Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #16

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Outline

- Average symbol rate
  - M-ary orthogonal signals
- A Union bound on the probability of error
- Synchronization

Probability of Error for M-ary Orthogonal Signals

- Each of M-ary orthogonal signals has equal energy.

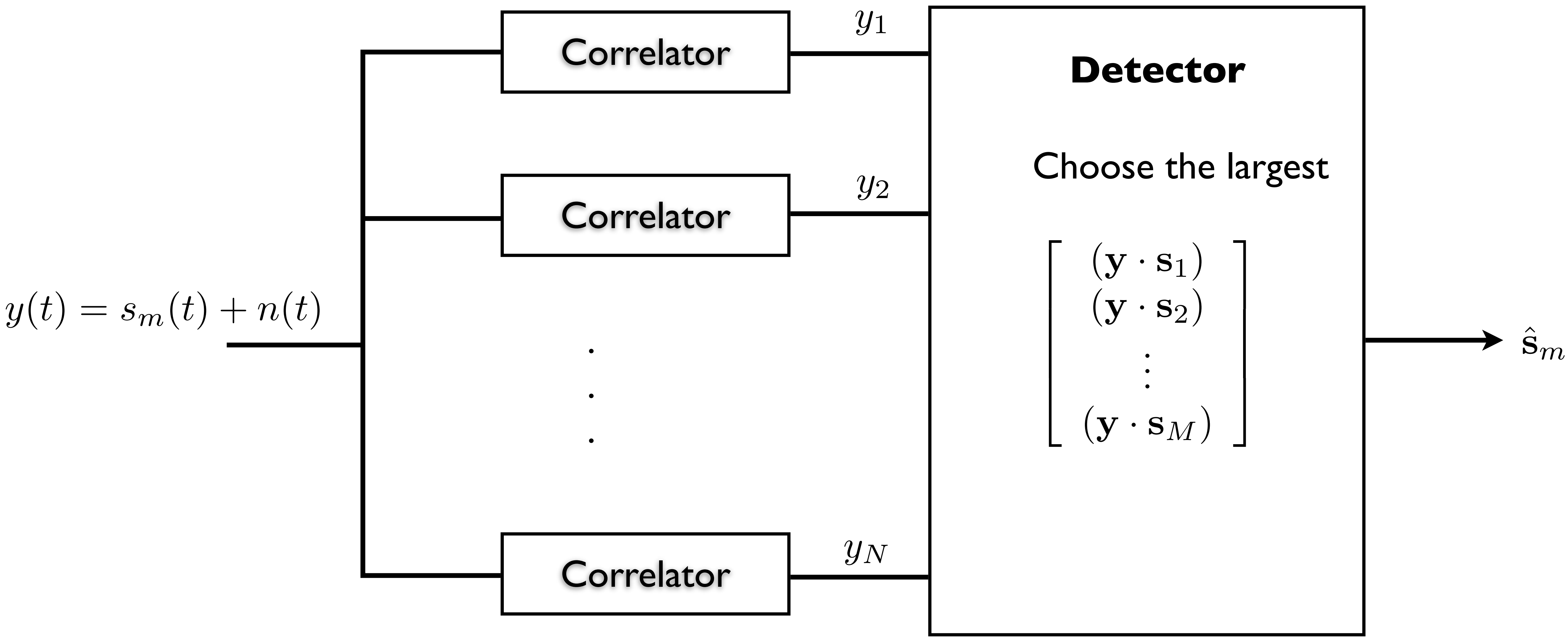
- For equal energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between the received vector $\mathbf{y}$ and each of the M possible transmitted signal vectors $\{\mathbf{s}_m\}$, i.e.,

$$C(y, \mathbf{s}_m) = \mathbf{y} \cdot \mathbf{s}_m = \sum_{k=1}^{M} y_k s_{mk}, \quad m = 1, 2, \ldots, M.$$

- To evaluate the probability of error, let us assume that the signal $\mathbf{s}_1$ is transmitted. Then the vector at the input to the detector is

$$\mathbf{y} = \left(\sqrt{\mathrm{E}_s} + n_1, n_2, n_3, \ldots, n_M\right),$$

  - where $n_1, n_2, n_3, \ldots, n_M$ are zero mean, mutually statistically independent Gaussian random variables with equal variance $N_0/2$.



