GEST 011, Newton's Clock & Heisenberg's Dice, Fall 2013

The  Story

Mahn-Soo Choi (Korea University)

October 5, 2013 (v5.1)

Image from `http://pi.ytmnd.com/`

http://www.poppyw.com/

하나,둘,셋,넷,다섯,여섯,일곱,여덟,아홉,열,열하나,열둘,· · ·

one,two,three,four,five,six,seven,eight,nine,ten,eleven,twelve,· · ·

http://www.poppyw.com/

하나,둘,셋,넷,다섯,여섯,일곱,여덟,아홉,열,열하나,열둘,$\cdots$

one,two,three,four,five,six,seven,eight,nine,ten,eleven,twelve,$\cdots$

1,2,3,4,5,6,7,8,9,10,11,12,$\cdots$

The Number Zero (0)

http://www.poppyw.com/

하나,둘,셋,넷,다섯,여섯,일곱,여덟,아홉,열,열하나,열둘,· · ·

one,two,three,four,five,six,seven,eight,nine,ten,eleven,twelve,· · ·

1,2,3,4,5,6,7,8,9,10,11,12,· · ·

십, 백, 천, 만; 억, 조, 경, . . .

10, 100, 1000, 10000, 10^8, 10^{12}, 10^{16}, . . .

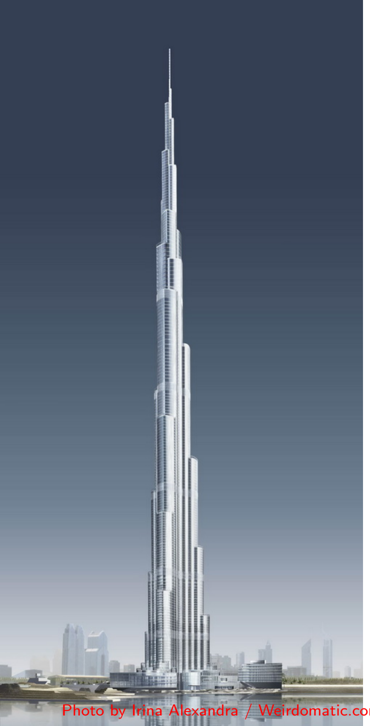


Photo by Irina Alexandra / Weirdomatic.com

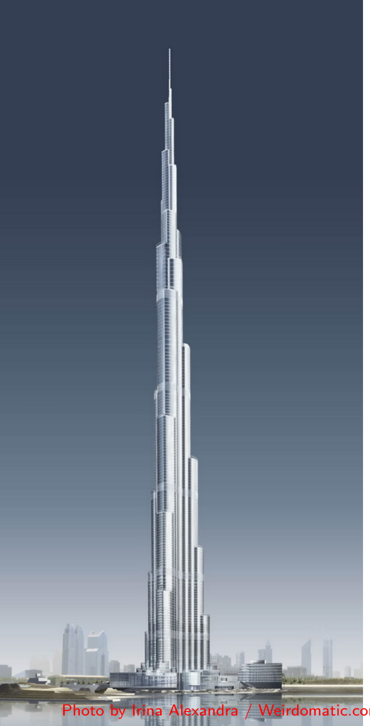


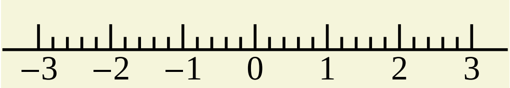
Photo by Irina Alexandra / Weirdomatic.com

European	American	
...	...	
3rd	4th	floor
2nd	3rd	floor
1st	2nd	floor
ground	1st	floor
1st	1st	basement floor
2nd	2nd	basement floor
3rd	3rd	basement floor
...	...	

