Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #20

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Outline

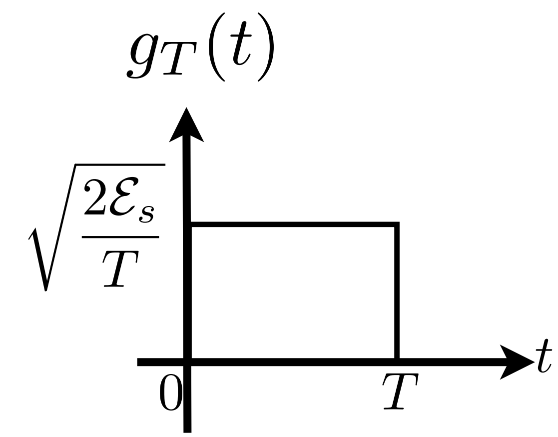
- PSK (phase-shift keying) modulation

Phase-Shifted Keying (PSK) Modulation

- PSK signal is in the form of

$$u_m(t) = g_T(t)\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1, \;\; 0 \le t \le T$$

$$= \quad \Re[s_{ml}(t)e^{j2\pi f_c t}]$$



- PSK signal has its equivalent low-pass signal

$$s_{ml}(t) = g_T(t)e^{j\theta_m}, \quad m = 0, 1, \ldots, M-1$$

where $\quad \theta_m = \frac{2\pi m}{M}$

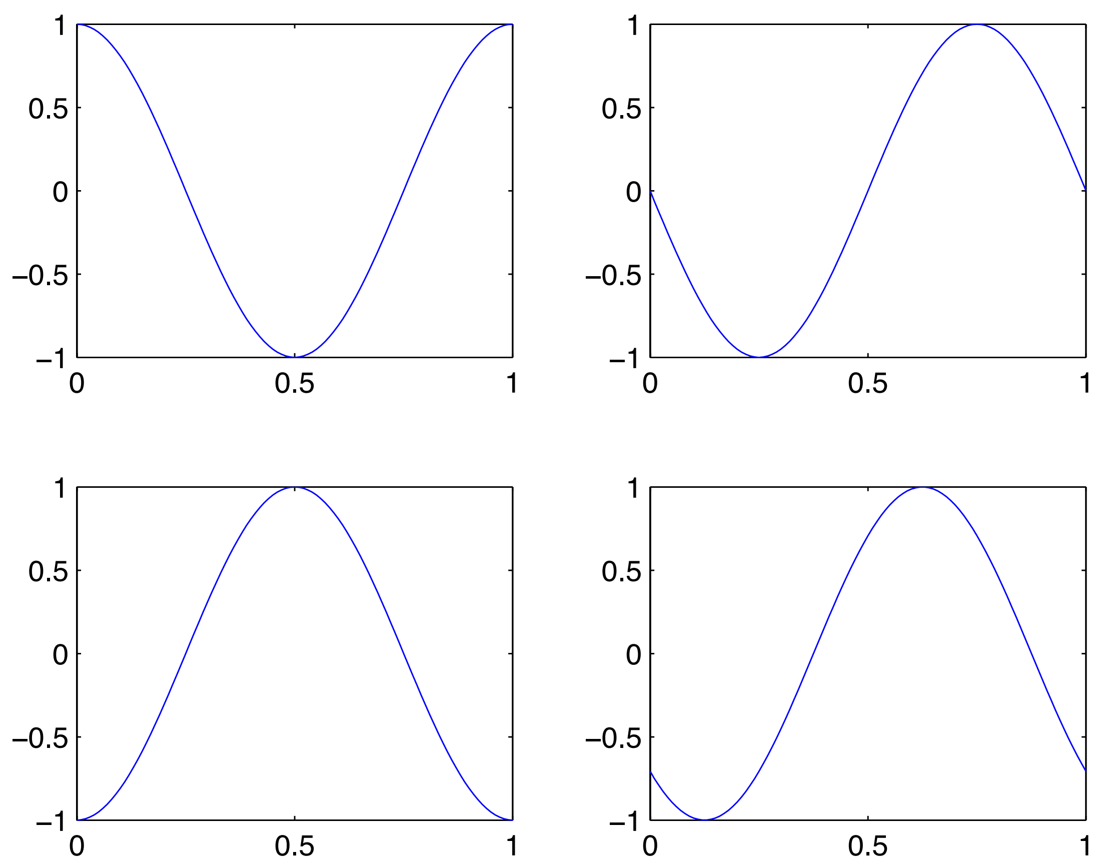
4-ary PSK or (QPSK)

