1. As derived in class, the angular momentum operator can be expressed in spherical polar coordinates as..

$$L_x = \frac{\hbar}{i}\left[-\sin\phi \frac{\partial}{\partial\theta} - \cot\theta \cos\phi \frac{\partial}{\partial\phi}\right]$$

$$L_y = \frac{\hbar}{i}\left[\cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi}\right]$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

Express $L^2 = L_x^2 + L_y^2 + L_z^2$ in spherical polar coordinates.

⇒ i) 가장 직접적인 방법.

$$L_x^2 = \left(\frac{\hbar}{i}\right)^2 \left[-\sin\phi \frac{\partial}{\partial\theta} - \cot\theta \cos\phi \frac{\partial}{\partial\phi}\right]^2$$

$$= \left(\frac{\hbar}{i}\right)^2 \Big\{ \left(-\sin\phi \frac{\partial}{\partial\theta}\right)\left(-\sin\phi \frac{\partial}{\partial\theta}\right) \quad \cdots ①$$

$$+ \left(-\sin\phi \frac{\partial}{\partial\theta}\right)\left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right) \quad \cdots ②$$

$$+ \left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right)\left(-\sin\phi \frac{\partial}{\partial\theta}\right) \quad \cdots ③$$

$$+ \left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right)\left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right) \Big\} \cdots ④$$

$$① = \left(-\sin\phi \frac{\partial}{\partial\theta}\right)\left(-\sin\phi \frac{\partial}{\partial\theta}\right)$$

$$= \sin^2\phi \frac{\partial^2}{\partial\theta^2}$$

$$\boxed{\begin{aligned} \cot\theta &= \frac{\cos\theta}{\sin\theta} \\ \frac{\partial}{\partial\theta}\cot\theta &= -\frac{\cos^2\theta}{\sin^2\theta} + \frac{-\sin\theta}{\sin\theta} \\ &= -\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} \\ &= -\csc^2\theta \end{aligned}}$$

$$② = +\sin\phi \left[\frac{\partial}{\partial\theta}\cot\theta\right] \cos\phi \frac{\partial}{\partial\phi}$$

$$+ \sin\phi \cot\theta \cos\phi \frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}$$

$$= -\sin\phi \cdot \csc^2\theta \cos\phi \frac{\partial}{\partial\phi}$$

$$+ \sin\phi \cot\theta \cos\phi \frac{\partial^2}{\partial\theta\partial\phi}.$$

$$③ = \left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right)\left(-\sin\phi \frac{\partial}{\partial\theta}\right)$$

$$= \cot\theta \cos\phi \left[\frac{\partial}{\partial\phi}\sin\phi\right]\frac{\partial}{\partial\theta}$$

$$+ \cot\theta \cos\phi \sin\phi \frac{\partial^2}{\partial\phi\partial\theta}$$

$$= \cot\theta \cos^2\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \sin\phi \frac{\partial^2}{\partial\phi\partial\theta}.$$

$$④ = \left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right)\left(-\cot\theta \cos\phi \frac{\partial}{\partial\phi}\right)$$

$$= \cot\theta \cos\phi \cot\theta \left[\frac{\partial}{\partial\phi}\cos\phi\right]\frac{\partial}{\partial\phi}$$

$$+ \cot\theta \cos\phi \cot\theta \cos\phi \frac{\partial^2}{\partial\phi^2}$$

$$L_x^2 = \left(\frac{\hbar}{i}\right)^2 \Big[ \sin^2\phi \frac{\partial^2}{\partial\theta^2}$$

$$- \csc^2\theta \sin\phi \cos\phi \frac{\partial}{\partial\phi}$$

$$+ \cot\theta \sin\phi \cos\phi \frac{\partial^2}{\partial\theta\partial\phi}$$

$$+ \cot\theta \cos^2\phi \frac{\partial}{\partial\theta}$$

$$+ \cot\theta \sin\phi \cos\phi \frac{\partial^2}{\partial\theta\partial\phi}$$

$$- \cot^2\theta \sin\phi \cos\phi \frac{\partial}{\partial\phi}$$

$$+ \cot^2\theta \cos^2\phi \frac{\partial^2}{\partial\phi^2} \Big].$$

$$L_y^2 = \left(\frac{\hbar}{i}\right)^2 \left[ \cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right]^2$$