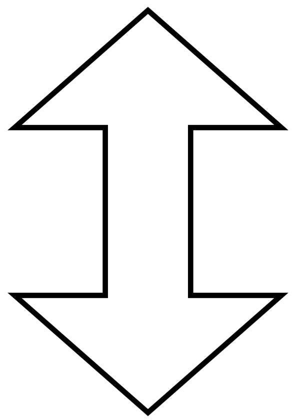
$N = M$ for orthogonal modulation

Signal representation in vector form

$$\begin{aligned}
\mathbf{s}_1 &= [\sqrt{\mathcal{E}_s}\ 0\ 0\ \cdots\ 0] \\
\mathbf{s}_2 &= [0\ \sqrt{\mathcal{E}_s}\ 0\ \cdots\ 0] \\
&\vdots \\
\mathbf{s}_M &= [0\ 0\ \cdots\ 0\ \sqrt{\mathcal{E}_s}]
\end{aligned}$$

Choose the largest

$$
\begin{aligned}
C(\mathbf{y}, \mathbf{s}_1) &= \sqrt{\mathcal{E}_s}(\sqrt{\mathcal{E}_s} + n_1); \\
C(\mathbf{y}, \mathbf{s}_2) &= \sqrt{\mathcal{E}_s} n_2; \\
&\vdots \\
C(\mathbf{y}, \mathbf{s}_M) &= \sqrt{\mathcal{E}_s} n_M.
\end{aligned}
$$



Choose the largest

$$
\begin{bmatrix} \sqrt{\mathcal{E}_s} + n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix}
$$

- Cross-correlation metric

$$\begin{aligned} C(\mathbf{y}, \mathbf{s}_1) &= \sqrt{\mathcal{E}_s}(\sqrt{\mathcal{E}_s} + n_1); \\ C(\mathbf{y}, \mathbf{s}_2) &= \sqrt{\mathcal{E}_s} n_2; \\ &\vdots \\ C(\mathbf{y}, \mathbf{s}_M) &= \sqrt{\mathcal{E}_s} n_M. \end{aligned}$$

  - Note that we can eliminate the scale factor $\sqrt{\mathcal{E}_s}$ for the comparisons.

- PDF of the first correlator output with the elimination of $\sqrt{\mathcal{E}_s}$.

$$f(y_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1 - \sqrt{\mathcal{E}_s})^2}{N_0}}$$

  - PDF's of the other M-1 correlator outputs

$$f(y_m) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{y_m^2}{N_0}}, \quad m = 2, 3, \ldots, M$$

- Correct decision event when $s_1(t)$ is transmitted

$$n_2 < y_1,$$

$$n_3 < y_1,$$

$$\vdots \quad \text{and}$$

$$n_M < y_1$$

- Correct decision probability when $s_1(t)$ is transmitted

$$P_{c|s_1} = \int_{-\infty}^{\infty} \Pr[n_2 < y_1,\, n_3 < y_1,\, \ldots,\, n_M < y_1 \mid s_1] f_{y_1}(y_1)\, dy_1$$

- Correct decision probability when $s_1(t)$ is transmitted

$$P_{c|s_1} = \int_{-\infty}^{\infty} \Pr[n_2 < y_1,\, n_3 < y_1,\, \ldots,\, n_M < y_1 \mid s_1] f_{y_1}(y_1)\, dy_1$$

  - For equally probable case,

$$P_c = \int_{-\infty}^{\infty} \Pr(n_2 < y_1,\, n_3 < y_1,\, \ldots,\, n_M < y_1 \mid y_1) f_{y_1}(y_1)\, dy_1$$

  - Note that

$$\begin{aligned} \Pr(n_M < y_1 | y_1) &= \int_{-\infty}^{y_1} f(y_m)\, dy_m, \quad m = 2, 3, \ldots, M \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{\frac{2y_1^2}{N_0}}} e^{-\frac{y_m^2}{2}}\, dy_m \\ &= 1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right). \end{aligned}$$

- Probability of correct decision

$$P_c = \int_{-\infty}^{\infty} \left[1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}} \right) \right]^{M-1} f_{y_1}(y_1)\, dy_1$$

- Probability of symbol error (Symbol error rate)

$$P_M = 1 - \int_{-\infty}^{\infty} \left[1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right)\right]^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-(y-\sqrt{\mathcal{E}_s})^2/N_0}\, dy_1$$

Change of variable $x = \frac{2y_1^2}{N_0}$

Then we have

$$P_M = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - Q\left(x\right)\right]^{M-1}\, e^{-(x-\sqrt{2\mathcal{E}_s/N_0})}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{1 - \left[1 - Q\left(x\right)\right]^{M-1}\right\}\, e^{-(x-\sqrt{2\mathcal{E}_s/N_0})}\, dx$$

