Chap. 9 Hamiltonian Mechanics

9.1 Legendre Transformation

The second form Euler-Lagrange eq. is

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}, \quad \text{where} \quad H = p\dot{x} - L\Big|_{(x,p)}$$

The transformation from Lagrangian to Hamiltonian in mathmetical terminology is Legendre Transformation.

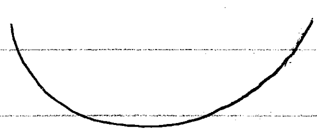
When we derive the E-L eq. by making use Lagrangian, we varied x, However, $\dot{x}$ was depended on x ;

$$\delta\dot{x} = \frac{d}{dt}\delta x$$

So, when we varied action,

$$\delta S = \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt \left(\quad \right) \underbrace{\delta x}_{\text{only using } x.}$$

Now, cosider the Legendre transformation, in other word, how $\dot{x}$ becom independent of x.

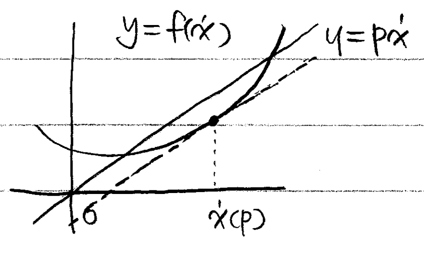


⇒ Convex function.

$f''(\dot{x}) > 0 \Leftrightarrow$ convex

$f'(\dot{x})$ is increasing as $\dot{x}$ increasing.

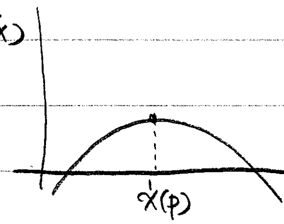
Let us conside a convex function. ; Drawing the straghit line that pass through the origin with slope p.



⇒ If the function is continuous and derivative is also continuous, then we should be able to fine the position s where in the middle where the slope e derivative has the same slope as thi straghit line. ⇒ called $\dot{x}(p)$

Let's define the function

$$y = F(p, \dot{x}) = p\dot{x} - f(\dot{x})$$



$\dot{x}(p)$ is the point where $F(p, \dot{x})$ has the maximum.

$$\Rightarrow \frac{\partial F(p, \dot{x})}{\partial \dot{x}} = p - \frac{\partial f}{\partial \dot{x}} = 0$$

$$\Rightarrow \frac{\partial f}{\partial \dot{x}} = p$$

Legendre transform $g(p)$ of the function $f(\dot{x})$ is

$$g(p) = F[p, \dot{x}(p)] = p\dot{x}(p) - f[\dot{x}(p)]$$

replace $\dot{x}$ by function of p.

$\Rightarrow$ independent of $\dot{x}$

Examples of convex functions

We just interest on kinetic energy only. So, consider Lagrangian,

$$L = T(\dot{x}) - U(x)$$

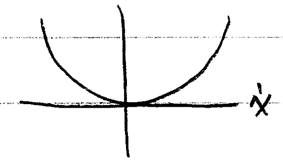
If we make Legendre transformation,

$$H = p\dot{x} - T(\dot{x}) + U(x) \Big|_{\dot{x} = \dot{x}(p) = \frac{p}{m}}$$

Still kinetic energy but now, function of p only

$$= \tilde{T}(p) + U(x)$$

ex1) $f(\dot{x}) = \dot{x}^2$; convex function $\Rightarrow$ can make Legendre transformati



$T(\dot{x}) = \frac{1}{2}m\dot{x}^2$

$F(p,\dot{x}) = p\dot{x} - \frac{1}{2}m\dot{x}^2$

Then take the derivative,

$$\frac{\partial F(p,\dot{x})}{\partial \dot{x}} = p - m\dot{x} = 0 \quad \Rightarrow \quad \dot{x}(p) = \frac{p}{m}$$

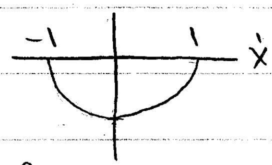
Therefore,

$$\tilde{T}(p) = p\dot{x}(p) - \frac{1}{2}m\dot{x}^2(p) = \frac{p^2}{m} - \frac{p^2}{2m} = \frac{p^2}{2m} \quad \square.$$

Let's make the table for Legendre transformation

$$\boxed{T(\dot{x}) = \frac{1}{2}m\dot{x}^2 \quad \underset{\dot{x}(p)=\frac{p}{m}}{\Rightarrow} \quad \tilde{T}(p) = \frac{p^2}{2m}}$$

ex2) $f(\dot{x}) = -\sqrt{1-\dot{x}^2}$; also convex function



$f(\dot{x}) = -m_0c^2\sqrt{1-(\dot{x}/c)^2}$,

where c is speed of light, m_0 is rest mass.