Negative Numbers

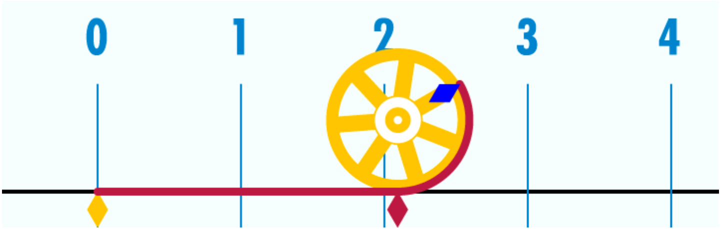
Photo by Irina Alexandra / Weirdomatic.com

European	American	
...	...	
3^{rd}	4^{th}	floor
2^{nd}	3^{rd}	floor
1^{st}	2^{nd}	floor
ground	1^{st}	floor
1^{st}	1^{st}	basement floor
2^{nd}	2^{nd}	basement floor
3^{rd}	3^{rd}	basement floor
...	...	



π and Circle

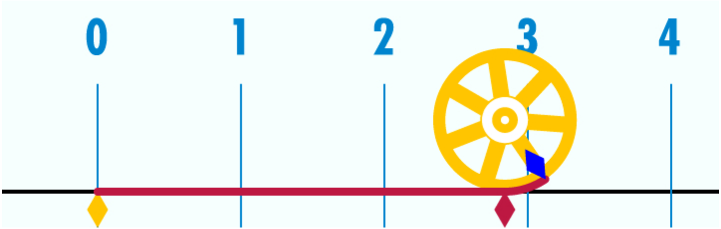
π, The Definition



$$\pi = \frac{(\textit{circumference})}{(\textit{diameter})}$$

Image courtesy of Wikipedia

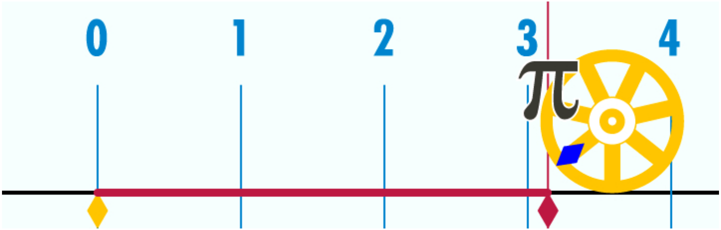
π, The Definition



$$\pi = \frac{(\textit{circumference})}{(\textit{diameter})}$$

Image courtesy of Wikipedia

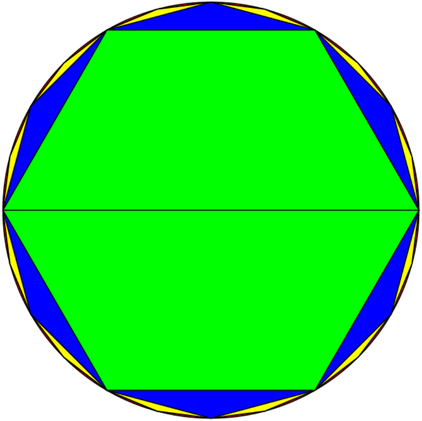
π, The Definition



$$\pi = \frac{(circumference)}{(diameter)}$$

Image courtesy of Wikipedia

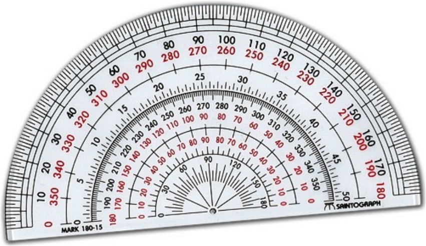
π, Its Value



Wikipedia

3.14159 26535 89793 23846 26433
83279 50288 41971 69399 37510
58209 74944 59230 78164 06286
20899 86280 34825 34211 7068
...