- Symbol error rate

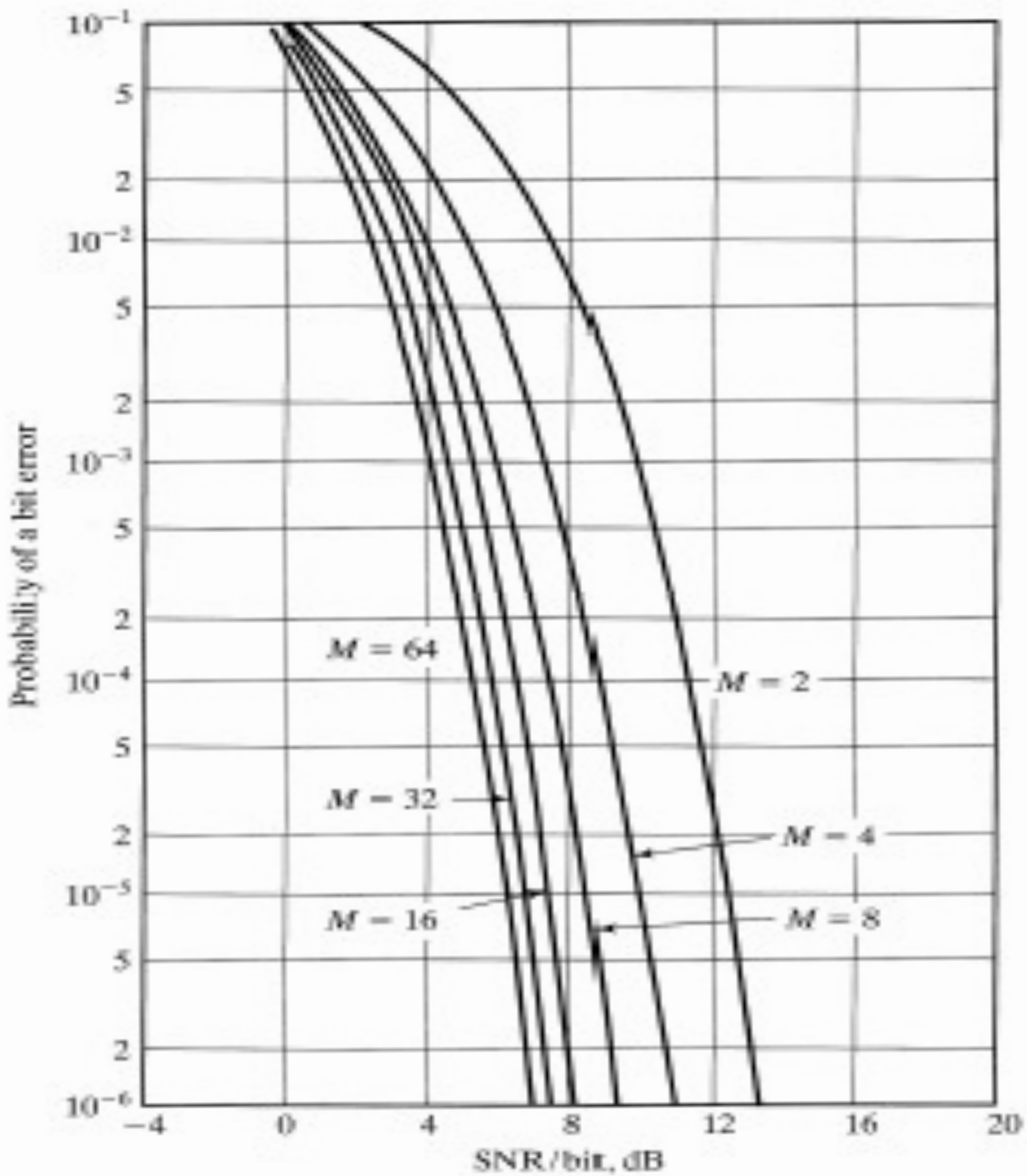


Figure 8.46 Probability of a bit error for optimum detection of orthogonal signals.

[Proakis and Salehi, "*Essentials of Communication Systems Engineering*", Prentice Hall, p.438]