$$= L_x^2 \Big|_{\phi \to \phi - \pi/2} \rightarrow \begin{bmatrix} \cos\phi \to +\sin\phi \\ \sin\phi \to -\cos\phi. \end{bmatrix}$$

$$= \left(\frac{\hbar}{i}\right)^2 \Big[ \cos^2\phi \frac{\partial^2}{\partial\theta}$$

$$+ \csc^2\theta \sin\phi \cos\phi \frac{\partial}{\partial\phi}$$

$$- \cot\theta \sin\phi \cos\phi \frac{\partial^2}{\partial\theta\partial\phi}$$

$$+ \cot\theta \sin^2\phi \frac{\partial}{\partial\theta}$$

$$- \cot\theta \sin\phi \cos\phi \frac{\partial^2}{\partial\theta\partial\phi}$$

$$+ \cot^2\theta \sin\phi \cos\phi \frac{\partial}{\partial\phi}$$

$$+ \cot^2\theta \sin^2\phi \frac{\partial^2}{\partial\phi^2} \Big]$$

$$L_z^2 = \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial\phi^2}.$$

$$\therefore L_x^2 + L_y^2 + L_z^2 = \left(\frac{\hbar}{i}\right)^2 \Big[ (\sin^2\phi + \cos^2\phi) \frac{\partial^2}{\partial\theta^2}$$

$$+ \cot\theta (\cos^2\phi + \sin^2\phi) \frac{\partial}{\partial\theta}$$

$$+ \cot^2\theta (\sin^2\phi + \cos^2\phi) \frac{\partial^2}{\partial\phi^2}$$

$$+ \frac{\partial^2}{\partial\phi^2} \Big]$$

$$= -\hbar^2 \left[ \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + (1+\cot^2\theta) \frac{\partial^2}{\partial\phi^2} \right]$$

$$= -\hbar^2 \left[ \frac{1}{\sin^2\theta} \cdot \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta} \cdot \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) \right].$$

$$= \vec{L}^2.$$

2 If $A$ is an operator such that..

10

$$[A, L_x] = [A, L_y] = 0.$$

Calculate $[A, L_z]$, and $[A, L^2]$.

$$\Rightarrow [A, L_z] = \left[A, \frac{1}{i\hbar}[L_x, L_y]\right] = \frac{1}{i\hbar}[A, L_xL_y - L_yL_x]$$

$$\left\{ [L_x, L_y] = i\hbar L_z \rightarrow L_z = \frac{1}{i\hbar}[L_x, L_y] \right\}$$

$$= \frac{1}{i\hbar}\left\{ AL_xL_y - AL_yL_x - \cancel{AL_x}\, L_xL_yA + L_yL_xA \right\}$$

$[A, L_x] = [A, L_y] = 0$ 이므로..

$AL_xL_y = L_xL_yA$ & $AL_yL_x = L_yL_xA$.

$= 0.$

$\therefore [A, L_z] = 0.$

$$[A, L^2] = [A, L_x^2 + L_y^2 + L_z^2]$$

$$= \left\{ [A, L_x]L_x + L_x[A, L_x] \right\} + (y_{''}) + (z_{''})$$

$$= 0.$$

$\boxed{3}$ Calculate the following:

10 $\langle l'm' | L_x | l m \rangle$, $\langle l'm' | L_y | l m \rangle$, $\langle l'm' | L_z | l m \rangle$,

$\langle l'm' | L_x^2 | l m \rangle$, $\langle l'm' | L_y^2 | l m \rangle$.

$\Rightarrow$ i) $L_\pm = L_x \pm i L_y$.

