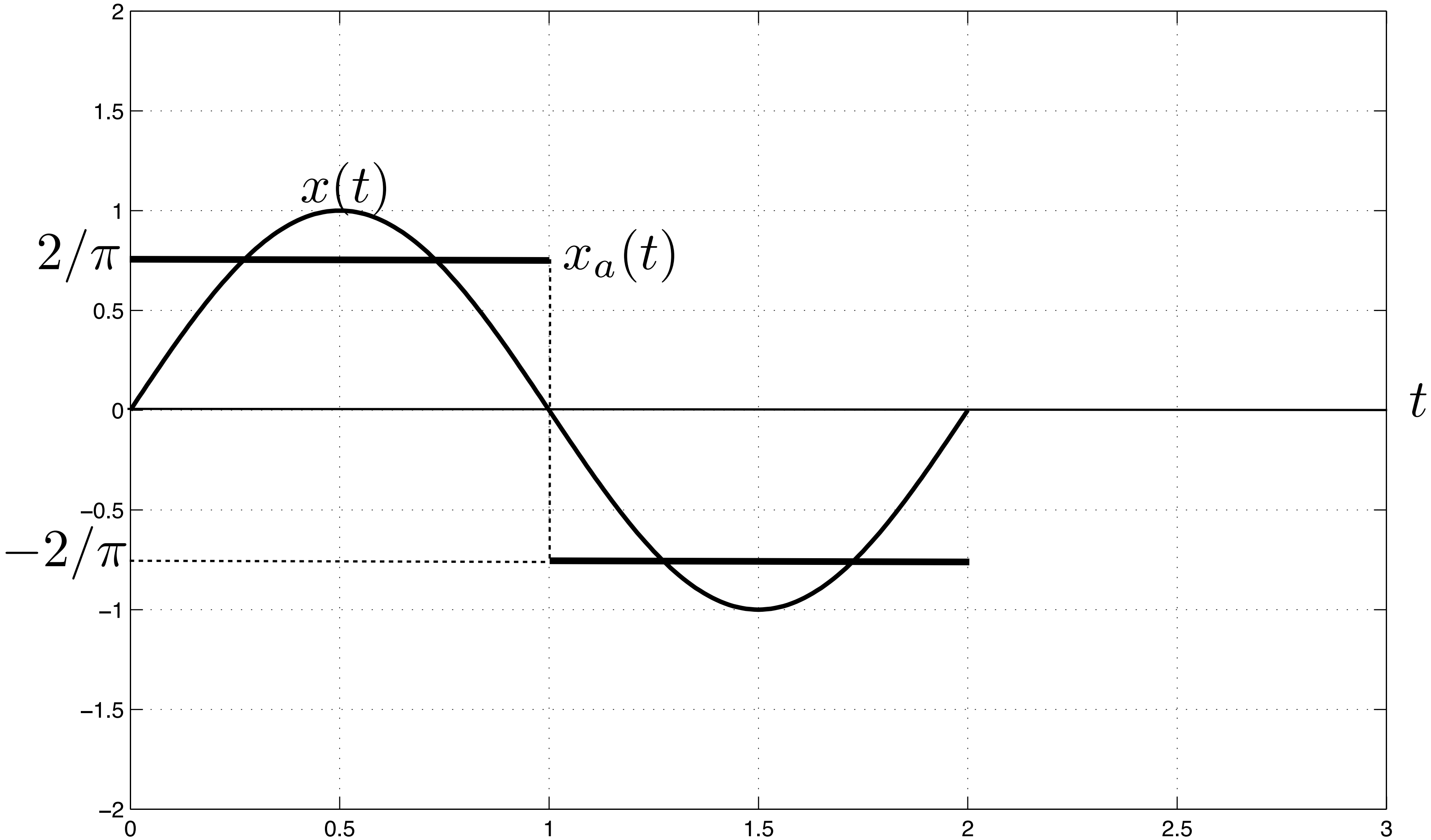
- The Fourier coefficients are calculated as