A Union Bound on the Probability of Error

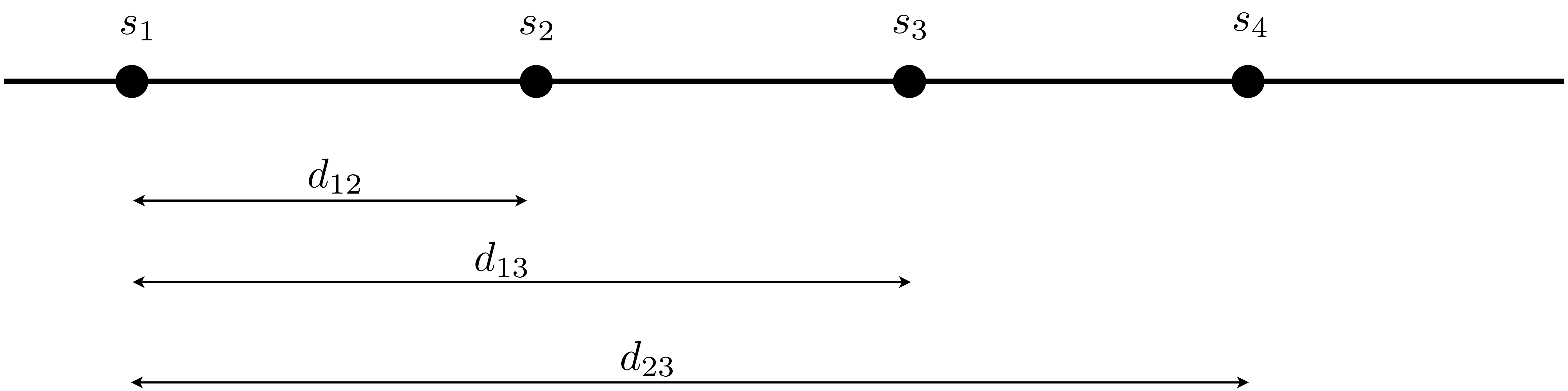
- BER of binary antipodal signals

$$P_2 = Q\left(\frac{d}{\sqrt{2N_0}}\right) \qquad \text{where} \quad d^2 = \int_{-\infty}^{\infty} (s_1(t) - s_2(t)^2 \, dt$$

- SER of M-ary PAM

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\mathcal{E}_{bav}}{(M^2-1)N_0}}\right).$$

4-ary PAM



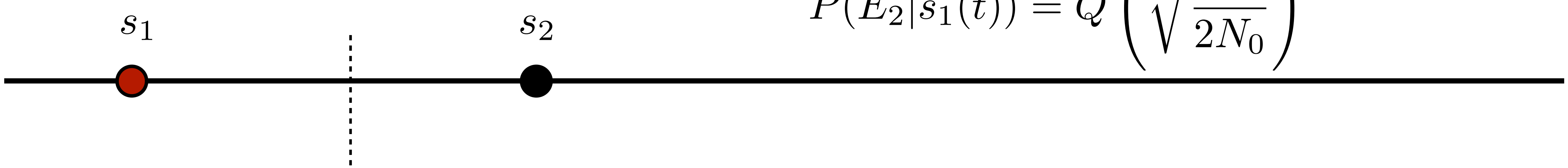
Define minimum distance

$$d_{\min} = \min_{\substack{1 \le m, m' \le M \\ m' m}} d_{mm'}.$$

Assume $s_1(t)$ is transmitted.