Angle with π

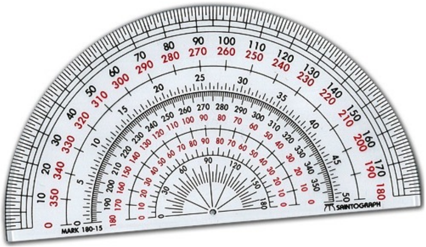


http://www.popular.com.sg/

$$\theta = \frac{2}{5} \times 180^\circ = 72^\circ$$

Angle with π

(Radian)



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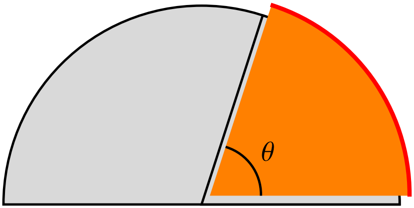
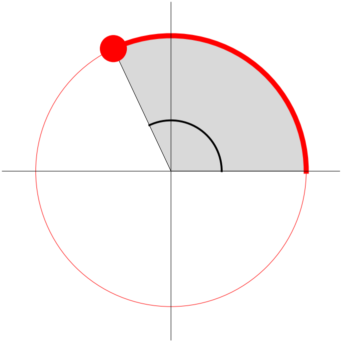


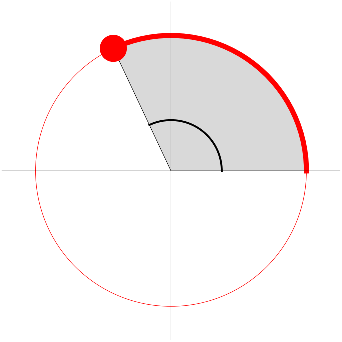
$$\theta = \frac{2}{5} \times 180^\circ = 72^\circ$$