Bit	$\theta_m = \frac{2\pi m}{M}$
0 0	0
0 1	$\pi/2$
1 1	π
1 0	$3\pi/2$

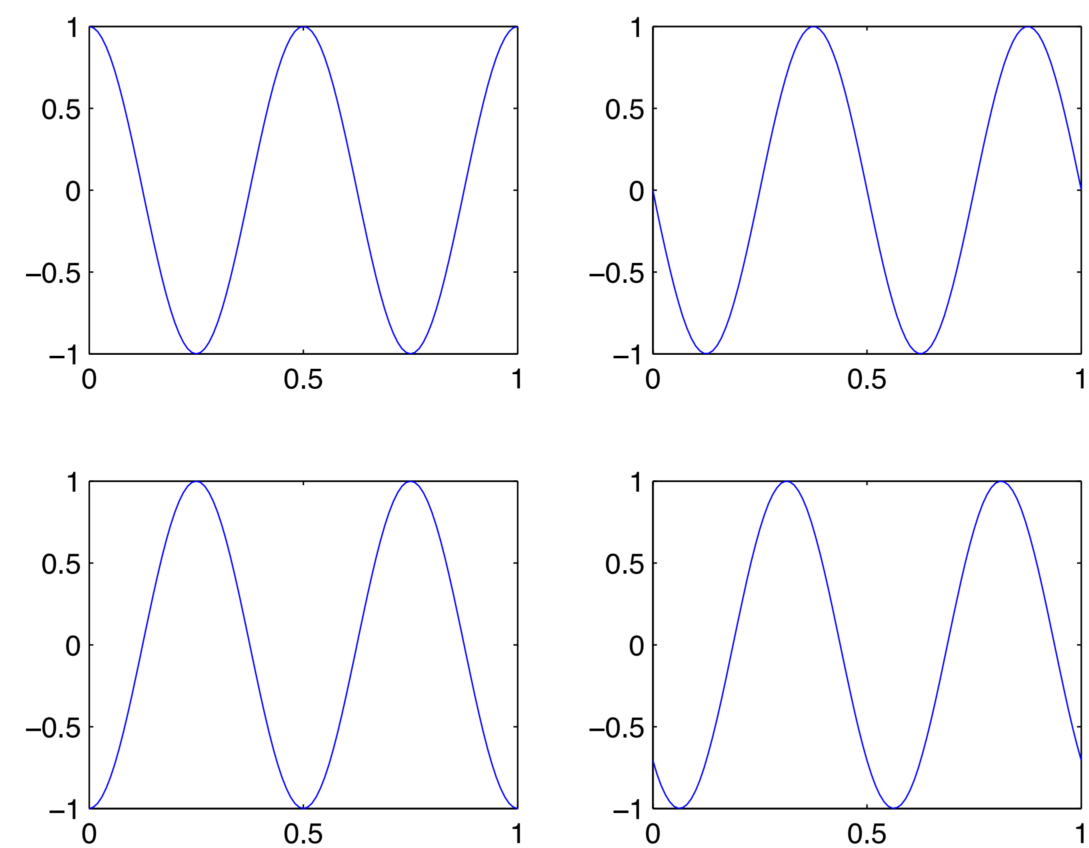
$$u_m(t) = g_T(t)\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right)$$