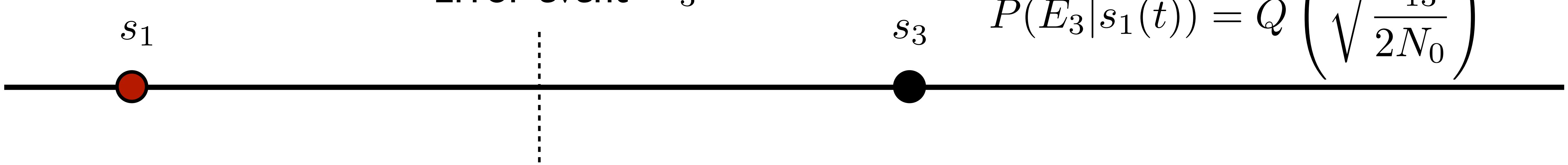
Error event E_2

$$P(E_2|s_1(t)) = Q\left(\sqrt{\frac{d_{12}}{2N_0}}\right)$$



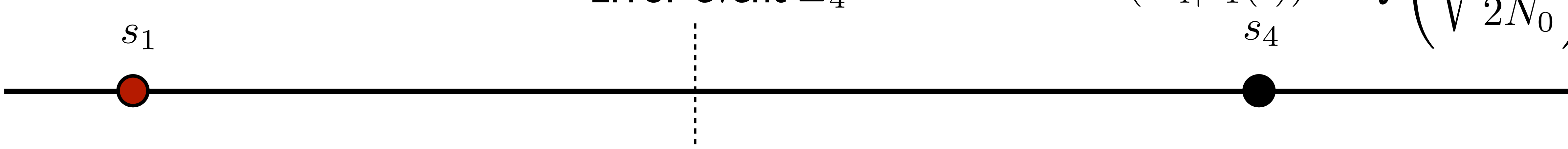
Error event E_3

$$P(E_3|s_1(t)) = Q\left(\sqrt{\frac{d_{13}}{2N_0}}\right)$$



Error event E_4

$$P(E_4|s_1(t)) = Q\left(\sqrt{\frac{d_{14}}{2N_0}}\right)$$



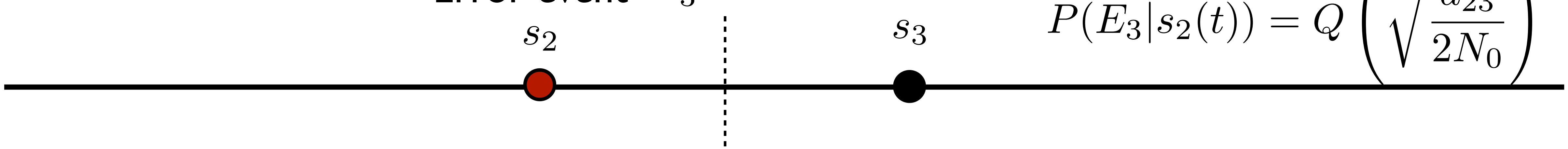
$$P(e|s_1(t)) = P(E_2 \cup E_3 \cup E_4) \leq P(E_2) + P(E_3) + P(E_4)$$

Assume $s_2(t)$ is transmitted.

Error event E_1

$$P(E_1|s_2(t)) = Q\left(\sqrt{\frac{d_{21}}{2N_0}}\right)$$

Error event E_3



$$P(E_3|s_2(t)) = Q\left(\sqrt{\frac{d_{23}}{2N_0}}\right)$$

Error event E_4

$$P(E_4|s_2(t)) = Q\left(\sqrt{\frac{d_{24}}{2N_0}}\right)$$

$$P(e|s_2(t)) = P(E_1 \cup E_3 \cup E_4) \leq P(E_1) + P(E_3) + P(E_4)$$

Assume that $s_m(t)$ is transmitted in an *M*-ary equiprobable signaling.

- A Union bound

$$P_m = P(\text{error}|s_m(t)\text{ sent}) = P\left(\bigcup_{i=1, i\neq m}^{M} E_i \middle| s_m(t)\text{ sent}\right) \leq \sum_{i=1, i\neq m} P(E_i|s_m(t)\text{ sent}).$$

where E_i : event that message i is detected at the receiver

- Necessary condition for $s_i(t)$ to be detected at the receiver when signal $s_m(t)$ is sent that $\mathbf{y}$ closer than to $\mathbf{s}_i$ than to $\mathbf{s}_m$:

$$P(E_i|s_m(t)\text{ sent}) \leq P[D(\mathbf{y},\mathbf{s}_m) < D(\mathbf{y},\mathbf{s}_m)]$$

$$= Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right)$$

$$\Longrightarrow \qquad P(E_i|s_m(t)\text{ sent}) \leq Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right)$$

- Hence, we have an upper bound

$$P_m \leq \sum_{i=1, i\neq m}^{M} Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right).$$

- Let us define the minimum distance as

$$d_{\min} = \min_{\substack{1 \le m, m' \le M \\ m'm}} d_{mm'}.$$

  - Thus for any *m*, $d_{mi} \ge d_{\min}$, so that we have

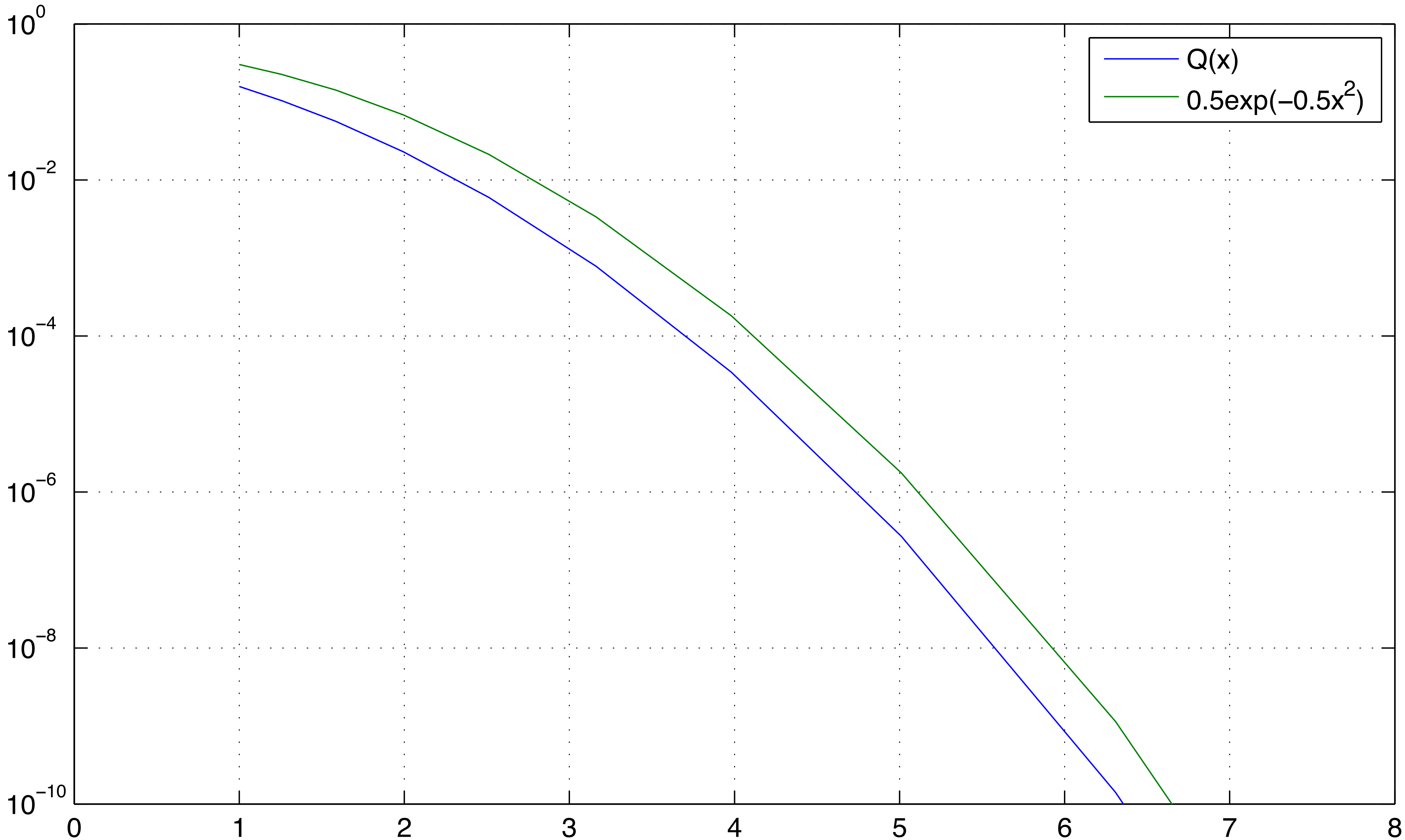
$$Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right) \le Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

