Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #12

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Outline

- Matched filter
- Optimum detection for binary antipodal signals
- M-ary PAM
- M-ary orthogonal signals

Technique of BER/SER Calculation

Assume a certain signal was transmitted, say $s_1(t)$

Calculate the conditional error probability, $P_2(e|s_1)$

Check if all the conditional probabilities are equal, that is, $P_2(e|s_1) = P_2(e|s_2)$, Then the average probability of error is

$$P_2 = P_2(e|s_1)P(s_1) + P_2(e|s_2)P(s_2)$$

For equally probable case, that is, $P(s_1) = P(s_2) = 1/2$

$$P_2 = \frac{1}{2}(P_2(e|s_1) + P_2(e|s_2)) = P_2(e|s_1)$$

Performance of Binary Orthogonal Signals

- Dimensionality of binary orthogonal signals
  - Two-dimensional transmit signals can be written as

$$s_m(t) = s_{m1}\phi_1(t) + s_{m2}\phi_2(t)$$

$$\mathbf{s}_m = [s_{m1} \;\; s_{m2}], \quad m = 1,\, 2$$

  - The output of the demodulator is also two-dimensional

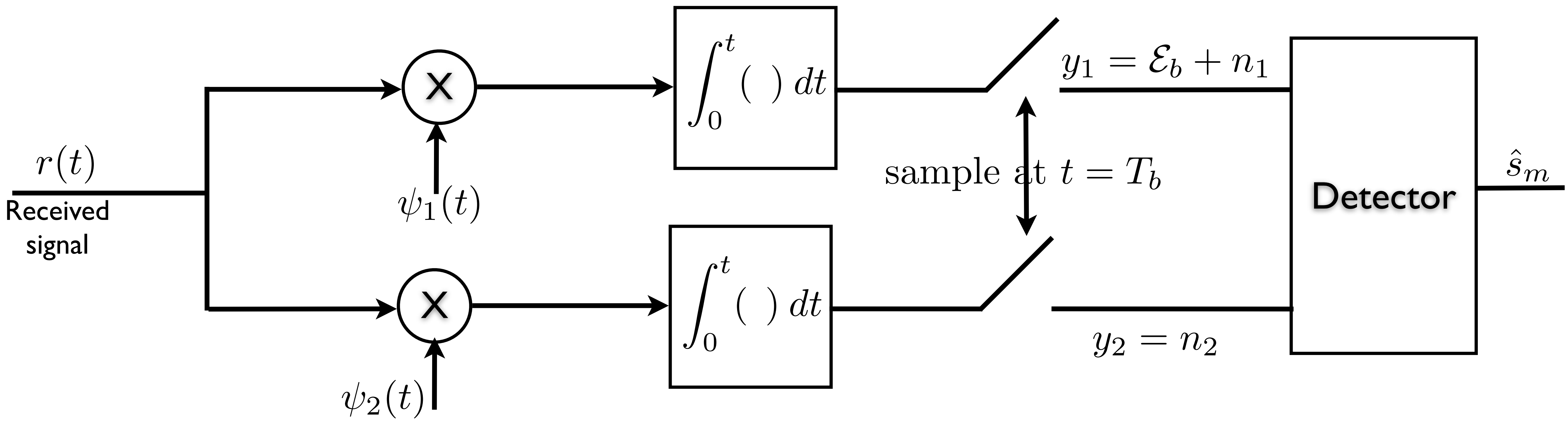
$$\mathbf{y} = [y_1 \;\; y_2]$$

- If $s_1(t)$ is transmitted, the demodulator outputs are

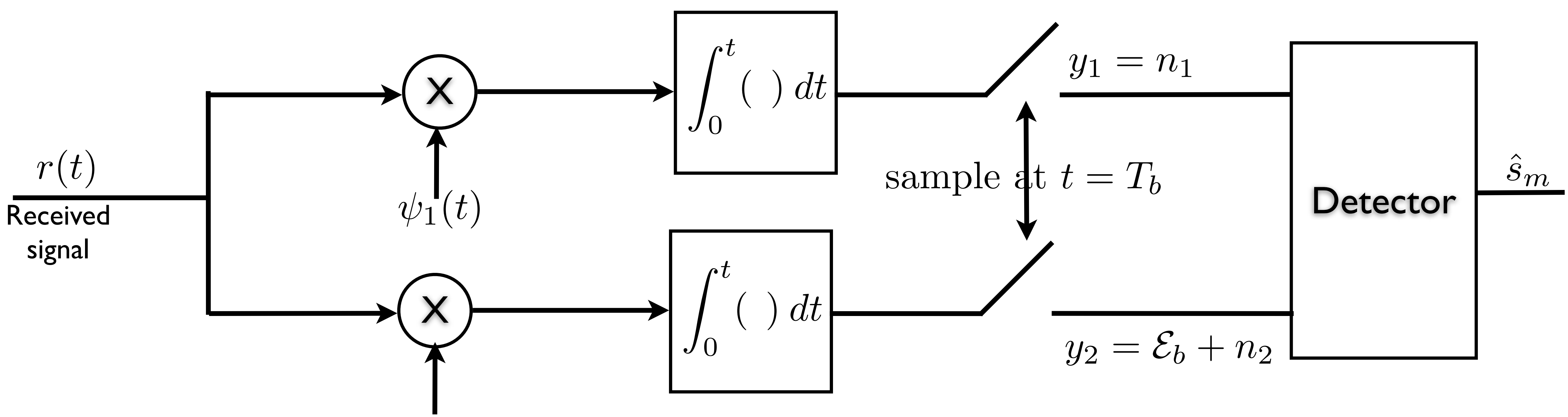
$$\begin{aligned} y_1 &= \mathcal{E}_b + n_1 \\ y_2 &= n_2 \end{aligned}$$

