Mobile Communications (KECE425)

Lecture Note 23

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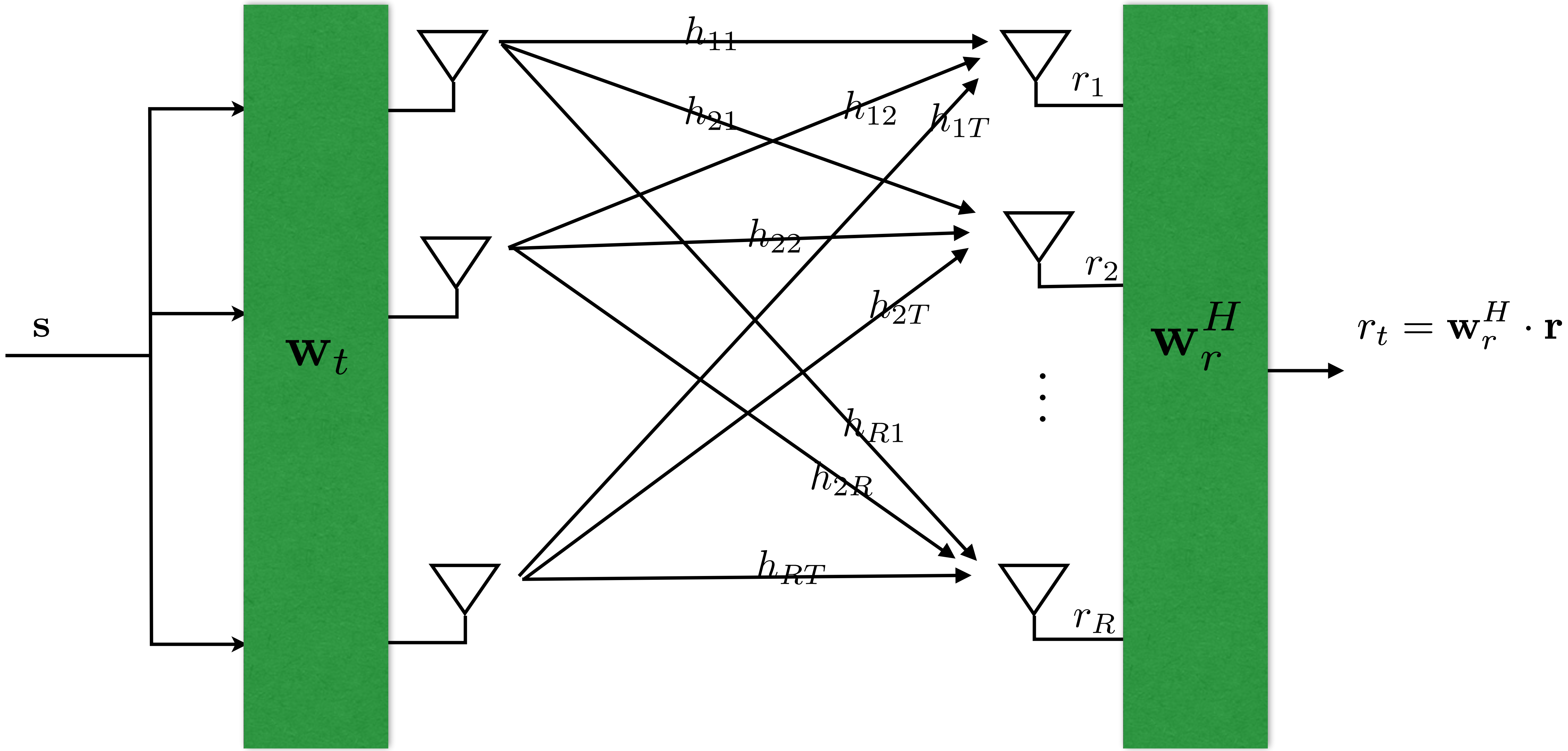
Summary

- Complexity issues of diversity systems
  - ADC and Nyquist sampling theorem
- Transmit diversity
  - Channel is known at the transmitter (Closed-loop transmit diversity: CLTD)
  - Channel is unknown at the transmitter (Space-time block coding: STBC)
- Transmit-Receive diversity (Maximal ratio transmission)
- Multi-user opportunistic diversity
- MIMO channel capacity