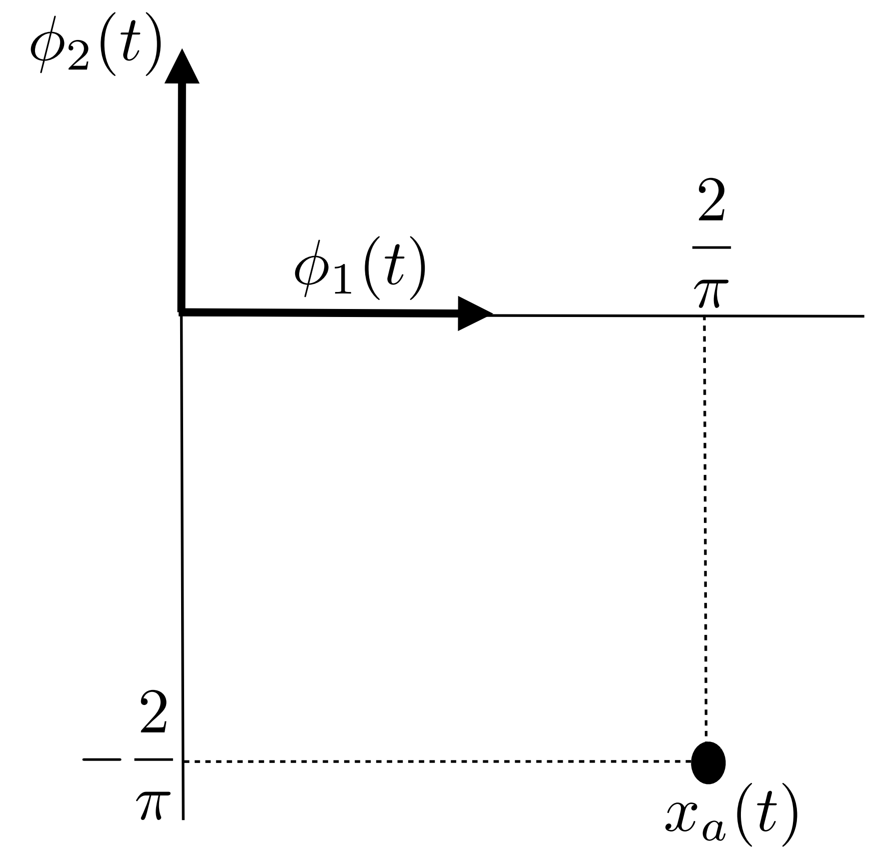
$$\begin{aligned} X_1 &= \int_0^2 \phi_1(t)\sin(\pi t)\,dt = \int_0^1 \sin(\pi t)\,dt = \frac{2}{\pi} \\ X_2 &= \int_0^2 \phi_2(t)\sin(\pi t)\,dt = \int_1^2 \sin(\pi t)\,dt = -\frac{2}{\pi} \end{aligned}$$

- Thus the generalized two-term Fourier series approximation for this signal is

$$x_a(t) = \frac{2}{\pi}\phi_1(t) - \frac{2}{\pi}\phi_2(t) = \frac{\pi}{2}\left[\text{rect}\left(t - \frac{1}{2}\right) - \text{rect}\left(t - \frac{3}{2}\right)\right]$$

- Space interpretation





- Minimum ISE

$$(\epsilon_N)_{\min} = \int_0^2 \sin^2(\pi t)\, dt - 2\left(\frac{2}{\pi}\right)^2 = 1 - \frac{8}{\pi^2} \approx 0.189$$

Complex Exponential Fourier Series

- Consider a signal $x(t)$ defined over the interval $(t_0, t_0 + T)$ with the definition

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

we define the complex exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \qquad t_0 \le t \le t + 0 + T_0$$

where

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt$$

- It can be shown to represent the signal $x(t)$ exactly in the interval $(t_0, t_0 + T_0)$, except at a point of jump discontinuity where it converges to the arithmetic mean of the left-hand and right-hand limits.

- Outside the interval $(t_0, t_0 + T_0)$, nothing is guaranteed.

- However, we note that the right-hand side of the complex exponential Fourier series is period with period T_0, since it is the sum of periodic rotating phasors with harmonic frequencies.