  - where n_1 and n_2 are statistically independent and identically distributed (I.I.D.) Gaussian random variable with zero mean and variance $\sigma_n^2 = N_0/2.$

- If $s_1(t)$ is transmitted,



- If $s_2(t)$ is transmitted,



- Decision rule to minimize the average probability of error
  - Compare y_1 with y_2
    - If $y_1 > y_2$ (equivalently $y_1 - y_2 > 0$) , declare $s_1(t)$ was transmitted.
    - Otherwise, declare $s_2(t)$ was transmitted.

- Probability of error
  - Assuming $s_1(t)$ is transmitted, the error occurs when $y_1 - y_2 < 0$.
  - Let

$$z = y_1 - y_2 = \sqrt{\mathcal{E}_b} + n_1 - n_2$$

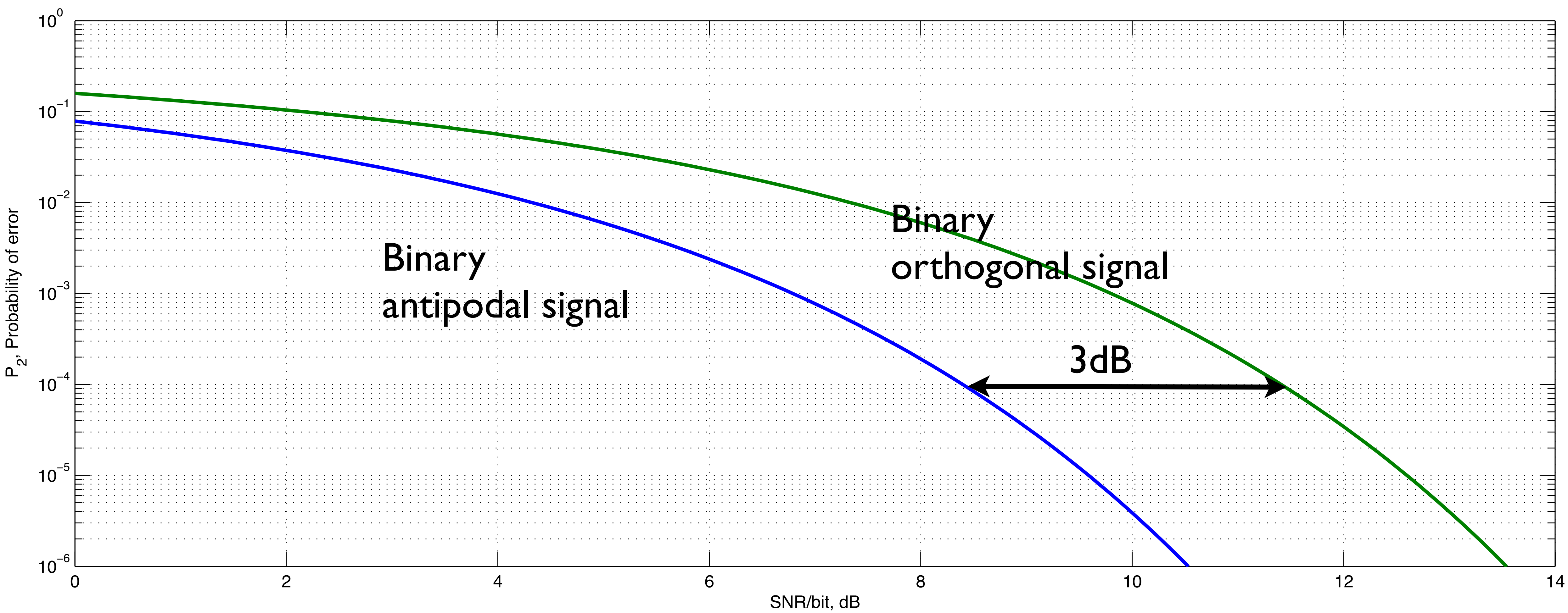
    - Then we can shown

$$z \sim \mathcal{N}(\sqrt{\mathcal{E}_b}, N_0) \quad \Longrightarrow \quad f(z) = \frac{1}{\sqrt{2\pi N_0}} e^{-(z-\sqrt{\mathcal{E}_b})^2/2N_0}$$

- Average probability of error

$$P_2 = P(z < 0) = \int_{-\infty}^{0} f(z)\,dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{\mathcal{E}_b/N_0}} e^{-x^2/2}\,dx = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$