Maximal Ratio Transmission (MRT)

- MRT is also called multiple input multiple output (MIMO)-MRC.



$$\mathbf{r} = \mathbf{H}\mathbf{w}_t s + \mathbf{n}$$

- MIMO channel can be represented in matrix form:

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1T} \\ h_{21} & h_{22} & \cdots & h_{2T} \\ \vdots & \vdots & \vdots & \vdots \\ h_{R1} & h_{R2} & \cdots & h_{RT} \end{bmatrix}$$

- Vector representation

$$\mathbf{w}_t = [w_{t1} \;\; w_{t2} \;\; \cdots \;\; w_{tT}]^T$$

$$\mathbf{w}_r = [w_{r1} \;\; w_{r2} \;\; \cdots \;\; w_{rR}]^T$$

$$\mathbf{n} = [n_1 \;\; n2 \;\; \cdots \;\; n_R]^T$$

- Received signal:

$$\begin{aligned} r_1 &= (w_{t,1}h_{11} + w_{t,2}h_{12} + \cdots + w_{t,T}h_{1T})s + n_1 \\ r_2 &= (w_{t,1}h_{21} + w_{t,2}h_{22} + \cdots + w_{t,T}h_{2T})s + n_2 \\ &\vdots \\ r_R &= (w_{t,1}h_{R1} + w_{t,2}h_{R2} + \cdots + w_{t,T}h_{RT})s + n_R \end{aligned}$$

  - Received signal in vector form:

$$\mathbf{r} = \mathbf{H}\mathbf{w}_t s + \mathbf{n}$$

- Combined signal:

$$r_t = \mathbf{H}\mathbf{r}$$

- Optimal receive weight vector $\mathbf{w}_r$ can be easily shown to be given as

$$\mathbf{w}_r^H = c(\mathbf{H}\mathbf{w}_t)^H = c\mathbf{w}_t^H\mathbf{H}^H$$

  where $(\cdot)^H$ denote the Hermitian operation.

  - In this case, the received signal can be written as

$$\begin{aligned} r_t &= \mathbf{w}_r^H\mathbf{r} \\ &= \mathbf{w}_r^H(\mathbf{H}\mathbf{w}_t + \mathbf{n}) \\ &= c\mathbf{w}_t^H\mathbf{H}^H\mathbf{H}\mathbf{w}_t s + c\mathbf{w}_t^H\mathbf{H}^H\mathbf{n} \end{aligned}$$

- SNR of the received signal

  - Received signal can be written as

$$r_t = c\mathbf{w}_t^H \mathbf{H}^H \mathbf{H} \mathbf{w}_t s + c\mathbf{w}_t^H \mathbf{H}^H \mathbf{n}$$

  - SNR of r_t

$$\gamma_t = \frac{1}{\sigma_n^2} \mathbf{w}_t^H \mathbf{H}^H \mathbf{H} \mathbf{w}_t$$

- Optimal transmit weight vector, $\mathbf{w}_t^{\mathrm{opt}}$

$$\begin{aligned}
\mathbf{w}_t^{\mathrm{opt}} &= \max_{\mathbf{w}_t} \gamma_t \\
&= \max_{\mathbf{w}_t} \frac{1}{\sigma_n^2} \mathbf{w}_t^H \mathbf{H}^H \mathbf{H} \mathbf{w}_t \\
&= \max_{\mathbf{w}_t} \mathbf{w}_t^H \mathbf{H}^H \mathbf{H} \mathbf{w}_t
\end{aligned}$$

- Find the optimal weight vector $\mathbf{w}_t$ to maximize the SNR γ_t.

$$\mathbf{w}_t^{\text{opt}} = \max_{\mathbf{w}_t} \mathbf{w}_t^H \mathbf{H}^H \mathbf{H} \mathbf{w}_t$$

  - We can solve this problem by making use of Rayleigh-Ritz theorem.

- Rayleigh-Ritz theorem

$$\mathbf{x}^H \mathbf{A} \mathbf{x} \leq \lambda_{\max} ||\mathbf{x}||^2$$

where $\mathbf{A}$ is the Hermitian matrix, $\mathbf{x}$ is an y non-zero complex vector and $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{A}$.