  - Hence, we obtain an upper bound as

$$P_m \le \sum_{i=1, i\neq m}^{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

  - Also note that

$$Q(x) \le \frac{1}{2}e^{-\frac{x^2}{2}}$$



Q(x)
0.5exp(−0.5x2)
10^0
10^−2
10^−4
10^−6
10^−8
10^−10
0
1
2
3
4
5
6
7
8

- Therefore, we can rewrite the upper bound as

$$P_m \leq \sum_{i=1, i\neq m}^{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \leq \frac{M-1}{2} e^{-\frac{d_{\min}^2}{4N_0}}$$

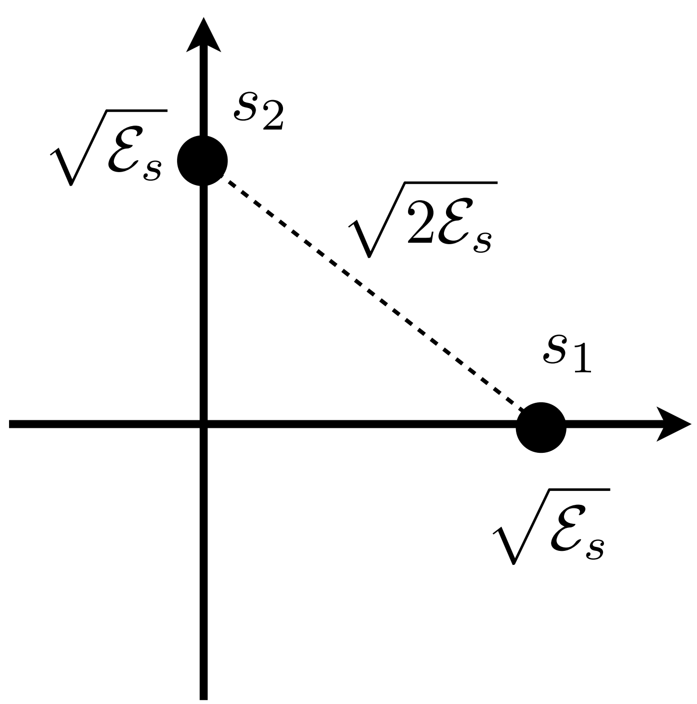
A union bound on the error probability of M-ary orthogonal signaling

- M-ary orthogonal signals are equidistant with

$$d_{mm'}^2 = ||\mathbf{s}_m - \mathbf{s}_{m'}||^2 = 2\mathcal{E}_s$$

- Therefore,

$$d_{\min} = \sqrt{2\mathcal{E}_s}.$$



$$P_2 = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right) \leq \frac{1}{2}e^{-\frac{\mathcal{E}_s}{2N_0}}$$