$$\theta = \frac{2}{5} \times \pi$$

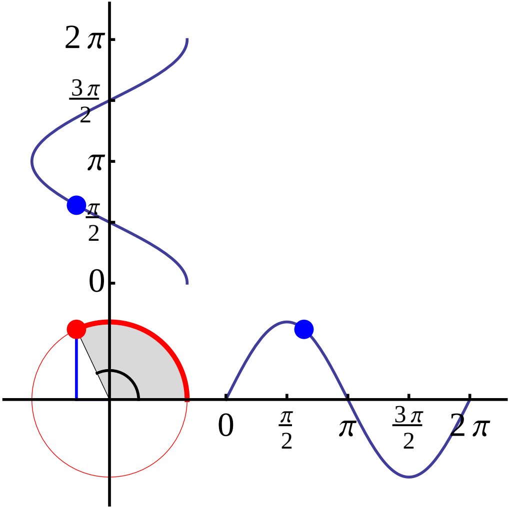
π and Rotation



π and Rotation



$x(\theta) = \cos(\theta)$
$y(\theta) = \sin(\theta)$



The Imaginary Number i

$$x^2 = 1 \quad \Rightarrow \quad x = ?$$

$$x^2 = 4 \quad \Rightarrow \quad x = ?$$

$$x^2 = 9 \quad \Rightarrow \quad x = ?$$

$$x^2 = 144 \quad \Rightarrow \quad x = ?$$

$$x^2 = 150 \quad \Rightarrow \quad x = ?$$

$$\vdots$$

Square & Square Root $\sqrt{x}$

$$x^2 = 1 \quad \Rightarrow \quad x = \sqrt{1} = 1$$

$$x^2 = 4 \quad \Rightarrow \quad x = \sqrt{4} = 2$$

$$x^2 = 9 \quad \Rightarrow \quad x = \sqrt{9} = 3$$

$$x^2 = 144 \quad \Rightarrow \quad x = \sqrt{144} = 12$$

$$x^2 = 150 \quad \Rightarrow \quad x = \sqrt{150} \approx 12.2474$$

$$\vdots$$

$$x^2 = -1 \quad \Rightarrow \quad x = ?$$

$$x^2 = -4 \qquad \Rightarrow \quad x = ?$$

$$x^2 = -9 \qquad \Rightarrow \quad x = ?$$

$$x^2 = -144 \qquad \Rightarrow \quad x = ?$$

$$x^2 = -150 \qquad \Rightarrow \quad x = ?$$

The Imaginary Number i

$$x^2 = -1 \quad \Rightarrow \quad x = i$$

$$x^2 = -4 \qquad \Rightarrow \quad x = i\sqrt{4} = 2i$$

$$x^2 = -9 \qquad \Rightarrow \quad x = i\sqrt{9} = 3i$$

$$x^2 = -144 \qquad \Rightarrow \quad x = i\sqrt{144} = 12i$$

$$x^2 = -150 \qquad \Rightarrow \quad x = i\sqrt{150} \approx 12.2474i$$

The Set of Complex Numbers

$\mathbb{C}$

$$1 + i = ?$$

Addition

$1 + i = ?$

$$(1 + x) + (2 + 3x) = 3 + 4x$$
$$3(1 + 2x) = 3 + 6x$$

Addition

$$1 + i = ?$$

$$(1 + i) + (2 + 3i) = 3 + 4i$$
$$3(1 + 2i) = 3 + 6i$$

$$(1+i)(1-2i) = ?$$

Multiplication

$$(1 + i)(1 - 2i) =?$$

$$(1 + i)(1 - 2i) = 1 \cdot 1 + i \cdot 1 + 1 \cdot (-2i) + i \cdot (-2i)$$

Multiplication

$$(1+i)(1-2i) =?$$

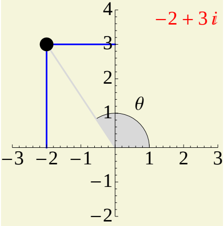
$$(1+i)(1-2i) = 3 - i$$

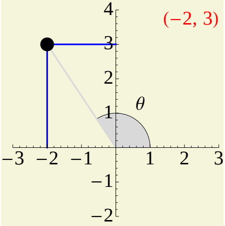
$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Complex Plane

$$\mathbb{C} \to \mathbb{R}^2$$

$$z \equiv x + iy \mapsto (x, y)$$





Definition (Conjugate)

$$z^* = x - iy$$

Definition (Magnitude)

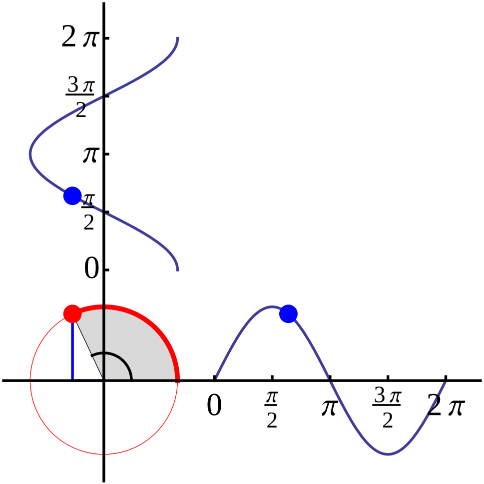
$$|z| \equiv \sqrt{x^2 + y^2} = \sqrt{zz^*}$$

Definition (Polar Form)