$$L_+ | l m \rangle = \hbar \sqrt{l(l+1) - m(m+1)}\, | l, m+1 \rangle$$

$$L_- | l m \rangle = \hbar \sqrt{l(l+1) - m(m-1)}\, | l, m-1 \rangle.$$

$$\cancel{L_+ + L_y =}$$

$$L_+ + L_- = 2 L_x \qquad \rightarrow \qquad L_x = \frac{1}{2}(L_+ + L_-)$$

$$L_+ - L_- = 2 i L_y \qquad\qquad L_y = \frac{1}{2i}(L_+ - L_-)$$

$$L_x^2 = \frac{1}{4}(L_+ + L_-)^2 = \frac{1}{4}\left[(L_+)^2 + L_+L_- + L_-L_+ + (L_-)^2\right]$$

$$L_y^2 = -\frac{1}{4}(L_+ - L_-)^2 = -\frac{1}{4}\left[(L_+)^2 - L_+L_- - L_-L_+ + (L_-)^2\right].$$

ii) $$L_x | l m \rangle = \frac{1}{2}(L_+ + L_-) | l m \rangle$$

$$= \cancel{\frac{1}{2}\hbar\left[\sqrt{l(l+1) - m(m+1)} + \sqrt{l(l+1) - m(m-1)}\right] | l}$$

$$= \frac{\hbar}{2}\left[\sqrt{l(l+1) - m(m+1)}\, | l\ m+1 \rangle + \sqrt{l(l+1) - m(m-1)}\, | l\ m-1 \rangle\right]$$

$\therefore \langle l' m' | L_x | l m \rangle$

$$= \frac{\hbar}{2} \Big[ \sqrt{l(l+1) - m(m+1)} \, \langle l' m' | l \; m+1 \rangle$$

$$+ \sqrt{l(l+1) - m(m-1)} \, \langle l' m' | l \; m-1 \rangle \Big]$$

$$= \frac{\hbar}{2} \Big[ \sqrt{l(l+1) - m(m+1)} \, \delta_{ll'} \delta_{m', m+1}$$

$$+ \sqrt{l(l+1) - m(m-1)} \, \delta_{l'l} \delta_{m', m-1} \Big].$$

$\langle l' m' | L_y | l m \rangle$

$$= \frac{\hbar}{2i} \Big[ \sqrt{l(l+1) - m(m+1)} \, \delta_{ll'} \delta_{m', m+1}$$

$$- \sqrt{l(l+1) - m(m-1)} \, \delta_{l'l} \delta_{m', m-1} \Big].$$

$$\langle l' m' | L_z | l m \rangle = m\hbar \, \langle l' m' | l m \rangle$$

$$= m\hbar \, \delta_{l'l} \delta_{m'm}$$

iii) $(L_+)^2 |l\, m\rangle = \hbar L_+ \sqrt{l(l+1)-m(m+1)}\, |l\ m+1\rangle$

$= \hbar^2 \sqrt{l(l+1)-m(m+1)}\sqrt{l(l+1)-(m+1)(m+2)}\, |l\ m+2\rangle.$

$(L_-)^2 |l m\rangle = \hbar^2 \sqrt{l(l+1)-m(m-1)}\sqrt{l(l+1)-(m-1)(m-2)}\, |l\ m-2\rangle$

$L_+L_- |l m\rangle = \hbar L_+ \sqrt{l(l+1)-m(m-1)}\, |l\ m-1\rangle$

$= \hbar^2 \sqrt{l(l+1)-m(m-1)}\sqrt{l(l+1)-m(m-1)}\, |l\ m\rangle.$

$L_-L_+ |l m\rangle = \hbar^2 \sqrt{l(l+1)-m(m+1)}\sqrt{l(l+1)-m(m+1)}\, |l m\rangle.$

$(L_+L_- - L_-L_+)|l m\rangle$

$= [l(l+1)+l(l+1)-m(m-1)-m(m+1)]|l m\rangle$

$= [2l(l+1)-2m^2]|l m\rangle$

$\therefore \langle l' m' | L_x^2 | l m\rangle$

$= \frac{\hbar^2}{4}\sqrt{l(l+1)-m(m+1)}\sqrt{l(l+1)-(m+1)(m+2)}\ \delta_{l'l}\,\delta_{m',m+2}$

$+ \frac{\hbar^2}{2}[l(l+1)-m^2]\ \delta_{l'l}\,\delta_{mm'}$

$+ \frac{\hbar^2}{4}\sqrt{l(l+1)-m(m-1)}\sqrt{l(l+1)-(m-1)(m-2)}\ \delta_{l'l}\,\delta_{m',m-2}.$

$$\langle l' m' | L_y^2 | l\, m \rangle$$

$$= -\frac{\hbar^2}{4}\sqrt{l(l+1)-m(m+1)}\sqrt{l(l+1)-(m+1)(m+2)}\;\delta_{l'l}\,\delta_{m',m+2}$$

$$+\frac{\hbar^2}{2}\left[\,l(l+1)-m^2\,\right]\delta_{l'l}\,\delta_{m'm}.$$

$$-\frac{\hbar^2}{4}\sqrt{l(l+1)-m(m-1)}\sqrt{l(l+1)-(m-1)(m-2)}\;\delta_{l'l}\,\delta_{m',m-2}.$$

$\boxed{4}$ Assume that a particle is in an eigenstate $|l\, m\rangle$.