- Previously we obtain an upper bound as

$$P_m \le \sum_{i=1, i\neq m}^{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

where $d_{\min} = \sqrt{2\mathcal{E}_s}$ for orthogonal signals

- Union bound

$$P_M \le \frac{M-1}{2} e^{-\frac{\mathcal{E}_s}{2N_0}} \le M e^{-\frac{\mathcal{E}_s}{2N_0}}.$$

- Using $M = 2^k$ and $\mathcal{E}_s = k\mathcal{E}_b$, we have

$$P_M \le 2^k e^{-k\mathcal{E}_b/2N_0} = e^{-k(\mathcal{E}_b/N_0 - 2\ln 2)/2}.$$

$$P_M \le 2^k e^{-k\mathcal{E}_b/2N_0} = e^{-k(\mathcal{E}_b/N_0 - 2\ln 2)/2}.$$

$$\lim_{k\to\infty} P_M = \lim_{k\to\infty} e^{-k(\mathcal{E}_b/N_0 - 2\ln 2)/2} = 0$$

- As $k \to \infty$ or $M \to \infty$, the probability of error approaches zero exponentially, provided that $\mathcal{E}_b/N_0$ is greater than $2\ln 2$, i.e.,

$$\frac{\mathcal{E}_b}{N_0} > 2\ln 2 = 1.39 \approx 1.42\,\mathrm{dB}$$

  - The simple upper bound on the probability of error implies that as long as SNR>1.42dB, we can achieve an arbitrary low P_M.

  - However, this union bound is not a very tight upper bound at low SNR values.

  - In fact, by more elaborate bounding techniques, it can be shown that $P_M \to 0$ as $k \to \infty$ provided that

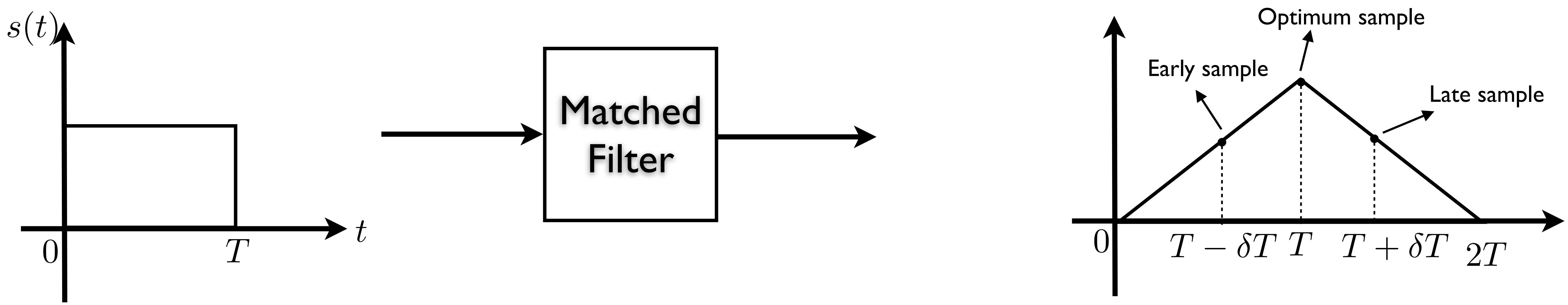
$$\frac{\mathcal{E}_b}{N_0} > \ln 2 = 0.693 \approx -1.6\,\mathrm{dB}$$

- Hence, -1.6 dB is the minimum required SNR/bit to achieve an arbitrarily small probability of error in the limit as $k \to \infty$ ($M \to \infty$).

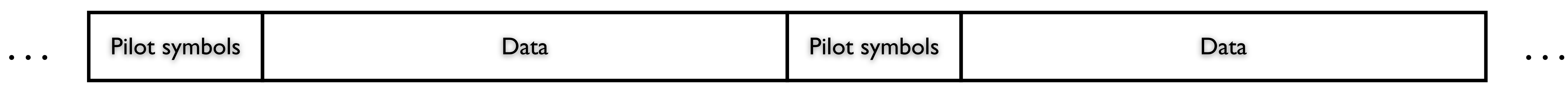
    - This minimum SNR/bit (-1.6 dB) is called the *Shannon limit* for an additive white Gaussian noise channel.