- Equality holds if and only if $\mathbf{x}$ is the eigenvector corresponding to $\lambda_{\max}$.

- Based on Rayleigh-Ritz theorem, we can find the optimal weight vector $\mathbf{w}_t^{\text{opt}}$, we can find the optimal weight vector as

$$\mathbf{w}_t^{\text{opt}} = \sqrt{\Omega} \mathbf{U}_{\max}$$

where $\mathbf{U}_{\max}$ is the eigenvector corresponding to the largest eigenvalue of the quadratic form $\mathbf{F} = \mathbf{H}^H \mathbf{H}$ and $\mathbf{U}_{\max}^H \mathbf{U}_{\max} = \mathbf{I}$

- Combined SNR with the optimum weight vector

$$\gamma_t = \frac{\Omega \lambda_{\max}}{\sigma_n^2}$$

- Example for $T = 3$ and $R = 3$
  - Assume that channel matrix for a certain duration within coherence time is given as

$$H = \begin{bmatrix} 1.9e^{j\frac{\pi}{5}} & 0.2e^{j\frac{\pi}{10}} & 0.6e^{-j\frac{\pi}{6}} \\ 0.5e^{-j\frac{\pi}{4}} & 1.6e^{-j\frac{\pi}{18}} & 0.5e^{-j\frac{\pi}{6}} \\ 1.4e^{-j\frac{\pi}{10}} & 1.8e^{j\frac{\pi}{10}} & 0.8e^{j\frac{\pi}{16}} \end{bmatrix}$$

$$= \begin{bmatrix} 1.5371 + j1.1168 & 0.1902 + j0.0618 & 0.5196 - j0.3 \\ 0.3536 - j0.3536 & 1.5757 - j0.2778 & 0.4330 - j0.25 \\ 1.3315 - j0.4326 & 1.7119 + j0.5562 & 0.7846 + j0.1561 \end{bmatrix}$$

- Then we can calculate the Wishart matrix F as

$$
\begin{aligned}
F &= \mathbf{H}^H\mathbf{H} \\
&= \begin{bmatrix} 1.9e^{-j\frac{\pi}{5}} & 0.4e^{-j\frac{\pi}{4}} & 1.4ev^{j\frac{\pi}{10}} \\ 0.2e^{-j\frac{\pi}{10}} & 1.6e^{j\frac{\pi}{18}} & 1.8e^{-j\frac{\pi}{10}} \\ 0.6e^{j\frac{\pi}{6}} & 0.5e^{j\frac{\pi}{6}} & 0.8e^{-j\frac{\pi}{16}} \end{bmatrix} \begin{bmatrix} 1.9e^{j\frac{\pi}{5}} & 0.2e^{j\frac{\pi}{10}} & 0.6e^{-j\frac{\pi}{6}} \\ 0.5e^{-j\frac{\pi}{4}} & 1.6e^{-j\frac{\pi}{18}} & 0.5e^{-j\frac{\pi}{6}} \\ 1.4e^{-j\frac{\pi}{10}} & 1.8e^{j\frac{\pi}{10}} & 0.8e^{j\frac{\pi}{16}} \end{bmatrix}
\end{aligned}
$$

$$
= \begin{bmatrix} 5.82 & 3.0554 + j1.8227 & 1.6824 - j0.4295 \\ 3.0554 - j1.8227 & 5.84 & 2.2621 - j0.5320 \\ 1.6824 + j0.4295 & 2.2621 + j0.532 & 1.25 \end{bmatrix}
$$

- Now let us find the eigenvectors and eigenvalues of F

$$F\mathbf{x} = \lambda\mathbf{x} \qquad \text{or equivalently} \qquad (F - \lambda I)\mathbf{x} = 0$$

where λ is the eigenvector corresponding to the eigenvector $\mathbf{x}$.