1° Show that..

$$\langle L_x \rangle = \langle L_y \rangle = 0$$

Also show that..

$$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{1}{2}\left[\,l(l+1)\hbar^2 - m^2\hbar^2\,\right].$$

$\Rightarrow$ 2번 결과 이용 ??

$$l' m' \to l\, m.$$

$$\langle L_x \rangle = \langle l\, m | L_x | l\, m \rangle = 0$$

$$\langle L_y \rangle = 0.$$

$$\langle L_x^2 \rangle = \langle l\, m | L_x^2 | l\, m \rangle = \frac{\hbar^2}{2}\left[\,l(l+1)-m^2\,\right].$$

$$\langle L_y^2 \rangle = \langle l\, m | L_y^2 | l\, m \rangle = \frac{\hbar^2}{2}\left[\,l(l+1)-m^2\,\right].$$

$\boxed{5}$ A system is given by the superposition of

15 angular momentum eigenstates $|l\ m\rangle$ as..

$$|\psi\rangle = a|11\rangle + b|10\rangle + c|1\,{-1}\rangle.$$

Calculate the expectation values of $L_x$ and $L_y$ in this state.

$\Rightarrow$ ~~3번 결과 이용~~

~~$|11\rangle$ state 확률 $|a|^2$~~.

$$\langle L_x\rangle = \langle\psi|L_x|\psi\rangle$$

$$= \left(a^*\langle 11| + b^*\langle 10| + c^*\langle 1\,{-1}|\right) L_x \left(a|11\rangle + b|10\rangle + c|1\,{-1}\rangle\right)$$

같은 state들끼리 계산은 0. ( 3번결과 )

$\nearrow$ (complex conjugate)

$$\therefore \text{c.c.}\Big[\ a^*b\ \langle 1 1|L_x|1 0\rangle + a^*c\ \langle 1 1|L_x|1 -1\rangle.$$

$$+ b^*a\langle 1 0|L_x|1 1\rangle \;/\; + \overset{\text{c.c.}}{}\ b^*c\ \langle 1 0|L_x|1 -1\rangle$$

$$+ c^*a\langle 1 -1|L_x|1 1\rangle + c^*b\ \langle 1 -1|L_x|1 0\rangle.\ \Big]\ \text{c.c.}$$

$$= \therefore 2\mathrm{Re}\Big[\ a^*b\ \langle 1 1|L_x|1 0\rangle$$

$$+ b^*c\ \langle 1 0|L_x|1 -1\rangle.$$

$$+ c^*a\ \langle 1 -1|L_x|1 1\rangle\ \Big].$$

2번 이용

$$= 2\mathrm{Re}\Big[\ a^*b \cdot \frac{\hbar}{2} \cdot \sqrt{1\cdot 2 - 0\cdot 1}\ .$$

$$+ b^*c \cdot \frac{\hbar}{2}\sqrt{1\cdot 2 - (-1)\cdot 0}$$

$$\cancel{+ c^*a \cdot \frac{\hbar}{2} \cdot \times 0}\ \Big]$$

2칸 차이!

$$= 2\mathrm{Re}\Big[\ a^*b \cdot \frac{\hbar}{2}\sqrt{2} + b^*c \cdot \frac{\hbar}{2}\sqrt{2}\ \Big]$$

$$= 2\left(\frac{\hbar}{2}\sqrt{2}\right)\mathrm{Re}\left[a^*b + b^*c\right] = \sqrt{2}\,\hbar\,\mathrm{Re}\left[a^*b + b^*c\right]$$