> If $x(t)$ is periodic with period T_0, the Fourier series is an accurate representation for $x(t)$ for all t (except at points of discontinuity).

- A useful observation about a complete orthonormal-series expansion of a signal is that the series is unique.

  - For example, if we somehow find a Fourier expansion for a signal $x(t)$, we know that no other Fourier expansion for that $x(t)$ exists, since $\{e^{jn\omega_0 t}\}$ forms a complete set.

Example

- Consider the signal $x(t) = \cos(\omega_0 t) + \sin^2(2\omega_0 t)$ where $\omega_0 = 2\pi/T_0$. Find the complex exponential Fourier series.

- Solution: Using trigonometric identities and Euler's theorem, we obtain

$$
\begin{aligned}
x(t) &= \cos(\omega_0 t) + \frac{1}{2} - \frac{1}{2}\cos(4\omega_0 t) \\
&= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2} - \frac{1}{4}e^{j\omega_0 t} - \frac{1}{4}e^{-j\omega_0 t}
\end{aligned}
$$

- Hence,

$$
\begin{aligned}
X_0 &= \frac{1}{2} \\
X_1 &= \frac{1}{2} = X_{-1} \\
X_4 &= \frac{1}{4} = X_{-4}
\end{aligned}
$$

Symmetry Properties of Fourier Coefficients

- Assuming $x(t)$ is real. Then we can show

$$X_n^* = X_{-n}$$

  - Writing $X_n = |X_n| e^{j\angle X_n}$, we have

$$|X_n| = |X_{-n}| \text{ and } \angle X_n = -\angle X_{-n}$$

- Using Euler's theorem, Fourier coefficient can be rewritten

$$\begin{aligned} X_n &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t)\, dt - \frac{j}{T} \int_{t_0}^{t_0+T_0} x(t) \sin(n\omega_0 t)\, dt \end{aligned}$$

Trigonometric Form of the Fourier Series

- Recall the Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$$

- Assuming $x(t)$ real, we can regroup the complex exponential Fourier series by paris of terms of the form

$$\begin{aligned} X_n e^{jn\omega_0 t} + X_{-n}e^{-jn\omega_0 t} &= |X_n|e^{j(n\omega_0 t + \angle X_n)} + |X_{-n}|e^{-j(nomega_0 t + \angle X_n)} \\ &= 2|X_n|\cos(n\omega_0 t + \angle X_n) \end{aligned}$$

  - Hence, we can rewrite the Fourier series as

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n|\cos(n\omega_0 t + \angle X_n)$$

- Using the trigonometric identity given as

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

  - we can rewrite Fourier series as

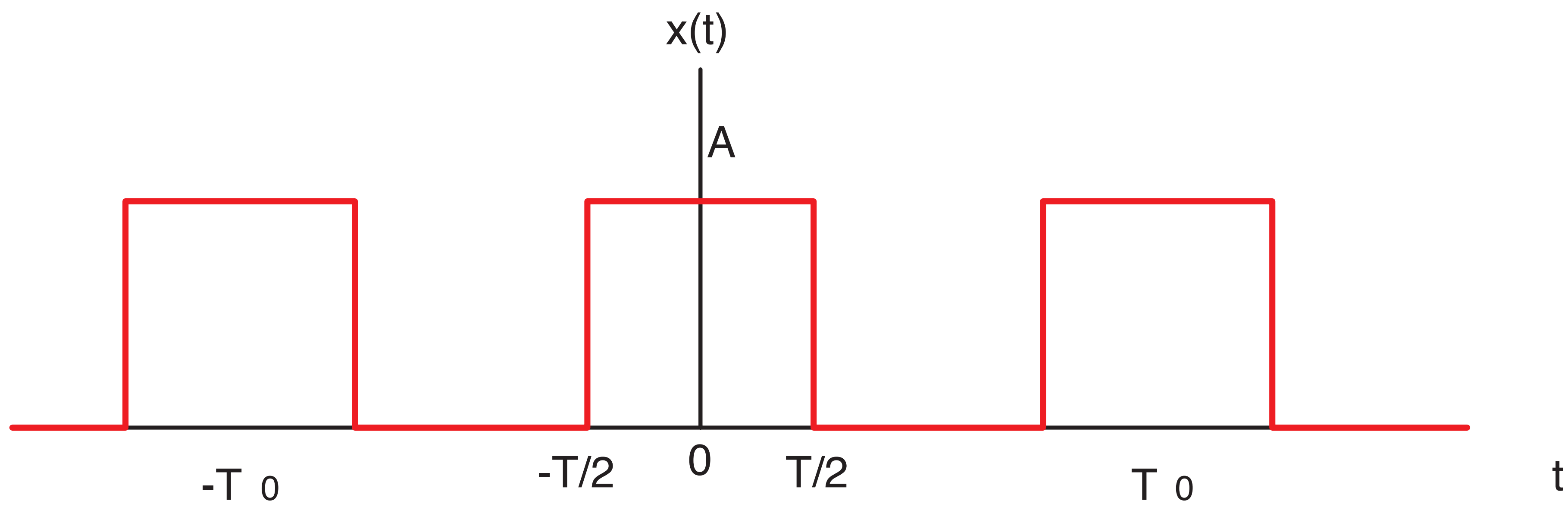
$$x(t) = X_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

