# Quantum Mechanics II

## Assignment 4

Due: October 22 (Tuesday), 2013

1. As claimed in class, compute explicitly Eq. (14-25) to obtain Eq. (14-27) in Gasiorowicz..

2. *Exercise in variational techniques*

   Consider a one-dimensional harmonic oscillator with the Hamiltonian

   $$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

   Take the trial wave function for the ground state as

   $$\psi(x) = \begin{cases} (a^2 - x^2)^2, & |x| \leq a, \\ 0, & |x| > a, \end{cases} \tag{2}$$

   where $a$ is the variational parameter. Compute the ground energy as a function of $a$, and minimize it. Compare it with the true value of the ground-state energy. Do not forget the normalization of the wave function.

3. Last semester, we showed that, in one dimension, an attractive square well has at least one bound state no matter how weak the potential is. Use the Rayleigh-Ritz variational method to prove that this is a general property of *any* potential which is purely attractive. Do this by using the trial function

   $$\psi(x) = e^{-\alpha x^2}, \tag{3}$$

   and showing that $\alpha$ can always be so chosen that the expectation value of the energy $E(\alpha)$ is negative. (Why does this constitute a proof?)

4. A system with unperturbed eigenstates and energies $\phi_n$ and $E_n$, respectively, is subject to a time-dependent perturbation

$$V(t) = \frac{A}{\sqrt{\pi}\tau} e^{-t^2/\tau^2}, \tag{4}$$

where $A$ is a time-*independent* operator.

(a) If initially ($t = -\infty$) the system is in its ground state $\phi_0$, show that, to first order, the probability amplitude that at $t = \infty$ the system will be in its $m$th state ($m \neq 0$) is

$$c_m = -\frac{iA_{m0}}{\hbar} e^{-i(E_0 - E_m)^2 \tau^2 / 4\hbar^2}, \tag{5}$$

where

$$A_{m0} = \langle \phi_m | A | \phi_0 \rangle. \tag{6}$$

(b) The limit $\tau^2 (E_1 - E_0)^2 / 4\hbar^2 \gg 1$ is called the *adiabatic limit*. Discuss the behavior of the system as $t$ progresses from minus to plus infinity in the adiabatic limit. Why do all transition probabilities tend to zero in this limit?

(c) Next consider the limit of an impulsive perturbation, $\tau = 0$. Show that the probability $P$ that the system makes any transition whatsoever out of the ground state is

$$P = \frac{1}{\hbar^2} \Big[ (A^2)_{00} - (A_{00})^2 \Big] = \frac{1}{\hbar^2} \Big[ \langle \phi_0 | A^2 | \phi_0 \rangle - \langle \phi_0 | A | \phi_0 \rangle^2 \Big]. \tag{7}$$

Hint: Find the transition probability to the $m$th state and sum over all excited states using the methods of matrix algebra.

(d) Show that the impulsive perturbation of part (c) is equivalent to

$$V(t) = A\delta(t). \tag{8}$$