Symbol Synchronization

- Matched filter output



- Data frame structure

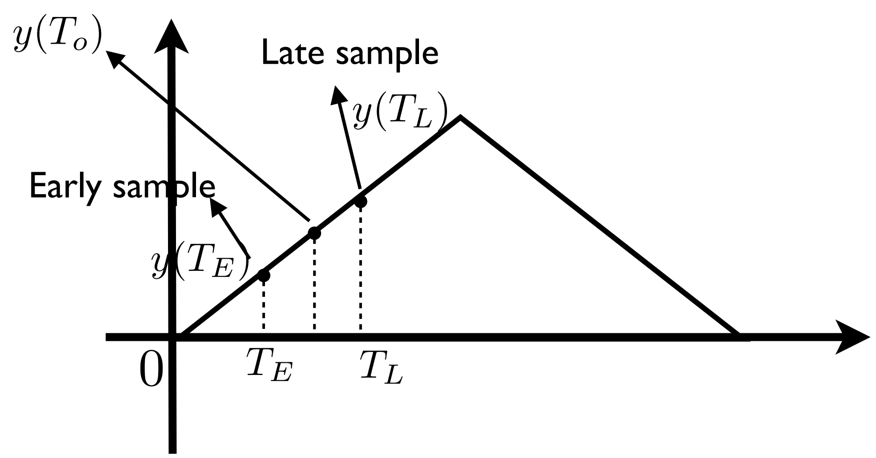


  - Pilot symbols are known symbols and are used for synchronization.

    Example of pilot symbols: $1\ 0\ 1\ 0\ 1\ \cdots\ 0$

Early-Late Gate Synchronizer

- Idea



Check if

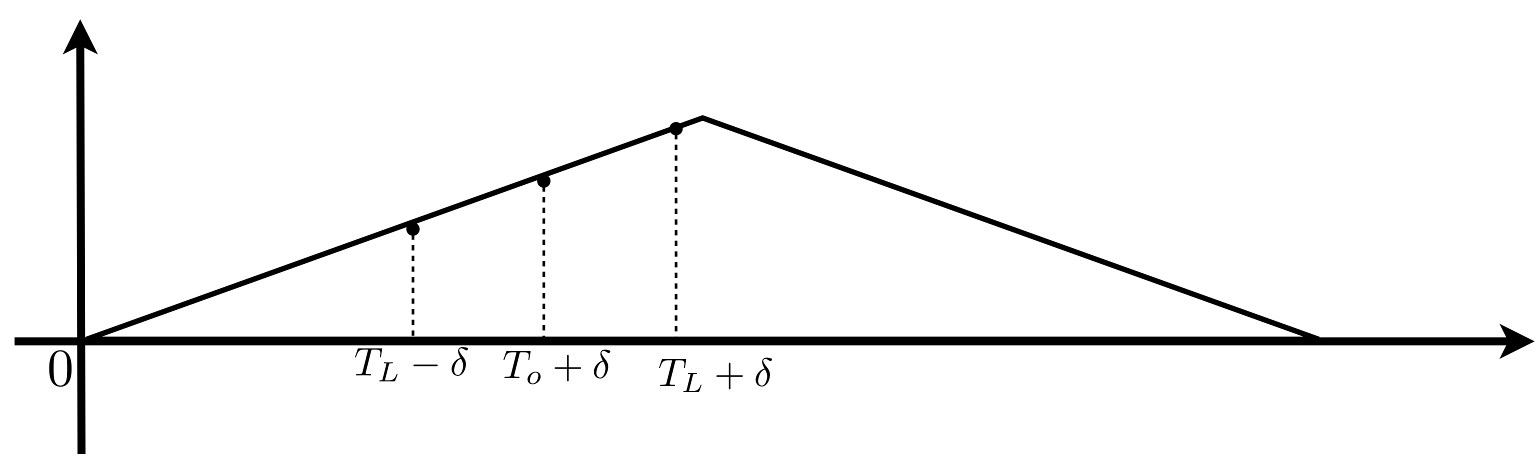
$$|y(T_L) - y(T_E)| > \lambda_{Th}$$ → Sampling error

or

$$|y(T_L) - y(T_E)| < \lambda_{Th}$$ → No sampling error

- Sampling error case, i.e., $|y(T_L) - y(T_E)| > \lambda_{Th}$

If $y(T_L) - y(T_E) > \lambda_{Th}$, move the sampling time by δ into the *late* direction.



If $y(T_E) - y(T_L) > \lambda_{Th}$, move the sampling time by δ into the *early* direction.

Minimum Mean-Square-Error Method

- Transmit signal

$$v(t) = \sum_{n=-\infty}^{\infty} a_n g_T(t - nT)$$

- Received signal at the output of the matched filter at the receiver

$$y(t) = \sum_{n=-\infty}^{\infty} a_n x(t - nT - \tau_0) + w(t)$$

where $x(t) = g_T(t) * g_R(t)$

- MSE

$$\text{MSE} = E\{[y_m(\tau_0) - a_m]^2\}$$

where $$y_m(\tau_0) = \sum_{n=-\infty}^{\infty} a_n x(mT - nT - \tau_0) + w(mT)$$

- The minimum of MSE with respect to the timing phase τ_0 is found by differentiating MSE with respect to τ_0

$$\sum_m [y_m(\tau_0) - a_m] \frac{dy_m(\tau_0)}{d\tau_0} = 0.$$