$$m = 0, 1, \ldots, M-1, \;\; 0 \le t \le T$$

$f_c = 1, \; T = 1$



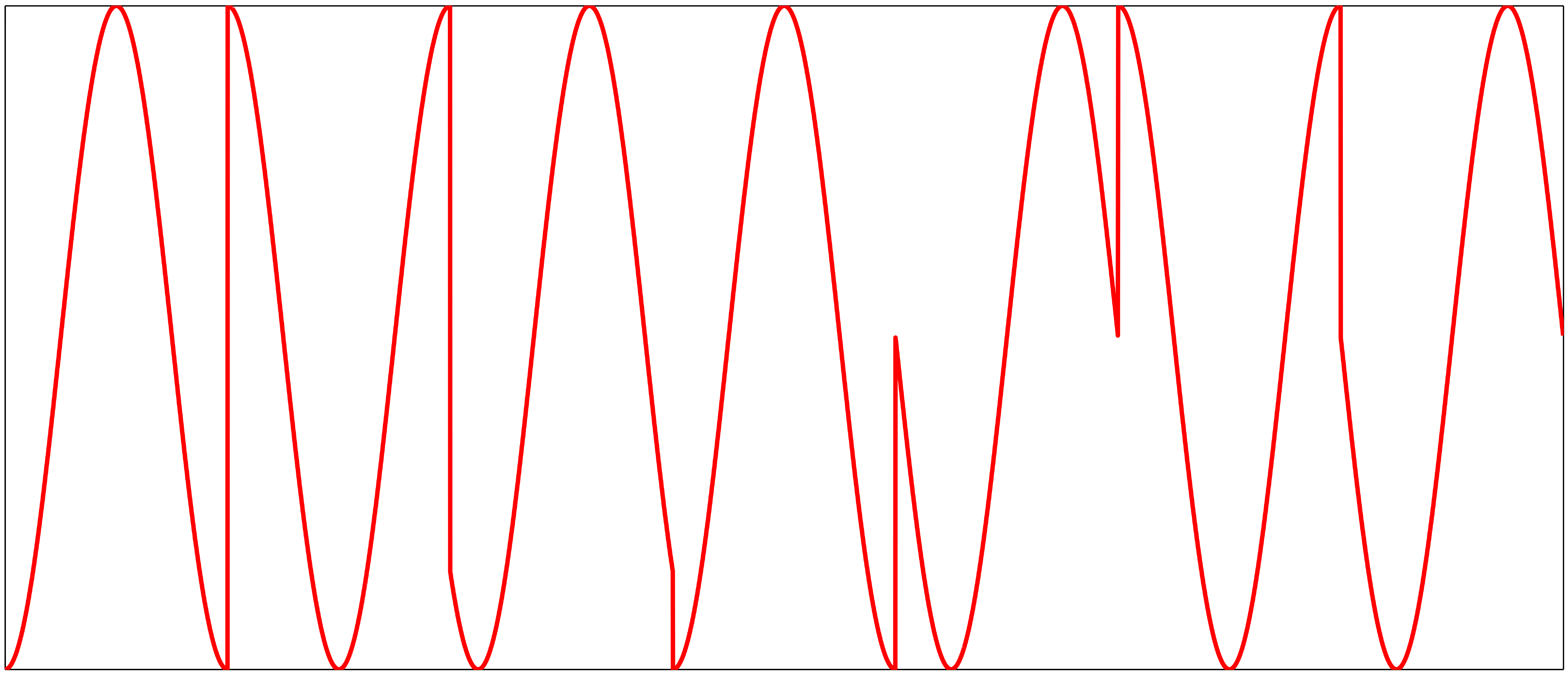
$f_c = 2, \; T = 1$



$$u_m(t) = \sqrt{\frac{2\mathcal{E}_s}{T}} \cos\left(2\pi f_c t + \frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1, \quad 0 \le t \le T$$

Bit stream: 1 0 0 0 1 1 1 0 0 1 0 0 0 1

s_4 s_1 s_3 s_4 s_2 s_1 s_2



- PSK signal waveform

$$u_m(t) = \sqrt{\frac{2\mathcal{E}_s}{T}} \cos\left(2\pi f_c t + \frac{2\pi m}{M}\right)$$

$$= \Re[s_{ml}(t) e^{j2\pi f_c t}]$$

where

$$s_{ml}(t) = g_T(t) e^{j\theta_m}, \quad m = 0, 1, \ldots, M-1$$

where $\theta_m = \frac{2\pi m}{M}$

$$s_{ml}(t) = g_T(t)\cos(\theta_m) + j g_T(t)\sin(\theta_m)$$

Recall

$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$ if $s_l(t) = x(t) + jy(t)$