$$z = r(cos\theta + i \sin\theta)$$

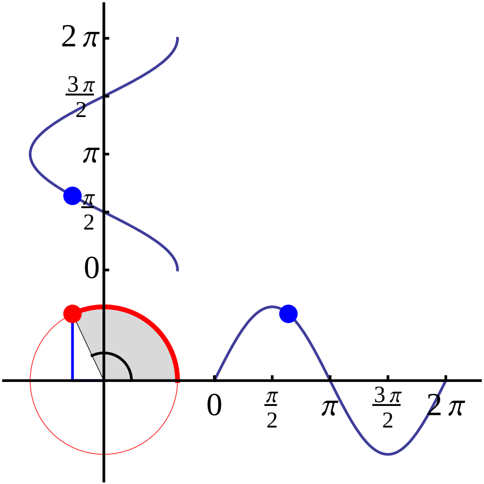
i and Rotation

i and Rotation I



$$x(\theta) = \cos(\theta)$$
$$y(\theta) = \sin(\theta)$$

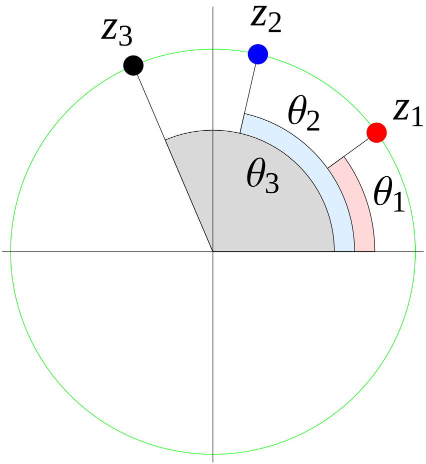
i and Rotation I



$$x(\theta) = \cos(\theta)$$
$$y(\theta) = \sin(\theta)$$

$$z(\theta) = \cos(\theta) + i\sin(\theta)$$

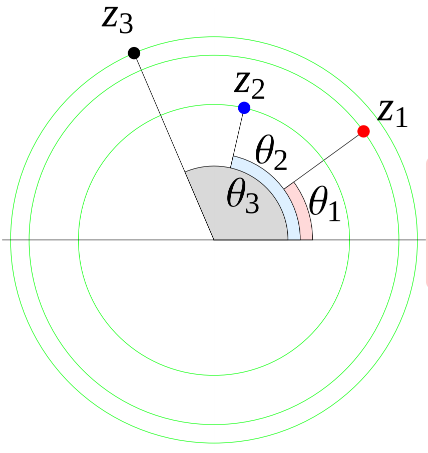
i and Rotation II



$$z_3 = z_1 z_2$$
$$\theta_3 = \theta_1 + \theta_2$$

Geometry by Complex Numbers

(product of complex numbers)

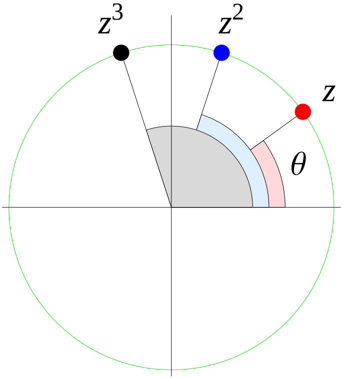


$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

i and Rotation III

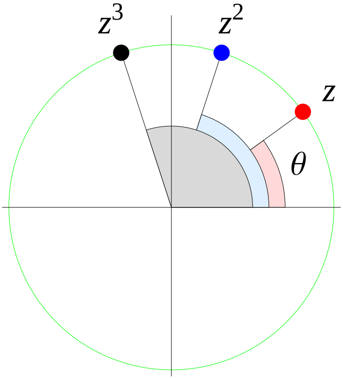
(De Moivre's formula)

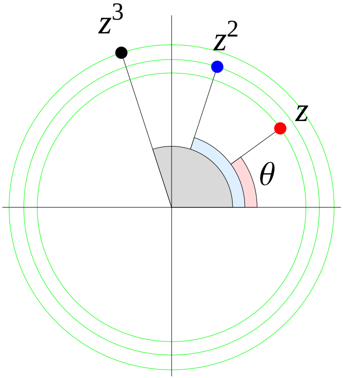


$$[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$$

i and Rotation III

(De Moivre's formula)





$$[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$$

$$|z^n| = |z|^n$$

$$\arg(z^n) = n\arg(z)$$

The Mathematical Constant e

$$e \equiv \lim_{N \to \infty} \left(1 + \frac{1}{N}\right)^N$$

Compounded Interest

Let W_0 be the amount of money originally invested, and x the annual rate of interest. How mouch do you earn after one year?

- When the interest is compounded yearly?

$$W_0 \to W_0(1 + x)$$

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- When the interest is compounded every half a year?

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- When the interest is compounded every half a year?

$$W_0 \to W_0(1 + x/2)^2$$

- When the interest is compounded quarterly?

$$W_0 \to W_0(1 + x/4)^4$$

Compounded Interest

Let W_0 be the amount of money originally invested, and x the annual rate of interest. How mouch do you earn after one year?