Define

$$F(p,\dot{x}) = p\dot{x} + m_0c^2\sqrt{1-(\dot{x}/c)^2}$$

and take the derivative,

$$\frac{\partial F}{\partial \dot{x}} = p + m_0\cancel{c^2}\frac{-\dot{x}/\cancel{c^2}}{\sqrt{1-(\dot{x}/c)^2}} = 0$$

$$\Rightarrow p = \frac{m_0\dot{x}}{\sqrt{1-(\dot{x}/c)^2}} = m\dot{x}, \quad \text{where } m = \frac{m_0}{\sqrt{1-(\dot{x}/c)^2}}$$

$$\Rightarrow p = \frac{m_0\dot{x}(p)}{\sqrt{1-\left(\frac{\dot{x}(p)}{c}\right)^2}}$$

$$p = \frac{m_0 \dot{x}(p)}{\sqrt{1-(\dot{x}(p)/c)^2}}$$

Let substitute $\alpha = \dot{x}(p)/c$, then the momentum becomes

$$p = \frac{m_0 \alpha c}{\sqrt{1-\alpha^2}} \Rightarrow \frac{p}{m_0 c} = \frac{\alpha}{\sqrt{1-\alpha^2}}$$

$$\Rightarrow \left(\frac{p}{m_0 c}\right)^2 = \frac{\alpha^2}{1-\alpha^2} = \frac{-(1-\alpha^2)+1}{1-\alpha^2} = \frac{1}{1-\alpha^2} - 1$$

$$\Rightarrow \frac{p^2 + m_0^2 c^2}{(m_0 c)^2} = \frac{1}{1-\alpha^2}$$

$$\Rightarrow \frac{(m_0 c)^2}{p^2 + (m_0 c)^2} = 1 - \alpha^2$$

$$\Rightarrow \alpha^2 = 1 - \frac{(m_0 c)^2}{p^2 + (m_0 c)^2} = \frac{p^2}{p^2 + (m_0 c)^2}$$

$$\Rightarrow \frac{\dot{x}(p)}{c} = \frac{p}{\sqrt{p^2 + (m_0 c)^2}}$$

$$\Rightarrow \dot{x}(p) = \frac{pc}{\sqrt{p^2 + (m_0 c)^2}} \quad \square$$

Now, put $\dot{x}(p)$ into $F(p, \dot{x})$,

$$F(p, \dot{x}) = p\dot{x} + m_0 c^2 \sqrt{1 - \left(\frac{\dot{x}}{c}\right)^2}$$

$$= \frac{p^2 c}{\sqrt{p^2 + (m_0 c)^2}} + m_0 c^2 \sqrt{\frac{(m_0 c)^2}{p^2 + (m_0 c)^2}}$$

$$= \frac{p^2 c + m_0^2 c^3}{\sqrt{p^2 + (m_0 c)^2}}$$

$$= c\sqrt{p^2 + (m_0 c)^2}$$

$$= \sqrt{(pc)^2 + (m_0 c^2)^2}$$

Now, actualy that function $f(\dot{x})$ is Lagrangian.

$$L(\dot{x}) = -m_0 c^2 \sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}$$

and Hamiltonian is

$$H = \sqrt{(pc)^2 + (m_0 c^2)^2}$$

$\rightsquigarrow$ Free relativistic particle

If $p=0$, this means that $\dot{x}=0$, then the Hamiltonian becomes

$$H_{rest} = m_0 C^2$$

Chap. 9 Hamiltonian Mechanics

9.2 EOM in terms of Hamiltōian

$$\vec{F} = m\vec{a} = \frac{d}{dt}\vec{P} \Rightarrow \frac{d}{dt}\vec{P} - \vec{F} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) + \frac{\partial}{\partial x_i}U = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}_i}(T-U)\right] - \frac{\partial}{\partial x_i}(T-U) = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

We have Lagrangian and Hamiltonian are

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \Rightarrow -\frac{\partial L}{\partial t} = 0$$

$$H = \frac{p^2}{2m} + U(x) \qquad \frac{dH}{dt} = 0; \quad H \text{ is conserved}$$

Then variation of Hamiltonian is

$$dH = \frac{p_i}{m}dp_i + \frac{\partial U}{\partial x_i}dx_i$$

We have know that

$$\dot{x}(p) = \frac{p}{m} \rightarrow \boxed{\dot{x} = \frac{p}{m}}$$

Since,

$$\frac{\partial U}{\partial x} = -F = -\dot{p} \Rightarrow \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} = -\dot{p}$$

$$\Rightarrow \boxed{\dot{p} = -\frac{\partial H}{\partial x}}$$

$$H(p, x) \Rightarrow \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}, \quad \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} = -\dot{p}$$

$\Rightarrow$ We called Hamilton's eq.

Let combine these two.

$$\frac{\partial H}{\partial p} = \dot{x} = \frac{p}{m} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial H}{\partial p} = \frac{\dot{p}}{m} = -\frac{1}{m}\frac{\partial H}{\partial x}$$

$$\Rightarrow \underbrace{m\frac{d}{dt}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial x} = 0}_{\text{EOM}}$$

exactly same as the E-L eq.