- Then

$$u_m(t) = g_T(t) A_{mc} \cos(2\pi f_c t) - g_T(t) A_{ms} \sin(2\pi f_c t)$$

where

$$\begin{aligned} A_{mc} &= \cos\left(\frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1 \\ A_{ms} &= \sin\left(\frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1 \end{aligned}$$

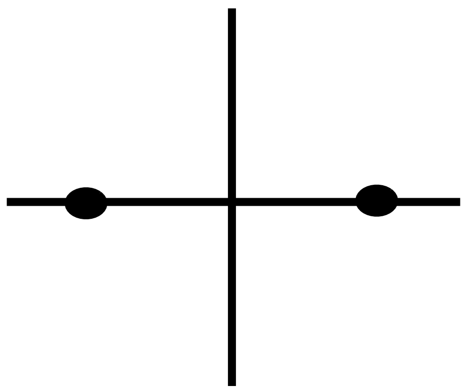
- Geometrical representation

$$u_m(t) = s_{m1}\psi_1(t) + s_{m2}\psi_2(t) \quad \Rightarrow \quad \mathbf{s}_m = \left(\sqrt{\mathcal{E}_s}\cos 2\pi m/M, \quad \sqrt{\mathcal{E}_s}\sin 2\pi m/M\right)$$

where

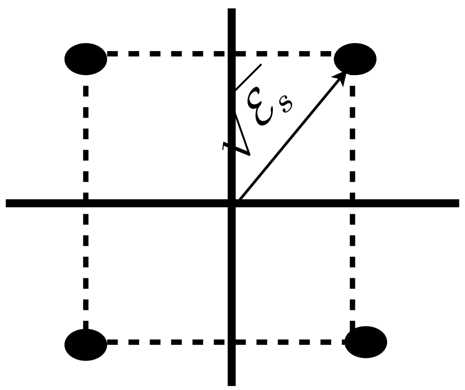
$$\begin{aligned} \psi_1(t) &= \sqrt{\frac{1}{\mathcal{E}_s}} g_T(t) \cos(2\pi f_c t) \\ \psi_2(t) &= -\sqrt{\frac{1}{\mathcal{E}_s}} g_T(t) \sin(2\pi f_c t) \end{aligned}$$