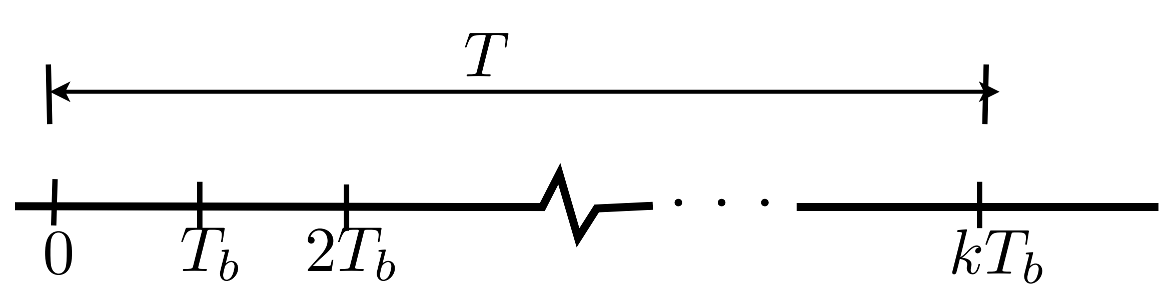
M-ary Pulse Modulation

- M-ary modulation
  - The binary sequence is subdivided into blocks of k bits, called symbols, and each block (or symbol) is represented by one of $M = 2^k$ signal waveforms, each of duration of T.
- Symbol (or signaling) rate
  - The number of signals (or symbols) transmitted per second

$$R_s = \frac{1}{T} \quad \text{symbols/sec}$$

- Bit rate

$$R_b = kR_s = \frac{k}{T} \quad \text{bits/sec}$$



- Bit interval

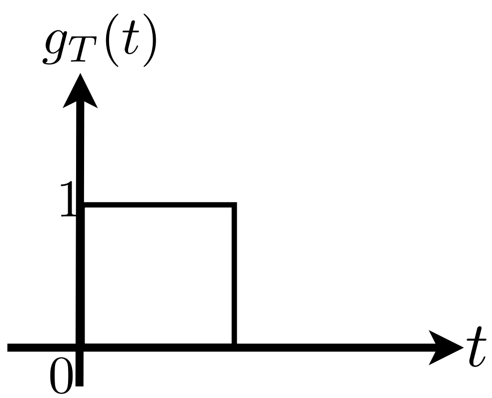
$$T_b = \frac{1}{R_b} = \frac{T}{k}$$

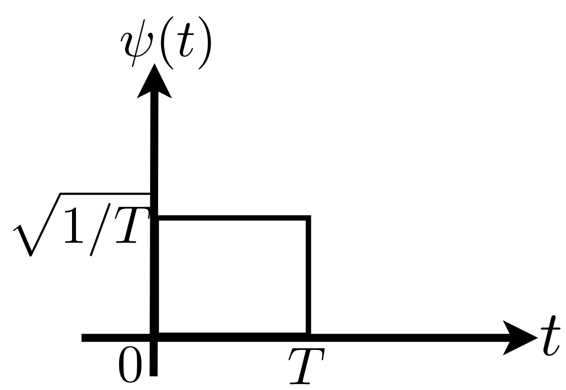
M-ary Pulse Amplitude Modulation (PAM)

- M-ary signal waveforms

$$s_m(t) = A_m g_T(t), \qquad 0 \le t \le T, \quad m = 1, 2, \ldots, M$$

$$= s_m \psi(t) \qquad \text{where } s_m = A_m \sqrt{T}$$





- Energy of each symbol

$$\mathcal{E}_m = \int_0^T s_m^2(t)\, dt = s_m^2 \int_0^T \psi^2(t)\, dt = s_m^2 = A_m^2 T$$

- Average energy

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m = \frac{T}{M} \sum_{m=1}^{M} A_m^2$$

- Signal amplitude

$$A_m = (2m-1-M)A, \quad m = 1, 2, \ldots, M$$

  - Signal amplitudes are symmetric about the origin and equally spaced by which there is no DC component and the average transmitted energy can be minimized.

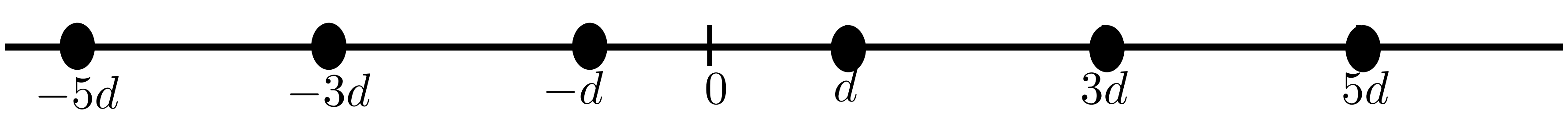
- Average energy

$$\mathcal{E}_{av} = \frac{A^2 T}{M} \sum_{m=1}^{M} (2m-1-M)^2 = \frac{A^2 T(M^2-1)}{3}$$