- When the interest is compounded yearly?

$$W_0 \to W_0(1 + x)$$

- When the interest is compounded every half a year?

$$W_0 \to W_0(1 + x/2)^2$$

- When the interest is compounded quarterly?

$$W_0 \to W_0(1 + x/4)^4$$

- When the interest is compounded monthly?

$$W_0 \to W_0(1 + x/12)^{12}$$

Compounded Interest

Let W_0 be the amount of money originally invested, and x the annual rate of interest. How mouch do you earn after one year?

- When the interest is compounded yearly?

$$W_0 \to W_0(1+x)$$

- When the interest is compounded every half a year?

$$W_0 \to W_0(1+x/2)^2$$

- When the interest is compounded quarterly?

$$W_0 \to W_0(1+x/4)^4$$

- When the interest is compounded monthly?

$$W_0 \to W_0(1+x/12)^{12}$$

- When the interest is compounded daily?

$$W_0 \to W_0(1+x/365)^{365}$$

$$\begin{aligned}
(1+1)^1 &= 2 \\
\left(1+\frac{1}{2}\right)^2 &= 2.25 \\
\left(1+\frac{1}{3}\right)^3 &= \frac{64}{27} \approx 2.37 \\
\left(1+\frac{1}{4}\right)^4 &= \frac{625}{256} \approx 2.44 \\
&\vdots
\end{aligned}$$

The Mathematical Constant e

$$(1+1)^1 = 2$$

$$\left(1+\frac{1}{2}\right)^2 = 2.25$$

$$\left(1+\frac{1}{3}\right)^3 = \frac{64}{27} \approx 2.37$$

$$\left(1+\frac{1}{4}\right)^4 = \frac{625}{256} \approx 2.44$$

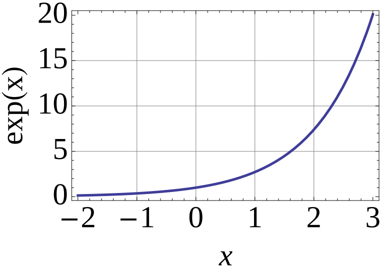
$$\vdots$$

$$\boxed{e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \approx 2.71828$$

The Exponential Function $\exp(x)$

$$\boxed{\exp(x) \equiv \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n}$$

The Exponential Function $\exp(x)$



$$\boxed{\exp(x) \equiv \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n} = e^x$$

$$e^2 = e \times e$$

$$e^3 = e \times e \times e$$

$$\vdots$$

$$e^{-2} = \frac{1}{e^2}$$

$$e^{-3} = \frac{1}{e^3}$$

$$\vdots$$

The Exponential Function $\exp(x)$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3{\cdot}2{\cdot}1} + \frac{x^4}{4{\cdot}3{\cdot}2{\cdot}1} + \cdots$$

$$\boxed{e^{a+b} = e^a\, e^b}$$

$$\boxed{\frac{dx}{dt} = x} \quad \Leftrightarrow \quad \boxed{x(t) = e^t}$$

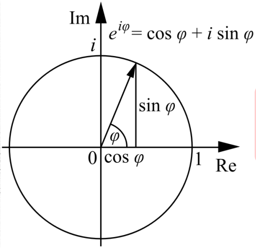
$$\boxed{e^x = y} \quad \Leftrightarrow \quad \boxed{x = \log y}$$

π, i, e

Euler's Formula

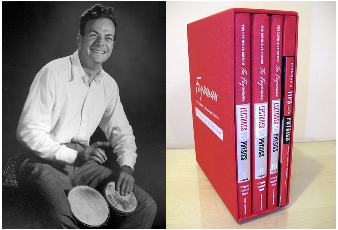
(the deep relation among "π", "e", "i")

Leonhard Euler (1707–1783)



$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Images courtesy of Wikipedia



Both images courtesy of Wikipedia

We summarize with this, the most remarkable formula in mathematics

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

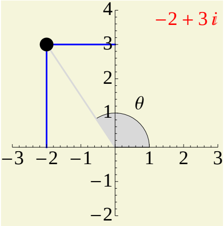
This is our jewel.