Chap. 9 Hamiltonian Mechanics

9.4 Canonical EOM

1st, defind conjugate momentum,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Let compute the total differential of Hamiltonian.

$$H = p_i \dot{q}_i - L(q_j, \dot{q}_k; t) \Big|_{\dot{q}_i = \dot{q}_i(p_i)}$$

$$dH = dp_i \dot{q}_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

$$= \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i + \overbrace{\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)}^{0} d\dot{q}_i - \frac{\partial L}{\partial t} dt$$

$$= \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt$$

Since,

$$dH(p_i, q_i; t) = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

Then, compare the two result,

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i}$$

⇒ exactly same as our previously result.

Next, let consider second form of E-L eq,

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$$\frac{dH(p_i, q_i; t)}{dt} = \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial t}$$

$$= \dot{q}_i \dot{p}_i + (-\dot{p}_i)\dot{q}_i + \left(-\frac{\partial L}{\partial t}\right)$$

$$\Rightarrow \frac{dH}{dt} = -\frac{\partial L}{\partial t} \ \square.$$

Chap. 9 Hamiltonian Mechanics

9.5 Modified Hamilton's Principle

Reason why we learned Lagrangian first was a when w applied the least action principle, it is natural to make use of Lagrangian. Because action was

$$S = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i ; t)$$

and varied action was

$$\delta S = \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i ,$$

because $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$.

Now, we would like to re-expressed this one in terms of Hamiltonian. Then recall Legendre transformation,

$$H(p_i, q_i) = p_i \dot{q}_i - L(q_i, \dot{q}_i ; t)$$

$$\Rightarrow L(q_i, \dot{q}_i ; t) = p_i \dot{q}_i - H(p_i, q_i)$$

Then the variation of action is

$$\delta S = \int_{t_1}^{t_2} dt\, \delta \left(p_i \dot{q}_i - H(p_i, q_i) \right)$$

$$= \int_{t_1}^{t_2} dt \left[\delta p_i\, \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right]$$

$$= \int_{t_1}^{t_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\frac{\partial H}{\partial q_i} + \dot{p}_i \right) \delta q_i \right]$$

Since,

$$\frac{\partial H}{\partial p_i} = \dot{q}_i , \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$\Rightarrow$ δp_i and δq_i varied independently, both are simultaneously zero!

Chap. 9 Hamiltonian Mechanics

9.6 Constants of Motion

The Second form of E-L eq

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Another important thing here, the E-L eq is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

and suppose Lagrangian independent of coordinate.

$\boxed{\frac{\partial L}{\partial q_i} = 0}$; we call q_i is cyclic coordinate. (ignorable)

$\Rightarrow \frac{d}{dt} P_i = 0$; momentum is conserved. (const. of motion)

Recall the canonical variable.

$\boxed{\dot{P}_i = -\frac{\partial H}{\partial q_i}}$ $\qquad \dot{q}_i = \frac{\partial H}{\partial P_i}$

So, the cyclic coordinate can be found two way

1) making use of Lagrangian ; $\frac{\partial L}{\partial q_i} = 0$
2) making use of Hamiltonian ; $\frac{\partial H}{\partial q_i} = 0$

Because at the least in the Cartecian coordinate system, kineti energy independent of coordinate. However, if we make use up generalized coordinate, for example angle, then coordinate dependenc appear in kinetic energy term. Therefore, if we use generali coordinate to find a force, it is just effective one, it is no a real force some time.

ex 1) $\frac{\partial L}{\partial q_k} = 0 \Rightarrow q_k$ is cyclic

$$\frac{\partial}{\partial q_k}\left[p_i \dot{q}_i - H(p_i, q_i)\right] = -\frac{\partial H(p_i, q_i)}{\partial q_k} = \dot{p}_k = 0$$

9.7 Poisson Braket

Define that

$$[f, g]_{PB} = \begin{vmatrix} \frac{\partial f}{\partial q_i} & \frac{\partial f}{\partial p_i} \\ \frac{\partial g}{\partial q_i} & \frac{\partial g}{\partial p_i} \end{vmatrix} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

We assumed that f and g are mechanical variable that are function of p_j and q_j.

1st of all, let us compute,

$$[f, f]_{PB} = 0$$

$$[g, f]_{PB} = -[f, g]_{PB} \quad ; \quad \text{anti-symmetric}$$

Thm 9.6 $\quad \frac{dF(p_i, q_i ; t)}{dt} = \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial F}{\partial t}$

But we know that Hamilton's eq ;

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i$$

Therefore,

$$\frac{dF(p_i, q_i ; t)}{dt} = \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial F}{\partial t}$$

$$= [F, H]_{PB} + \frac{\partial F}{\partial t}$$