- Signal constellation

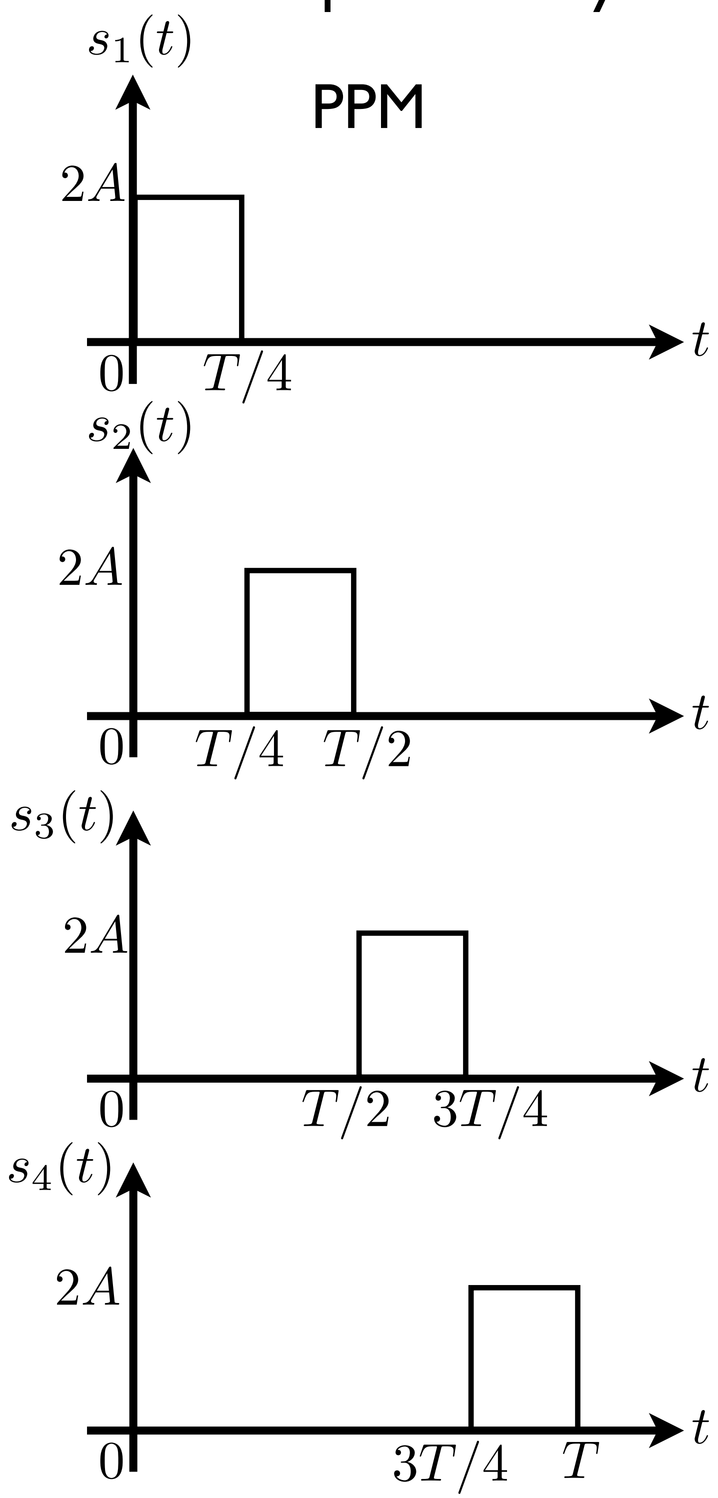
$$\begin{aligned} s_m &= A_m \sqrt{T} = A\sqrt{T}(2m-1-M) \\ &= (2m-1-M)d, \quad m = 1, 2, \ldots, M \end{aligned}$$