- PSK signal constellations



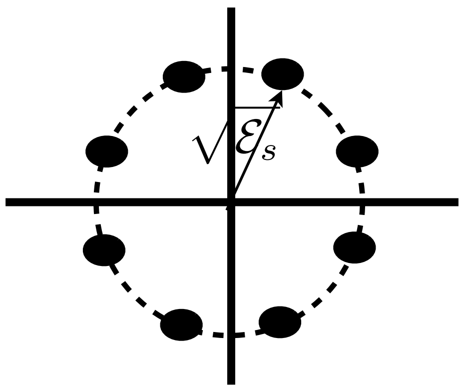
$M=2$

BPSK



$M=4$

QPSK



$M=8$

8-PSK

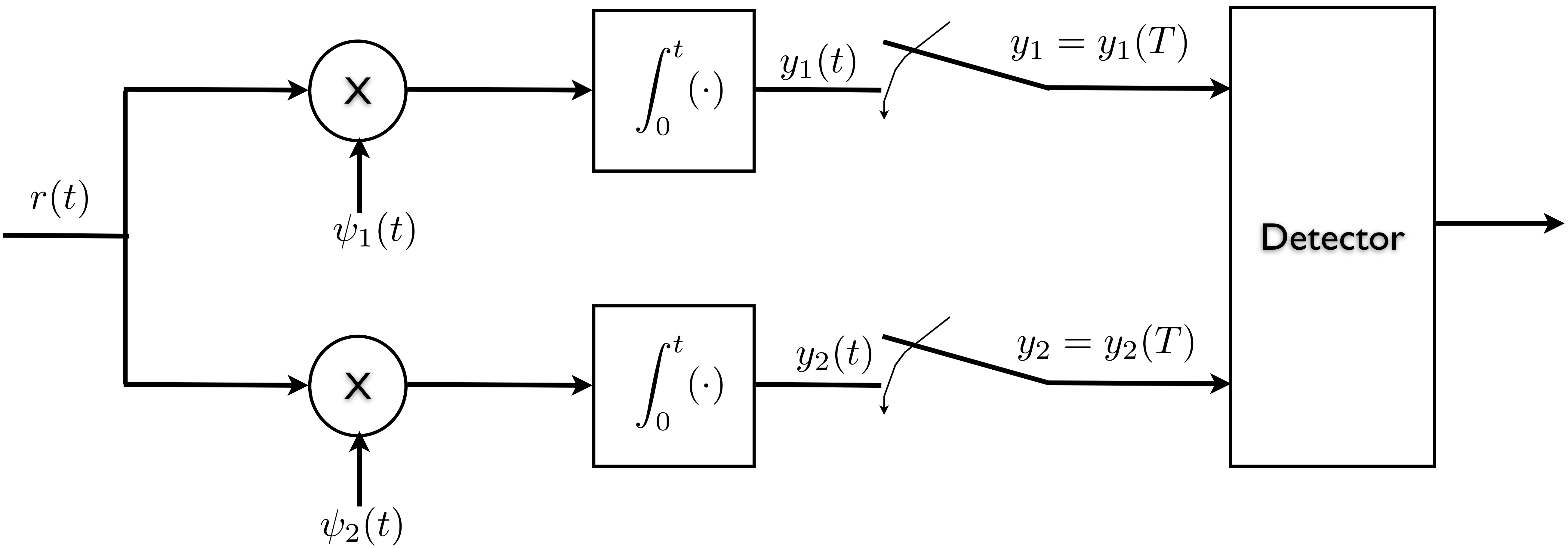
Euclidean distance

$$\begin{aligned} d_{mn} &= \sqrt{||\mathbf{s}_m - \mathbf{s}_n||^2} \\ &= \sqrt{2\mathcal{E}_s\left(1-\cos\frac{2\pi(m-n)}{M}\right)} \end{aligned}$$

Minimum Euclidean distance

$$d_{\min} = \sqrt{2\mathcal{E}_s\left(1-\cos\frac{2\pi}{M}\right)} = 2\sqrt{\mathcal{E}_s}\sin\frac{\pi}{M}$$

Demodulation of PSK signals



$$r(t) = u_m(t) + n(t)$$

$$y_k = y_k(T) = \int_0^T r(t)\psi_k(t)\,dt, \quad k = 1, 2$$

$$= s_{mk} + n_k$$

Noise components

$$n_1 = \int_0^T n(t)\psi_1(t)\,dt, \quad \text{and} \quad n_2 = \int_0^T n(t)\psi_2(t)\,dt$$

Mean of n_1

$$E[n_1] \quad = \quad E\left[\int_0^T n(t)\psi_1(t)\,dt\right] = \int_0^T E[n(t)]\psi_1(t)\,dt = 0$$

Variance of n_1

$$\text{var}[n_1] = E[n_1^2] = E\left[\int_0^T \int_0^T n(t)n(\lambda)\psi_1(t)\psi(\lambda)\,dtd\lambda\right] = \frac{N_0}{2}$$

In a similar way, we can obtain

$$E[n_2] = 0 \quad \text{and} \quad \text{var}[n_2] = E[n_2^2] = \frac{N_0}{2}$$

- Independence between

$$E[n_1 n_2] = E\left[\int_0^T \int_0^T n(t)n(\lambda)\psi_1(t)\psi_2(\lambda)\, dt d\lambda\right]$$

$$= \int_0^T \int_0^T E\left[n(t)n(\lambda)\right]\psi_1(t)\psi_2(\lambda)\, dt d\lambda$$

$$= \frac{N_0}{2}\int_0^T \psi_1(t)\psi_2(t)\, dt$$

$$= 0$$

- Output of two correlators

$$\begin{aligned} \mathbf{y} &= \mathbf{s}_m + \mathbf{n} \\ &= \left(\sqrt{\mathcal{E}_s}\cos 2\pi m/M + n_1, \quad \sqrt{\mathcal{E}_s}\sin 2\pi m/M + n_2\right) \end{aligned}$$

- Optimum detector
    - Choose the signal which gives the maximum correlation metrics

$$C(\mathbf{y}, \mathbf{s}_m) = \mathbf{y} \cdot \mathbf{s}_m, \quad m = 0, 1, \ldots, M-1$$