* λ can be found by solving the characteristic equation:

$$\det(F - \lambda I) = 0$$

$$F - \lambda I = \begin{bmatrix} 5.82 - \lambda & 3.0554 + j1.8227 & 1.6824 - j0.4295 \\ 3.0544 - j1.8227 & 5.84 - \lambda & 2.2621 - j0.5320 \\ 1.6824 + j0.4295 & 2.2621 + j0.532 & 1.25 - \lambda \end{bmatrix}$$

- We can calculate the determinant of $F - \lambda I$ as

$$\begin{aligned}\det(F-\lambda I) &= (5.82-\lambda)\det\begin{bmatrix} 5.84-\lambda & 2.2621-j0.5320 \\ 2.2621+j0.532 & 1.25-\lambda \end{bmatrix} \\ &\quad -(3.0554+j1.8227)\det\begin{bmatrix} 3.0544-j1.8227 & 2.2621-j0.5320 \\ 1.6824+j0.4295 & 1.25-\lambda \end{bmatrix} \\ &\quad +(1.6824-j0.4295)\det\begin{bmatrix} 3.0544-j1.8227 & 5.84-\lambda \\ 1.6824+j0.4295 & 2.2621+j0.532 \end{bmatrix}\end{aligned}$$

is the third-order polynomial of λ with the weight of real value in each order.

$$\begin{aligned}p(\lambda) &= (5.82-\lambda)\left[(5.84-\lambda)(1.25-\lambda)-(2.2621-j0.5320)(2.2621+j0.532)\right] \\ &\quad -(3.0544+j1.8227)\left[(3.0544-j1.8227)(1.25-\lambda)-(2.2621-j0.5320)(1.6824+j0.4295)\right] \\ &\quad +(1.6824-j0.4295)\left[(3.0544-j1.8227)(2.2621+j0.532)-(5.84-\lambda)(1.6824+j0.4295)\right] \\ &= -\lambda^3+a\lambda^2+b\lambda+c\end{aligned}$$

where a, b, and c are real constants.

- Solving $p(\lambda) = 0$, we have

$$\begin{aligned} \lambda_1 &= 10.246 \\ \lambda_2 &= 2.5885 \\ \lambda_3 &= 0.0755 \end{aligned}$$

∗ So the maximum eigenvalue is $\lambda_{\max} = 10.246$.

∗ The eigenvector corresponding to $\lambda_{\max}$ can be found as

$$F\mathbf{x} = \lambda_{\max}\mathbf{x}$$

Sbstituting F and $\lambda_{\max}$ we have

$$\begin{bmatrix} 5.82 & 3.0554 + j1.8227 & 1.6824 - j0.4295 \\ 3.0554 - j1.8227 & 5.84 & 2.2621 - j0.5320 \\ 1.6824 + j0.4295 & 2.2621 + j0.532 & 1.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 10.246 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then we have three simultaneous equations given as

$$\begin{aligned}(3.0554 + j1.8227)x_2 + (1.6824 - j0.4295)x_3 &= 4.426x_1 \\ (3.0554 - j1.8227)x_1 + (2.2621 - j0.5320)x_3 &= 4.406x_2 \\ (1.6824 + j0.4295)x_1 + (2.2621 + j0.532)x_2 &= 8.9960x_3\end{aligned}$$

Then we can find the maximum eigenvector given as

$$\mathbf{x} = \begin{bmatrix} 0.5888 + j0.3048 \\ 0.6874 \\ 0.2684 + j0.1258 \end{bmatrix}$$

where we normalize such that $||\mathbf{x}||^2 = 1$.

- Then, we have the weight vector in the transmitter

$$\mathbf{w}_t = \begin{bmatrix} 0.5888 + j0.3048 \\ 0.6874 \\ 0.2684 + j0.1258 \end{bmatrix}$$