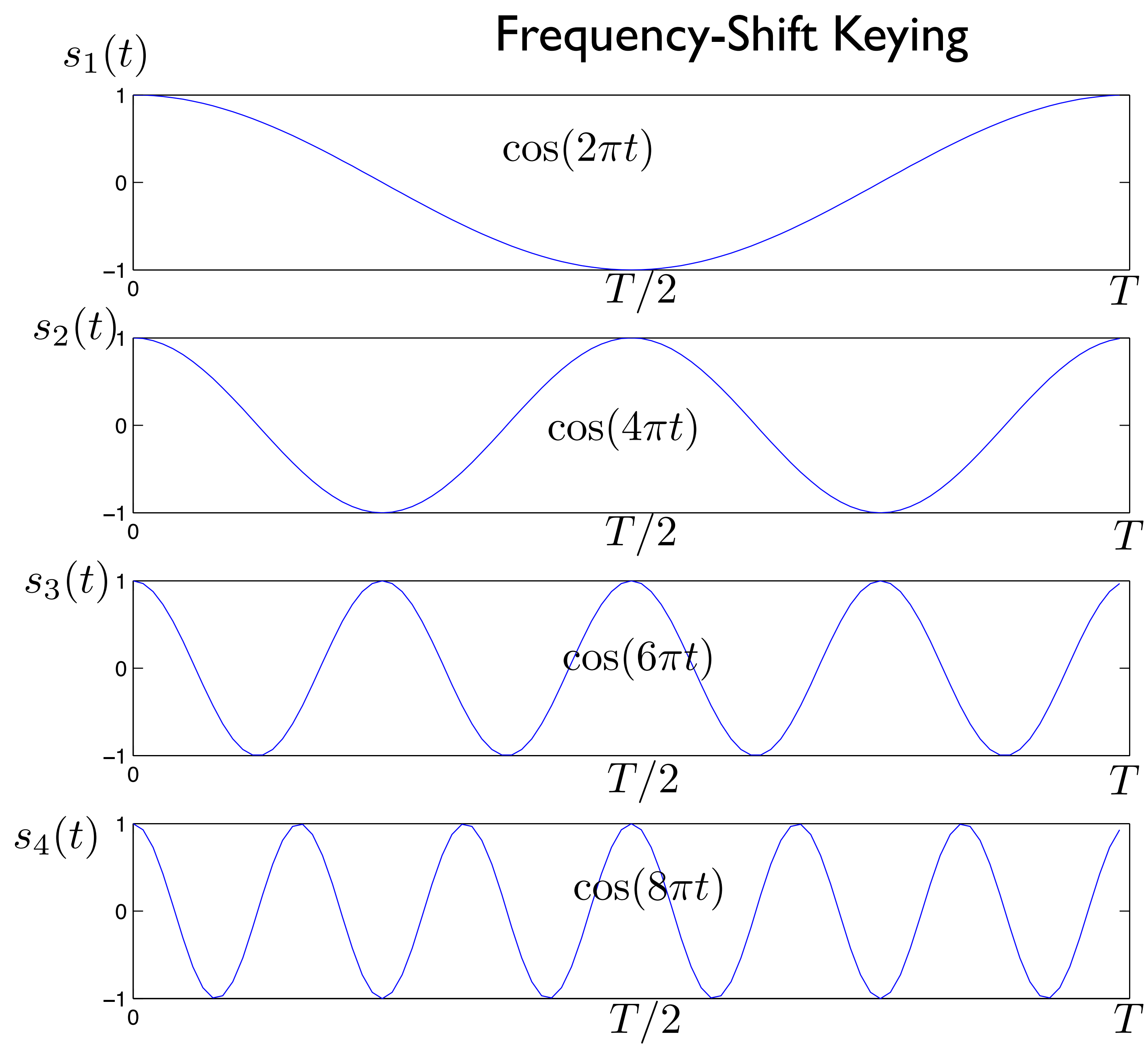
where we define $d = A\sqrt{T}$



M-ary Orthogonal Signals

- Example of 4-ary orthogonal signal waveforms





- Orthogonality condition for $\{s_m(t)\}_{m=1}^{M}$

$$\int_0^T s_i(t)s_j(t)\,dt = 0, \quad i \neq j$$

- Signal waveform expression

$$s_m(t) = \sqrt{\mathcal{E}_s}\psi_m(t), \quad m = 1, 2, \ldots, M$$

  - For PPM,

$$\psi_m(t) = g_T\left(t - \frac{(m-1)T}{M}\right), \quad \frac{(m-1)T}{M} \leq t \leq \frac{mT}{M}$$

  - For frequency shift keying,

$$\psi_m(t) = \sqrt{\frac{2}{T}}\cos(2\pi m t), \quad m = 1, 2, \ldots, M$$

- Dimensionality of M-ary orthogonal signals
  - Dimensionality is M
- Energy

$$\int_0^T s_m^2(t)\,dt = \mathcal{E}_s \int_0^T \psi_m^2(t)\,dt = \mathcal{E}_s, \quad \text{all } m$$

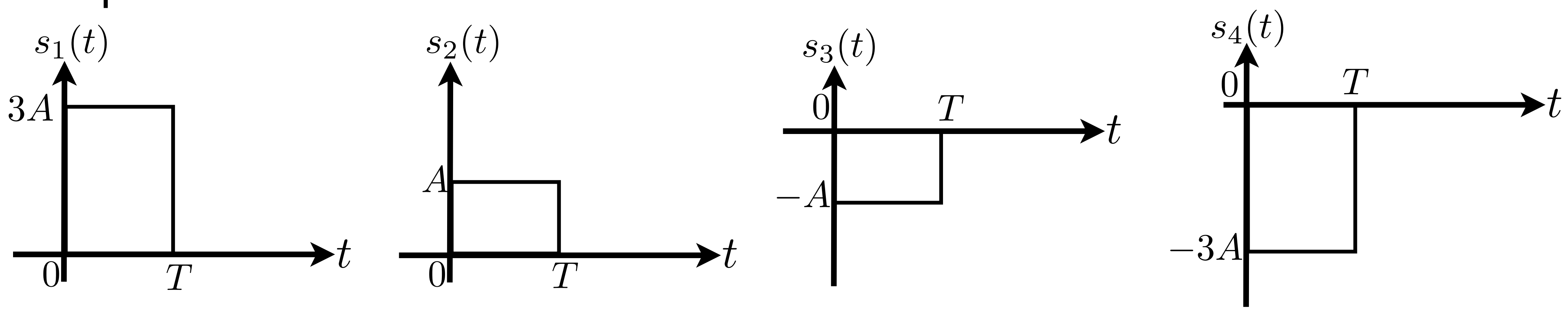
- Geometrical expression

$$\begin{aligned}
\mathbf{s}_1 &= (\sqrt{\mathcal{E}_s}, 0, 0, \ldots, 0) \\
\mathbf{s}_2 &= (0, \sqrt{\mathcal{E}_s}, 0, \ldots, 0) \\
&\vdots \\
\mathbf{s}_M &= (0, 0, 0, \ldots, \sqrt{\mathcal{E}_s})
\end{aligned}$$