where

$$\begin{aligned} A_n &= 2|X_n| \cos\angle X_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\, dt \\ B_n &= -2|X_n| \sin\angle X_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\, dt \end{aligned}$$

Example: Periodic Pulse Train

- Find the complex Fourier coefficients X_n



$$x(t) = \begin{cases} A, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{for the remainder of the period} \end{cases}$$

fundamental frequency: $f_0 = \dfrac{1}{T_0}$

- Complex Fourier coefficients X_n

$$
\begin{aligned}
X_n &= \int_{-T/2}^{T/2} A \exp(-j2\pi n f_0 t) dt \\
&= \frac{A}{-j2\pi n f_0} \exp(-j2\pi n f_0 t)\Big|_{t=-T/2}^{t=T/2} \\
&= A \frac{[\exp(-j\pi n f_0 T) - \exp(j\pi n f_0 T)]}{-j2\pi n f_0} \\
&= \frac{A}{\pi n f_0} \frac{[\exp(j\pi n f_0 T) - \exp(-j\pi n f_0 T)]}{j2} \\
&= \frac{A}{\pi n f_0} \sin(\pi n f_0 T) = AT\mathrm{sinc}(n f_0 T)
\end{aligned}
$$

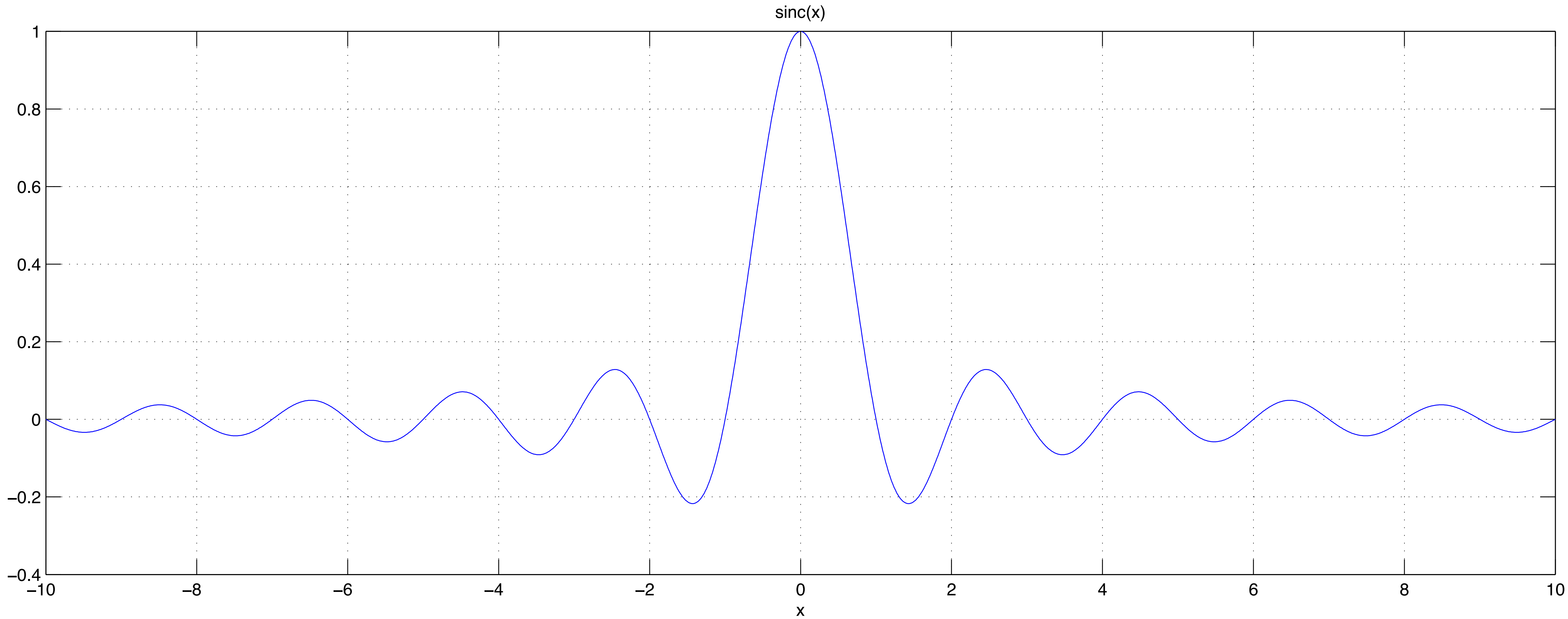
where we define sinc function as

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

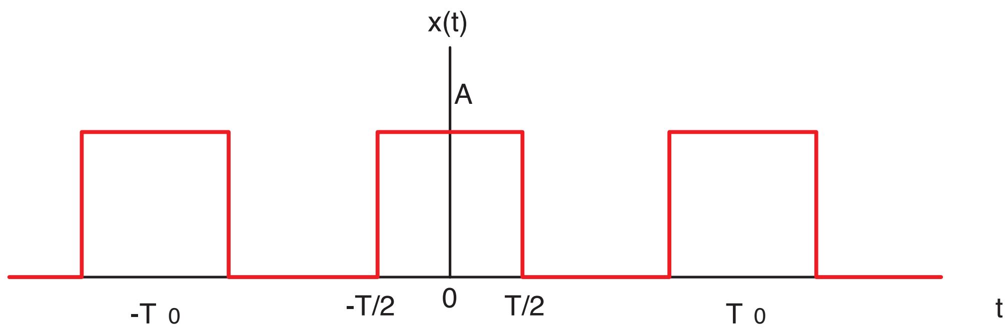
Sinc Function

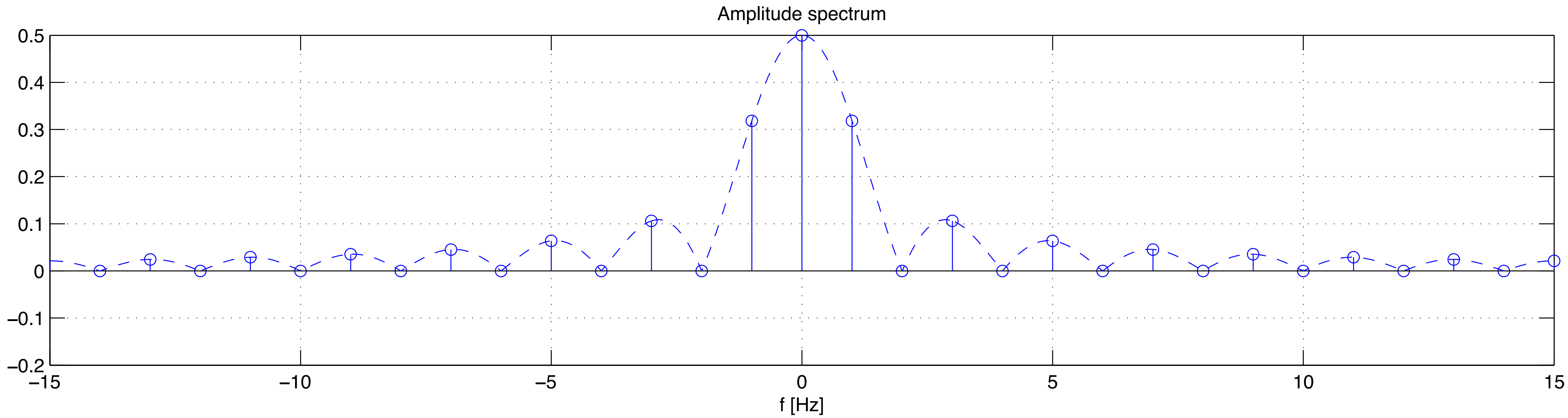
- Definition

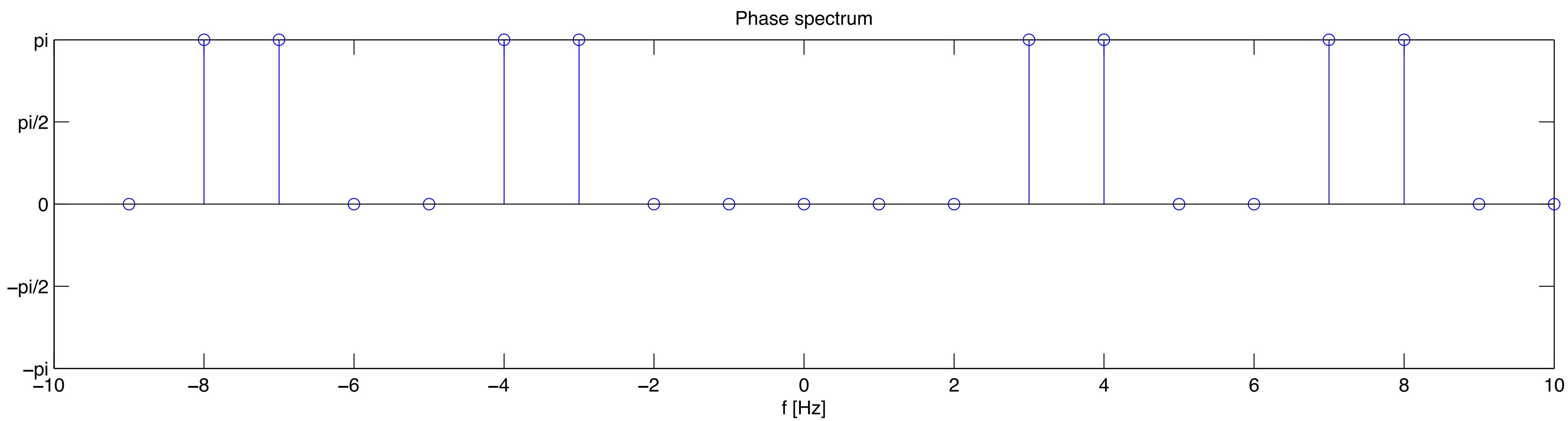
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Amplitude and Phase Spectrum







Fourier Transform

- Now we want to generalize the Fourier series to represent aperiodic signals using the Fourier series form given as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \qquad t_0 \le t \le t + 0 + T_0$$

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt$$

- Consider nonperiodic signal $x(t)$ but is an energy signal.
  - In the interval $|t| < \frac{1}{2}T_0$, we can represent $x(t)$ as

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(\lambda) e^{-j2\pi n f_0 \lambda}\, d\lambda \right] e^{jn2\pi n f_0 t}, \qquad |t| < \frac{T_0}{2}$$

    - where $f_0 = 1/T_0$.
  - To represent $x(t)$ for all time, we simply let $T_0 \to \infty$ such that

$$nf_0 = n/T_0 \to f, \quad 1/T_0 \to df, \quad \sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$

- Thus

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} \, d\lambda \right] e^{j2\pi f t} \, df$$

- Defining

$$X(f) = \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} \, d\lambda$$

we can rewrite

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$