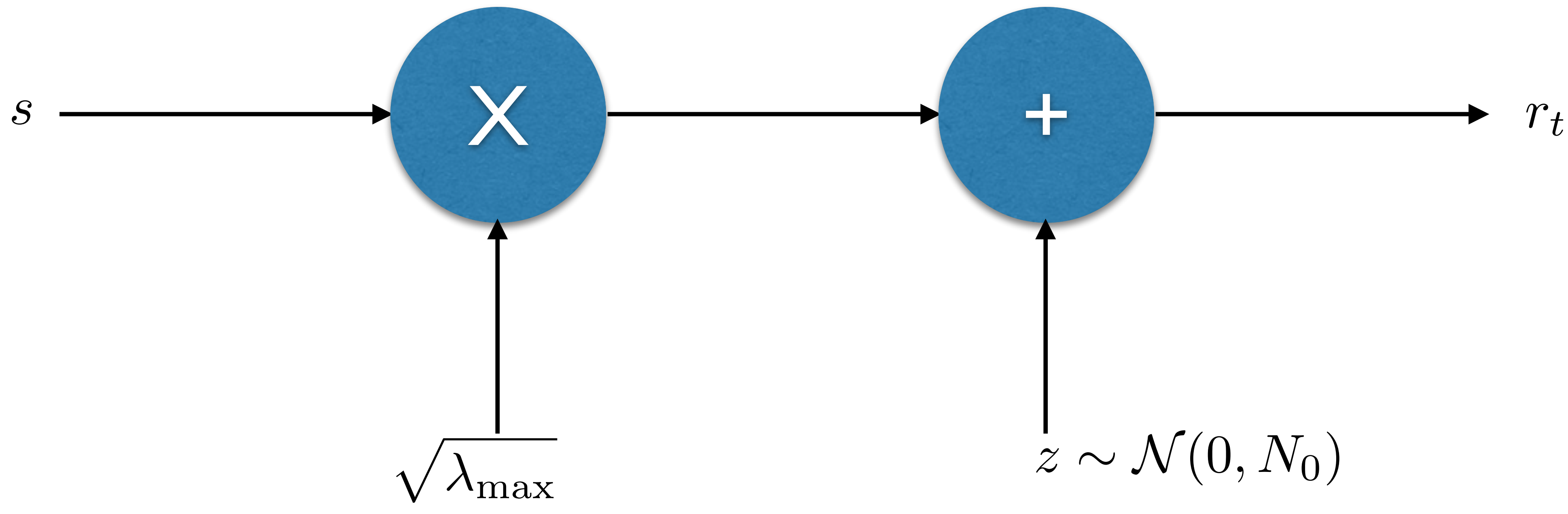
- The receive weight vector can be found as $\mathbf{w}_r^H = (\mathbf{H}\mathbf{w}_t)^H$ which can be calculated as

$$\mathbf{w}_r^H = \begin{bmatrix} 0.8725 + j1.1535 \\ 1.5468 + j0.3040 \\ 2.2836 + j0.6741 \end{bmatrix}^H$$

- Also note that the SNR of the combined signal can be written as

$$\gamma_t = \frac{\lambda_{\max} E_s}{N_0}$$

- In the previous example, 3×3 MIMO channel can be remodeled as 1×1 SISO eigen-channel (maximum eigen-channel, to be exactly speaking) with the channel $\sqrt{\lambda_{\max}}$



$$r_t = \sqrt{\lambda_{\max}} s + z$$

- Note that $\lambda_{\max}$ is the maximum eigenvalue of the matrix $F = \mathbf{H}^H \mathbf{H}$ and thus it is the random variable.

- BER/SER Performance of MIMO-MRC

$$
\begin{aligned}
P(E) &= \int_0^\infty P(E|\gamma_t) p_{\gamma_t}(\gamma_t)\, d\gamma \\
&= \int_0^\infty P\left(E \middle| \frac{\lambda_{\max} E_s}{N_0}\right) p_{\lambda_{\max}}(\lambda_{\max})\, d\lambda_{\max}
\end{aligned}
$$

  - Find the PDF of $\lambda_{\max}$ is not simple problem but it is already solved long time ago for both the Rayleigh channel and the Ricean channel cases.