- Euclidean distance between M signal vectors are mutually equidistant, i.e.,

$$d_{mn} = \sqrt{||\mathbf{s}_m - \mathbf{s}_n||^2} = \sqrt{2\mathcal{E}_s}, \quad \text{for all } m \neq n$$

- **Example of 4-PAM**



- Average energy

$$\mathcal{E}_{av} = 5A^2T = 5d^2 \qquad \text{where } d^2 = A^2T$$

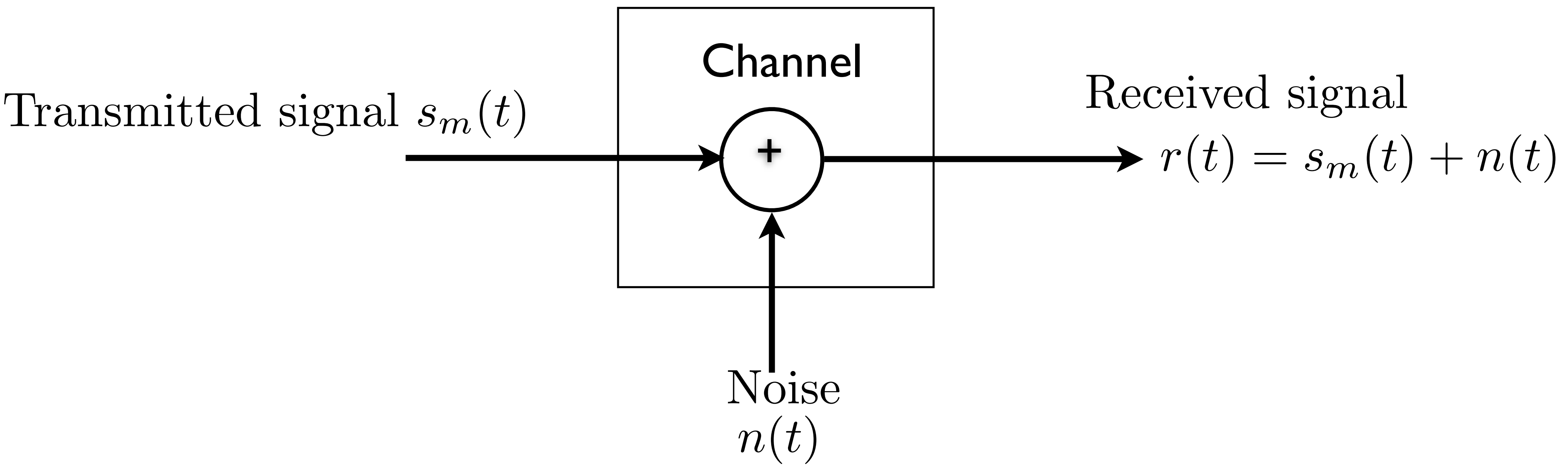
Optimum Receiver for M-ary Signals in AWGN

- Received signal over AWGN during the time interval $0 \leq t \leq T$

$$r(t) = s_m(t) + n(t), \quad 0 \leq t \leq T, \quad m = 1, 2, \ldots, M$$

where $M = 2^k$

- $n(t)$ is AWGN with PSD $S_n(f) = \frac{N_0}{2}$ [W/Hz]