    - Equivalently, we select the signal from the set $\{\mathbf{s}_m\}$ whose phase is closest to Θ, where

$$\Theta = \tan^{-1} \frac{y_2}{y_1}$$

Probability of Error for Phase-Coherent PSK Modulation

- Assume $\mathbf{s}_1$ is transmitted.

$$\mathbf{s}_1 = (\sqrt{\mathcal{E}_s}, \quad 0)$$

- Received signal vector

$$\begin{aligned} y_1 &= \sqrt{\mathcal{E}_s} + n_1; \\ y_2 &= n_2. \end{aligned}$$

$$\begin{aligned} y_1 &\sim \mathcal{N}\left(\sqrt{\mathcal{E}_s}, \frac{N_0}{2}\right) \\ y_2 &\sim \mathcal{N}\left(0, \frac{N_0}{2}\right) \end{aligned}$$

  - Joint PDF of y_1 and y_2

$$f_{\mathbf{y}}(y_1, y_2) = \frac{1}{2\pi\sigma_y^2} e^{-\frac{(y_1 - \sqrt{\mathcal{E}_s})^2 + y_2^2}{2\sigma_y^2}},$$

where $\sigma_y^2 = \frac{N_0}{2}$

Transformation of y_1 and y_2

$$V = \sqrt{y_1^2 + y_2^2},$$
$$\Theta = \tan^{-1}\frac{y_2}{y_1}.$$

- Joint PDF of V and Θ

$$f_{V,\Theta}(v,\theta) = \frac{v}{2\pi\sigma_y^2} e^{-(v^2+\mathcal{E}_s-2\sqrt{\mathcal{E}_s}v\cos\theta)/2\sigma_y^2}.$$

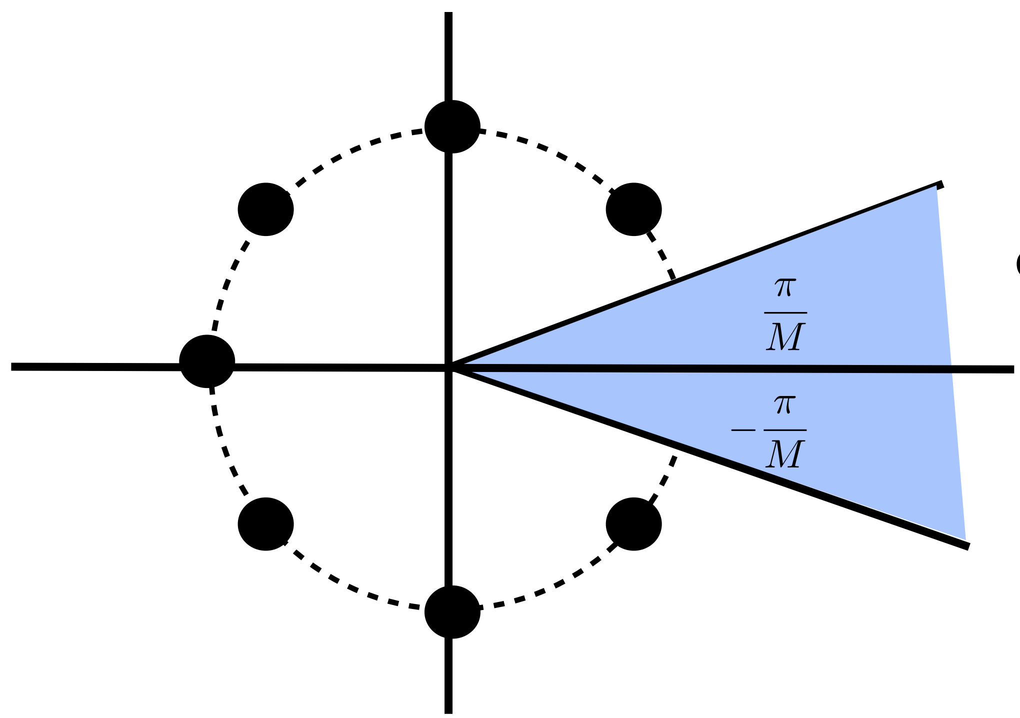
- Marginal PDF of Θ

$$f_\Theta(\theta) = \int_0^\infty f_{V,\Theta}(v,\theta)\,dv = \frac{1}{2\pi} e^{-\rho_s \sin^2\theta} \int_0^\infty v e^{-(v-\sqrt{2}\rho_s\cos\theta)^2/2}\,dv$$

where $\rho_s = \dfrac{\mathcal{E}_s}{N_0}$ is the SNR per symbol.