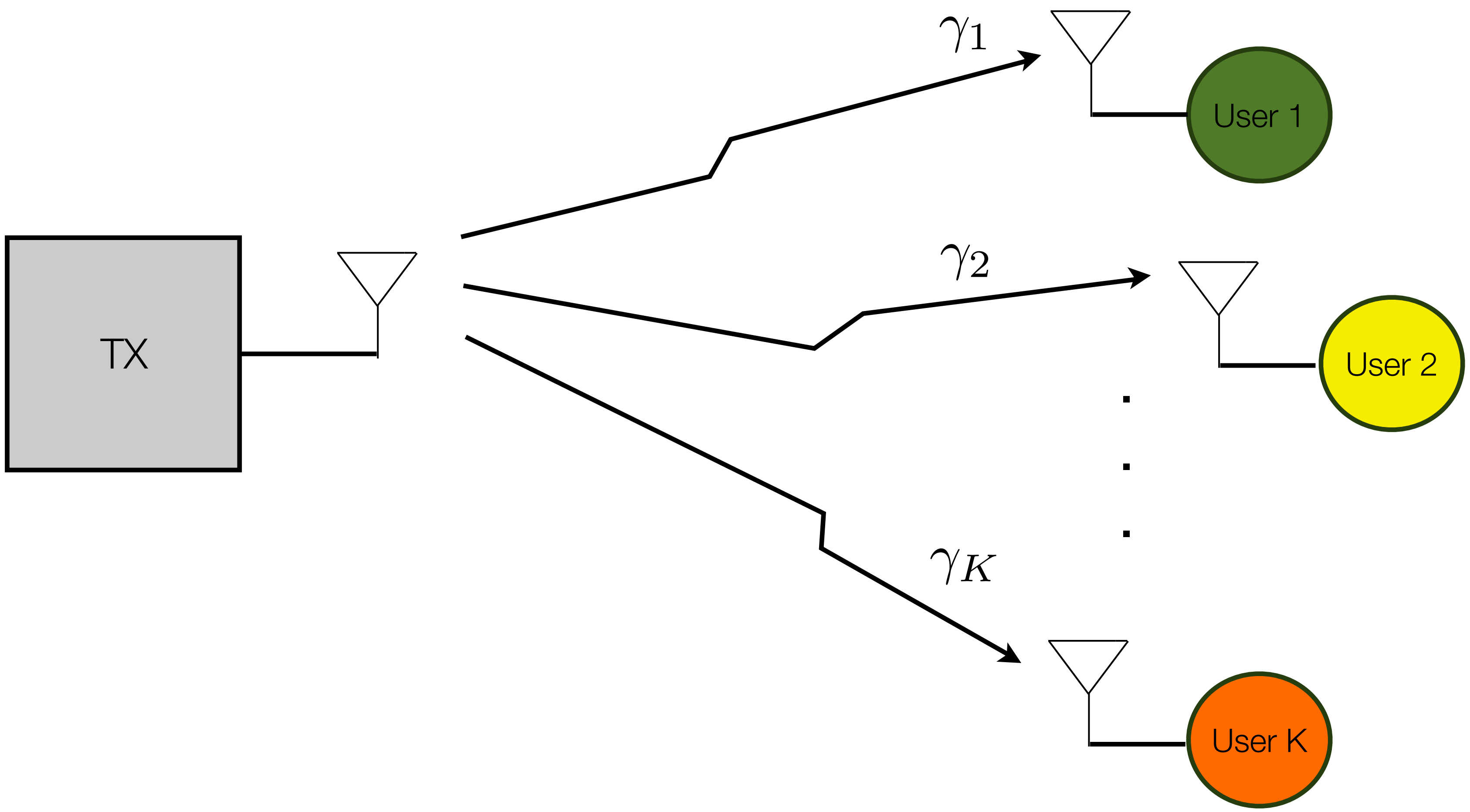
Multi-User Opportunistic Diversity

- We often need to select users if there are more than users to support the service, for a certain limited frequency (or/and time) resource.

- Example:
  - There are 50 MHz bandwidth for the service and each user takes 5 MHz bandwidth. In this case, we can support 10 users for a given time.
  - However, more than 50 users, saying 100 users, are willing to communicate at the same time, what is the best way to select users among 100 users?

- Multi-user opportunistic diversity scheme is simply to select the users with the strongest SNRs.

- Schematic concept of multi-user diversity (MUD).



  - Choose the user which has the largest SNR among K users.