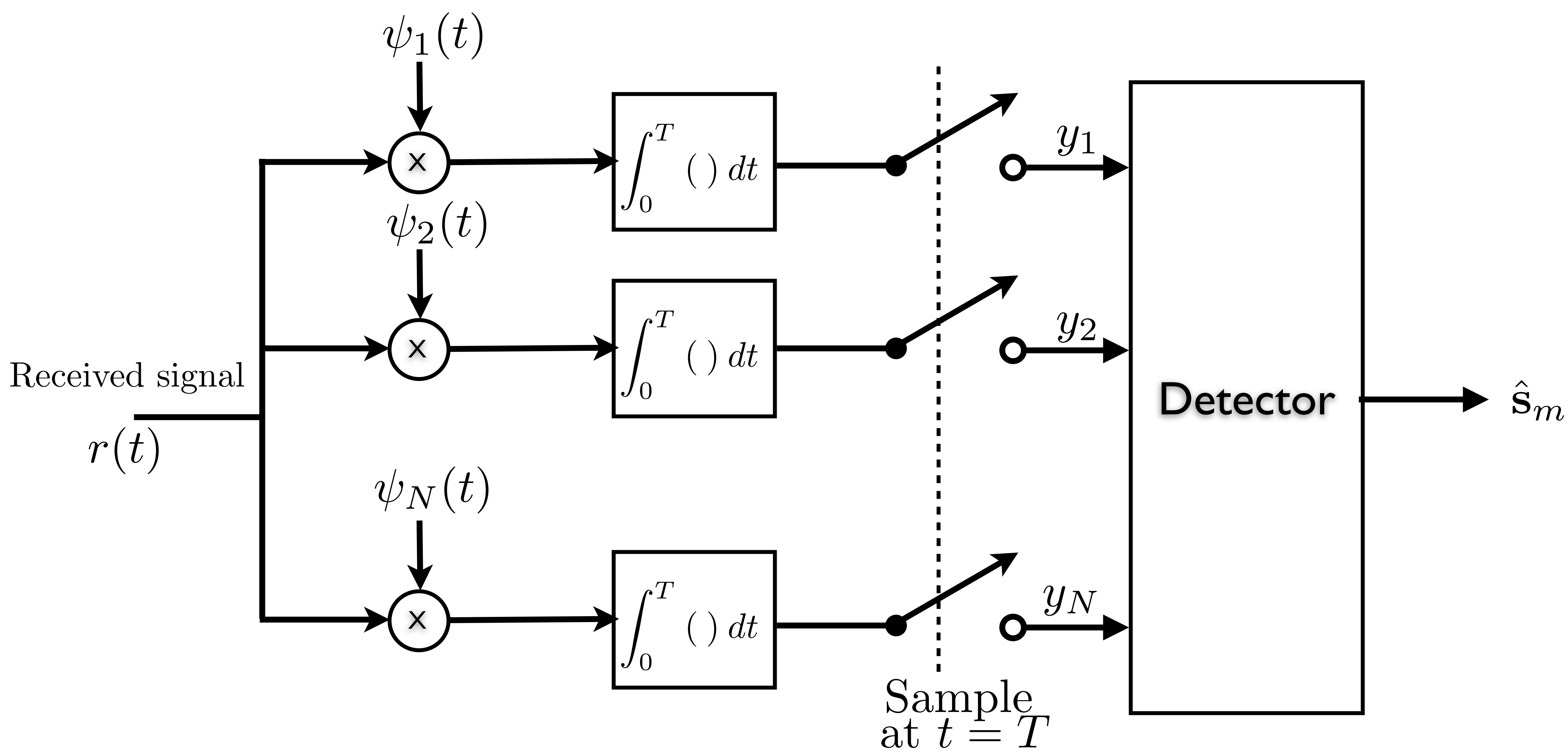
- Geometrical representation of M-ary signals

$$s_m(t) = \sum_{k=1}^{N} s_{mk}\psi_k(t), \quad 0 \le t \le T, \quad m = 1, 2, \ldots, M$$

where $\{s_{mk}\}$ are the coordinates of the signal vector

$$\mathbf{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN}), \quad m = 1, 2, \ldots, M$$

- Receiver with correlation type demodulator



- Correlator output at the end of the signal interval

$$
\begin{aligned}
y_k &= \int_0^T r(t)\psi_k(t)\,dt \\
&= \int_0^T [s_m(t) + n(t)]\psi_k(t)\,dt \\
&= \int_0^T \left(\sum_{n=1}^{N} s_{mn}\psi_n(t)\right)\psi_k(t)\,dt + \int_0^T n(t)\psi_k(t)\,dt \\
&= \sum_{n=1}^{N} s_{mn}\int_0^T \psi_n(t)\psi_k(t)\,dt + \int_0^T n(t)\psi_k(t)\,dt \\
&= s_{mk} + n_k \qquad\qquad \text{for } k = 1, 2, \ldots, N
\end{aligned}
$$

where $n_k = \int_0^T n(t)\psi_k(t)\,dt$

- In vector form we can express the output signal of the demodulator as

$$\mathbf{y} = \mathbf{s}_m + \mathbf{n}$$

Statistics of noise

$$n_k = \int_0^T n(t)\psi_k(t)\, dt$$ is Gaussian. Hence, we need to find the mean and variance for PDF.

- **Mean**

$$E[n_k] = \int_0^T E[n(t)]\psi_k(t)\, dt = 0$$

- **Covariance**

$$\begin{aligned} E[n_k n_j] &= \int_0^T \int_0^T E[n(t)n(\tau)]\psi_k(t)\psi_j(t)\, dt d\tau \\ &= \int_0^T \int_0^T \frac{N_0}{2}\frac{N_0}{2}\delta(t-\tau)\psi_k(t)\psi_j(\tau)\, dt d\tau \\ &= \frac{N_0}{2}\int_0^T \psi_k(t)\psi_j(t)\, dt \\ &= \frac{N_0}{2}\delta_{jk} \end{aligned}$$

where $$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

- Joint PDF of noise

$$f(\mathbf{n}) = \prod_{l=1}^{N} f(n_l) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{l=1}^{N} \frac{n_l^2}{N_0}}$$

- Statistic of the output signal at the demodulator

$y_k = s_{mk} + n_k$ which is also Gaussian given s_{mk}.