마찬가지로.. $\langle L_y \rangle$

$$\langle L_y \rangle = 2\mathrm{Re}\Big[ a^*b \langle 1\,1|L_y|1\,0\rangle + b^*c \langle 1\,0|L_y|1\,-1\rangle + \cancel{c^*a \langle 1\,-1|L_y|1\,1\rangle}^{\,0} \Big]$$

$$= 2\mathrm{Re}\Big[ a^*b \frac{\hbar}{2i}\sqrt{1\cdot 2 - 0\cdot 1} + b^*c \frac{\hbar}{2i}\sqrt{1\cdot 2 - (-1)\cdot 0} \Big]$$

$$= 2\cdot\frac{\hbar}{2}\sqrt{2}\ \mathrm{Re}\Big[\frac{1}{i}\cdot(a^*b + b^*c)\Big]$$

$$= \sqrt{2}\hbar\ \mathrm{Re}\big[-i(a^*b + b^*c)\big]$$

$$= \sqrt{2}\hbar\ \mathrm{Im}(a^*b + b^*c)$$

$$\therefore \begin{cases} \langle L_x \rangle = \sqrt{2}\hbar\ \mathrm{Re}\big[\cancel{ab^* + b^*c}\ a^*b + b^*c\big] \\ \langle L_y \rangle = \sqrt{2}\hbar\ \mathrm{Im}\big[a^*b + b^*c\big]. \end{cases}$$

$\boxed{6}$ The spherical harmonics $Y_{21}(\theta,\phi)$ is given by

15

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}.$$

Using the raising and lowering operators, compute explicitly $Y_{22}$ and $Y_{20}$.

$$\Rightarrow \left(\frac{L_\pm}{\hbar}\right) = \pm e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right).$$

$$\therefore \cancel{Y_{22}(\theta,\phi) =} \left(\frac{L_+}{\hbar}\right) Y_{21}(\theta,\phi)\cdot \cancel{N_{21}} \quad \cancel{\rightarrow \text{normalization}}$$

$$= + e^{+i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\cdot\left[-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}\right]$$

$$= -\sqrt{\frac{15}{8\pi}}\Big[\left(\cos\theta\cos\theta - \sin\theta\sin\theta\right)e^{i2\phi}$$

$$+ i\cot\theta\cdot\sin\theta\cos\theta\cdot e^{i2\phi}\cdot i\Big]$$

$$= +\sqrt{\frac{15}{8\pi}}\,\sin^2\theta\, e^{i2\phi}$$

$$\cancel{Y_{20}(\theta,\phi)} \; \cancel{=} \; \left(\frac{L_-}{\hbar}\right) Y_{21}(\theta,\phi)$$

$$= \left[ -e^{-i\phi} \left( \frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\phi} \right) \right] \left[ -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta \, e^{i\phi} \right]$$

$$= \sqrt{\frac{15}{8\pi}} \left[ \left( \cos\theta\cos\theta - \sin\theta\sin\theta \right) - i \frac{\cos\theta}{\cancel{\sin\theta}} \cdot \cancel{\sin}\theta\cos\theta \; i \right]$$

$$= \sqrt{\frac{15}{8\pi}} \left( 2\cos^2\theta - \sin^2\theta \right)$$

$$= \sqrt{\frac{15}{8\pi}} \left( 3\cos^2\theta - 1 \right)$$

$5$

$\frac{15}{24}$ $\cancel{3}\;6$

$$\therefore \left(\frac{L_+}{\hbar}\right) Y_{21}(\theta,\phi) = \sqrt{\frac{15}{8\pi}} \sin^2\theta \, e^{i2\phi}$$

$$\Downarrow$$

$$Y_{22}(\theta,\phi) = \frac{\left(\frac{L_+}{\hbar}\right) Y_{21}(\theta,\phi)}{\sqrt{2\cdot 3 - 1\cdot 2}} = \frac{1}{2}\sqrt{\frac{15}{8\pi}} \sin^2\theta \, e^{i2\phi}$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{i2\phi}$$

$$\left(\frac{L_-}{\hbar}\right) Y_{21}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\,(3\cos^2\theta - 1)$$

$$\therefore \quad Y_{20}(\theta,\phi) = \frac{1}{\sqrt{2\cdot 3 - 1\cdot 0}} \cdot \left(\frac{L_-}{\hbar}\right) Y_{21}(\theta,\phi)$$

$$= \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{15}{8\pi}}\,(3\cos^2\theta - 1)$$

$$= \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1).$$