Linear Stability Analysis (II)

For a Predator-Prey Model

Introduction

- For linear stability analysis, we assume that the amplitude of initial condition is sufficiently small (but not too small either).

- If the solution converges to specific value as both time step size and space step size go to zero, the scheme is convincing.

- Otherwise, i.e. if an oscillation is observed, the scheme is unbelievable.

Linearization

- For linear stability analysis, we start the linearization of our PDE by using the Taylor expansion with simple example as below :

$$c_t = c - c^3 + \epsilon^2 \Delta c$$

- For the linearization at c=0, we just omit the nonlinear term :

$$c_t = c + \epsilon^2 \Delta c$$

- We will use this one; however, we also find the linearization at the non-zero point for generalization.

Linearization

- In general case, i.e. the linearization at c = a which is not zero, we first take f(c) = c – c^3.

- Then, the Taylor expansion of f(c) is

$$\begin{aligned} f(c) &= f(a) + (c-a)f'(a) + \frac{1}{2}(c-a)^2 f''(\xi) \\ &\approx a - a^3 + (c-a)(1-3a^2) \end{aligned}$$

- So, our PDE becomes

$$c_t = a - a^3 + (c-a)(1-3a^2) + \epsilon \Delta c$$

Stability Analysis

- Let c = a(t)cos(kπx)

- Plugging this in linearized one, then we get

$$a'(t) = \left[1 - (\epsilon k\pi)^2\right] a(t)$$

- The assumption is originated from the Fourier series form. We use exponential function generally, however, the cosine function is used for the Neumann boundary condition.

$$c(x,t) = \sum_{k=0}^{\infty} a_k(t) e^{ik\pi x} = \sum_{k=0}^{\infty} a_k(t) \left[\cos(k\pi x) + \underbrace{i \sin(k\pi x)}_{=0}\right]$$

Stability Analysis

- Now, we compare two equations: one is for numerical simulation and the other is for linear stability analysis (LSA).

- Since LSA is derived from the Taylor expansion, they are same at very short time duration.

- From $a(t) = a(0)e^{\lambda t}$,

$$\log \frac{a(t)}{a(0)} = \log e^{\lambda t} = \lambda t$$

So, we can get λ.

Example

- For more practical example, we refer

  "An Efficient and Accurate Numerical Scheme for Turing Instability on a Predator-Prey Model", Ana Yun, Darae Jeong, Junseok Kim (2011)

- In this paper, the system of partial differential equations is given as

$$N_t = N(1-N) - \frac{\alpha NP}{P+N} + d_1 N_{xx}$$

$$P_t = -\frac{\epsilon P(\gamma + \delta\beta P)}{1+\beta P} + \frac{\epsilon NP}{P+N} + d_2 P_{xx}$$

It is a model of predator-prey simulation.

Example

- N and P are the prey and predator population densities, respectively.

- d_i's and greek alphabets are given coefficients.

- Let

$$f(N,P) = N(1-N) - \frac{\alpha NP}{P+N}$$

$$g(N,P) = -\frac{\epsilon P(\gamma + \delta\beta P)}{1+\beta P} + \frac{\epsilon NP}{P+N}$$

and N* and P* are equilibrium points for f and g, respectively.

Example

- Now, we use the Taylor's expansion for two variables to both N and P at (N*, P*).

$$N_t = f(N^*, P^*) + f_N(N^*, P^*)(N - N*) + f_P(N^*, P^*)(P - P^*) + d_1 N_{xx}$$

$$P_t = g(N^*, P^*) + g_N(N^*, P^*)(N - N*) + g_P(N^*, P^*)(P - P^*) + d_2 P_{xx}$$

- For simplicity, we let

$$N = N^* + a(t)\cos(k\pi x)$$

$$P = P^* + b(t)\cos(k\pi x)$$

and plug these in above equation.

Example

- Then, we get

$$a'(t)\cos(k\pi x) = f(N^*, P^*) + f_N(N^*, P^*)(a(t)\cos(k\pi x)) \\ + f_p(N^*, P^*)(b(t)\cos(k\pi x)) - (k\pi)^2 d_1 a(t)\cos(k\pi x)$$

$$b'(t)\cos(k\pi x) = g(N^*, P^*) + g_N(N^*, P^*)(a(t)\cos(k\pi x)) \\ + g_p(N^*, P^*)(b(t)\cos(k\pi x)) - (k\pi)^2 d_2 b(t)\cos(k\pi x)$$

- It can be solved by matrix decomposition as below: (The results can be shown in the paper)

$$X' = AX \rightarrow X' = PDP^{-1}X \rightarrow (P^{-1}X)' = D(P^{-1}X)$$

where D is diagonal matrix.