- Mean

$$E[y_k] = E[s_{mk} + n_k] = s_{mk}$$

- Variance

$$\text{var}[y_k] = N_0/2$$

- Conditional PDF of $\mathbf{y} = (y_1, y_2, \ldots, y_N)$

$$\begin{aligned} f(\mathbf{y}|\mathbf{s}_m) &= \prod_{k=1}^{N} f(y_k|s_{mk}) \\ &= \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N}(y_k - s_{mk})^2/N_0\right] \\ &= \frac{1}{(\pi N_0)^{N/2}} \exp[-||\mathbf{y} - \mathbf{s}_m||^2/N_0], \quad m = 1, 2, \ldots, M. \end{aligned}$$

Example of 4-PAM

- Received signal

$$r(t) = s_m(t) + n(t)$$

- Output of the demodulator

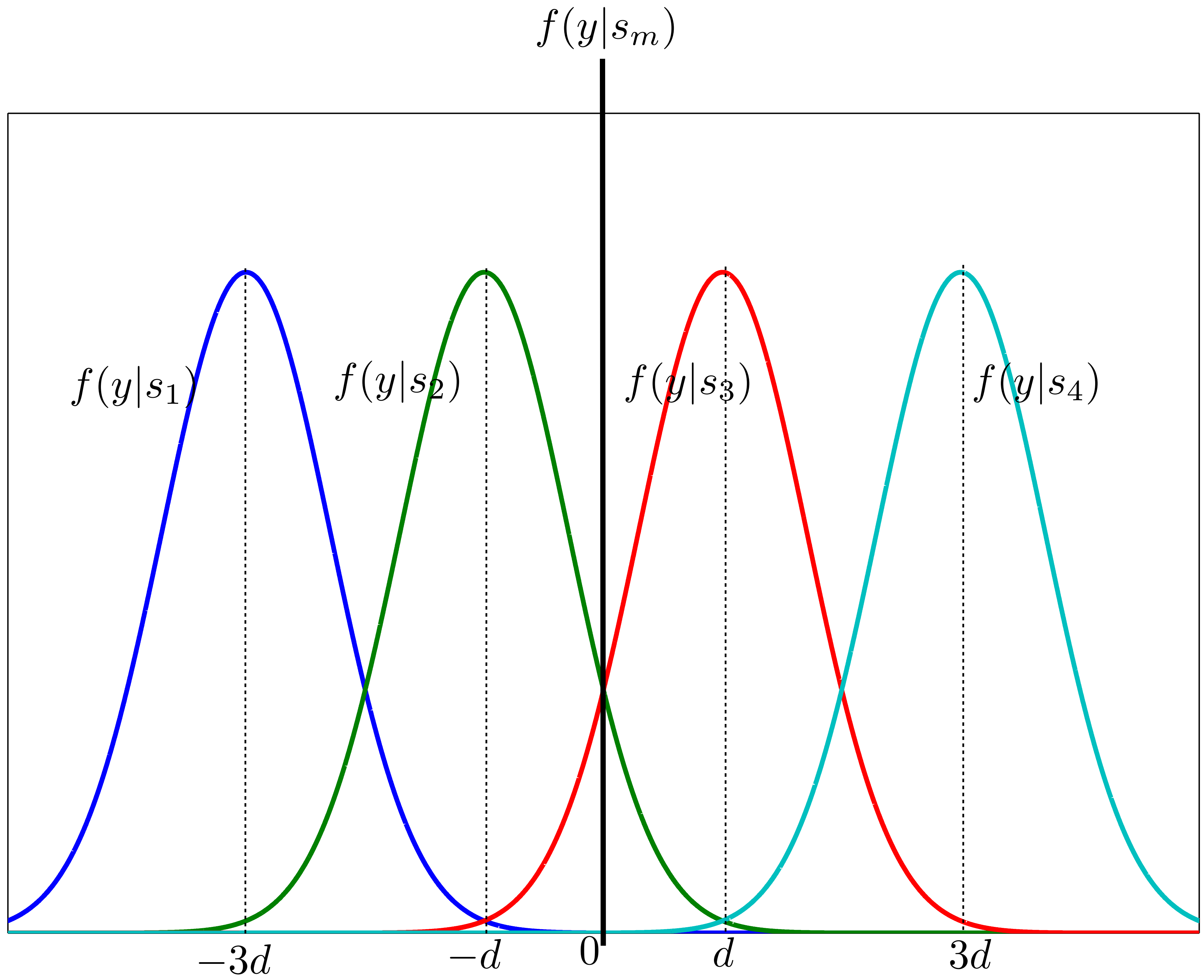
$$y(T) = \int_0^T r(t)\psi(t)\,dt = \int_0^T [s_m(t) + n(t)]\psi(t)\,dt = s_m + n,$$

where $n \sim \mathcal{N}(0, N_0/2)$

- PDF of $y(T)$

$$f(y|s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1, 2, \ldots, M$$

where $s_m = (2m - 1 - M)d.$


$f(y|s_m)$
$f(y|s_1)$
$f(y|s_2)$
$f(y|s_3)$
$f(y|s_4)$
$-3d$
$-d$
0
d
$3d$

Example of 4-PPM

- The signal vector

$$\mathbf{s}_1 = (\sqrt{\mathcal{E}_s}, 0, 0, 0)$$

- Received signal vector under $s_1(t)$

$$\mathbf{y} = \mathbf{s}_1 + \mathbf{n} = \left(\sqrt{\mathcal{E}_s} + n_1, n_2, n_3, n_4\right)$$

- Joint conditional PDF

$$f(\mathbf{y}|\mathbf{s}_1) = f(y_1, y_2, y_3, y_4|\mathbf{s}_1) = \frac{1}{(\pi N_0)^2} \exp\left[-\frac{(y_1 - \sqrt{\mathcal{E}_s})^2 + y_2^2 + y_3^2 + y_4^2}{N_0}\right]$$