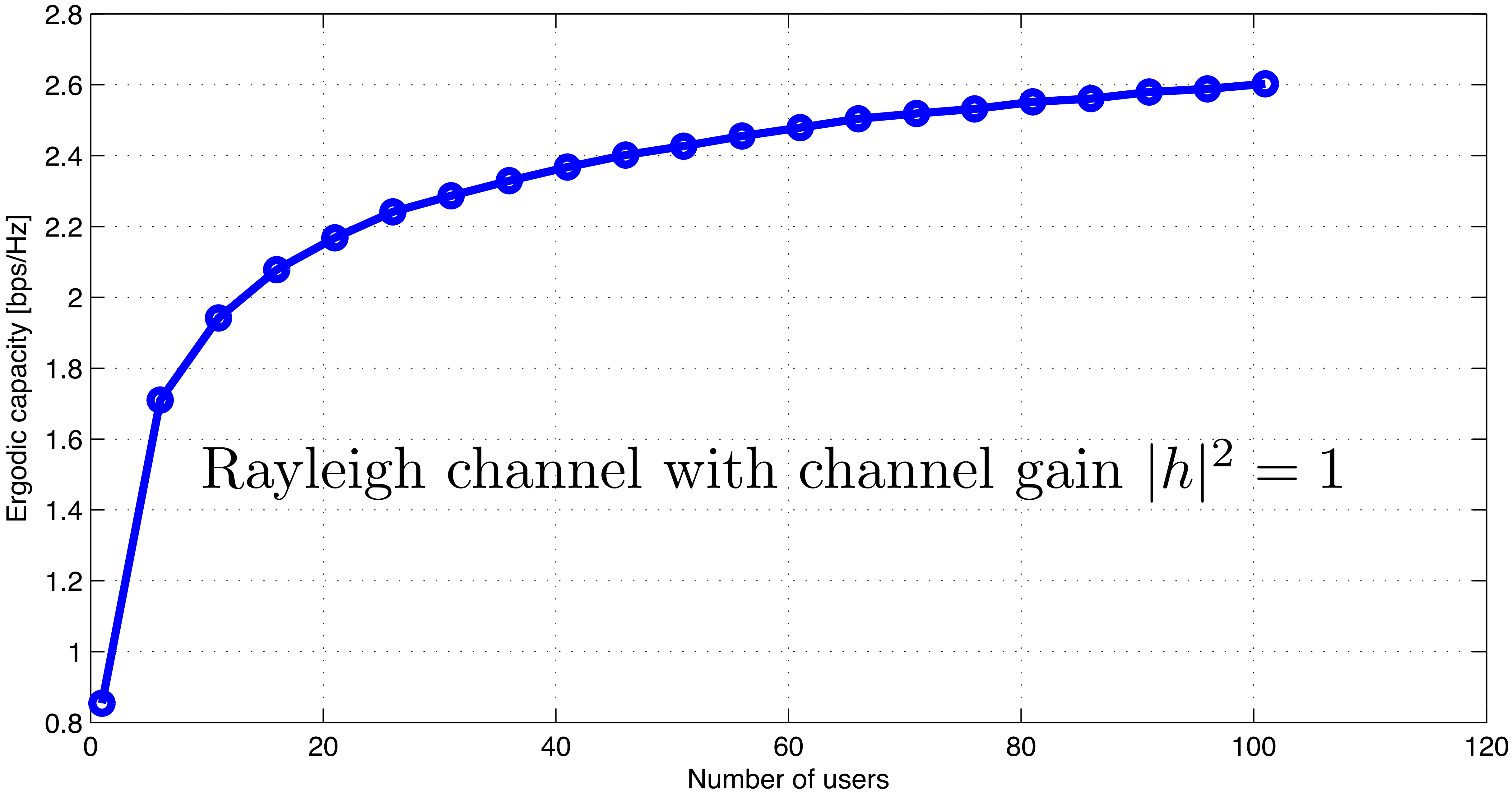
- If one user is selected out of K users at every selection period, the selected user k^* can be written as

$$k^* = \max_k(\gamma_1, \gamma_2, \cdots, \gamma_K)$$

- By doing this, we can improve the channel capacity such as

$$\begin{aligned} C &= E\left[\log_2(1+\gamma_k^*)\right] \\ &= \int_0^\infty \log_2\left(1+\gamma_k^*\right) p_{\gamma_{k^*}}\left(\gamma_{k^*}\right) d\gamma_{k^*} \end{aligned}$$

- Multi-user diversity gain



Channel Capacity in Diversity MIMO

$$C = \log_2 (1 + \gamma_t) \quad [\text{bps/Hz}]$$

Channel capacity is logarithmically increasing versus SNR which is very slow rate of increasing.

Degree of freedom is 1.

$$C = \log_2(1 + \gamma_t) \quad [\text{bps/Hz}]$$