- Symbol error rate

$$P_M = 1 - \int_{-\pi/M}^{\pi/M} f_\Theta(\theta)\, d\theta.$$



- BER of BPSK

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

- SER of QPSK

$$P_c = (1 - P_2)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2,$$

$$P_4 = 1 - P_c$$

$$P_4 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right] \approx 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right).$$

- Approximation of SER for large $\mathcal{E}_s/N_0$

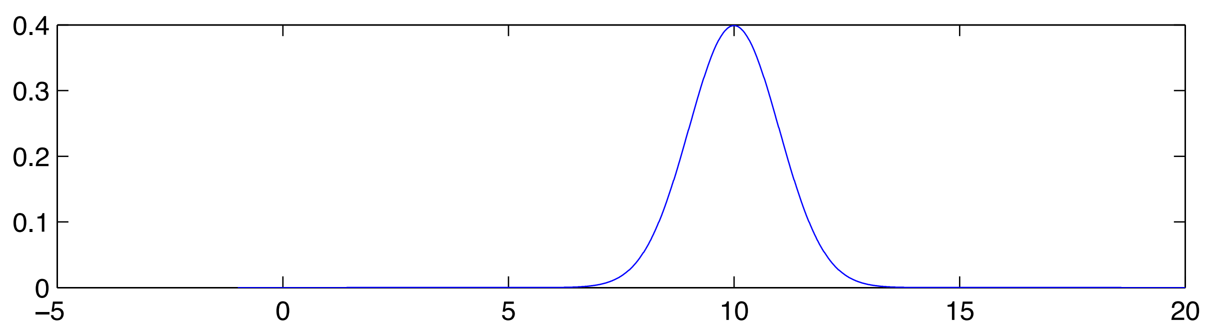
$$f_\Theta(\theta) = \frac{1}{2\pi} e^{-\rho_s \sin^2\theta} \int_0^\infty v e^{-(v-\sqrt{2}\rho_s\cos\theta)^2/2}\, dv$$

$$\approx \sqrt{\frac{\rho_s}{\pi}} \cos\theta e^{-\rho_s \sin^2\theta}$$

Note

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(v-m)^2}{2}}\, dv = 1 \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v e^{-\frac{(v-m)^2}{2}}\, dv = m$$

For high SNR per symbol, $\sqrt{2}\rho_s\cos\theta >> 1$



$$\int_{-\infty}^{\infty} v e^{-(v-m)^2/2}\, dv \approx \int_0^\infty v e^{-(v-m)^2/2}\, dv = \sqrt{2\pi} m$$

$$
\begin{aligned}
P_M &\approx 1 - \int_{-\pi/M}^{\pi/M} \sqrt{\frac{\rho_s}{\pi}} \cos\theta e^{-\rho_s \sin^2\theta}\, d\theta \\
&\approx \frac{2}{\sqrt{2\pi}} \int_{\sqrt{2\rho_s}\sin\pi/M}^{\infty} e^{-u^2/2}\, du \\
&= 2Q\left(\sqrt{2\rho_s}\sin\frac{\pi}{M}\right) \\
&= 2Q\left(\sqrt{2k\sin^2\left(\frac{\pi}{M}\right)\frac{\mathcal{E}_b}{N_0}}\right) \\
&\approx 2Q\left(\sqrt{\frac{2\pi^2\log_2 M}{M^2}\frac{\mathcal{E}_b}{N_0}}\right)
\end{aligned}
$$

change in variable:

$$u = 2\sqrt{\rho_s}\sin^2\theta$$

where $k = \log_2 M$

$\rho_s = k\rho_b$

$\sin\frac{\pi}{M} \approx \frac{\pi}{M}$ for large M