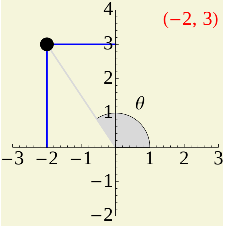
(Feynman, Leighton & Sands 1989, Section 22-6)

Complex Plane
(Revisited)

$$\mathbb{C} \to \mathbb{R}^2$$

$$z \equiv x + iy \mapsto (x, y)$$





Definition (Conjugate)

$$z^* = x - iy$$

Definition (Magnitude)

$$|z| \equiv \sqrt{x^2 + y^2} = zz^*$$

Definition (Polar Form)

$$z = r(cos\theta + i\sin\theta)$$

Complex Plane

(Revisited)

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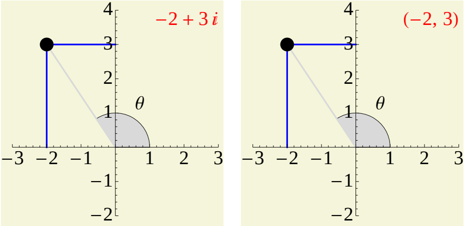

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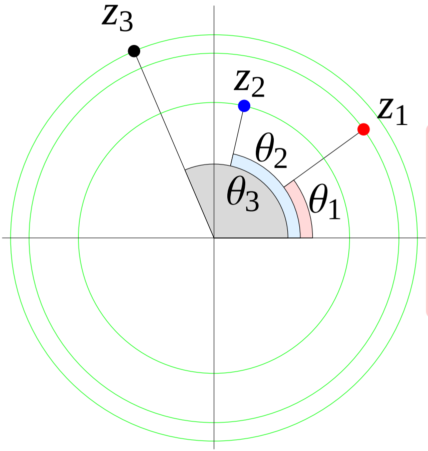
Definition (Polar Form)

$$z = r\, e^{i\theta}$$

Geometry by Complex Numbers

(phasors)

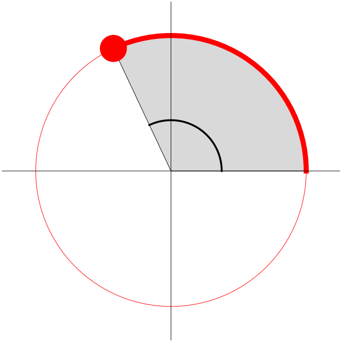


$$|z_1 z_2| = |z_1||z_2|$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$r_1 e^{i\theta_1}\ r_2 e^{i\theta_2} = r_1 r_2\ e^{i(\theta_1+\theta_2)}$$

Rotation, Oscillation, and Wave

(Summary)

Rotation

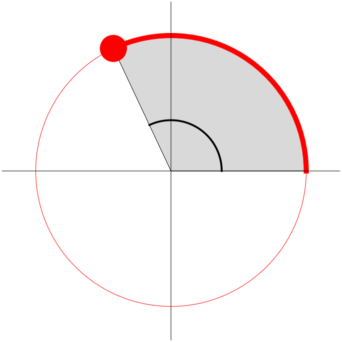


$$\theta = \omega t$$

ω= "angular frequency"

- It rotates ☐ radian per one second.
- It takes ☐ seconds per one rotation.

Rotation



$$\theta = \omega t$$

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$$z(t) = \exp(i\omega t) \equiv e^{i\omega t}$$

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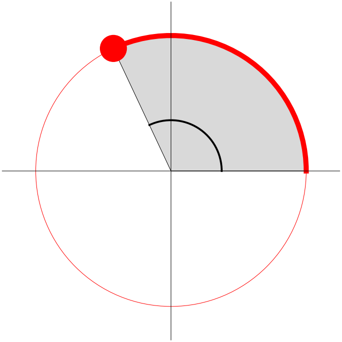
Rotation



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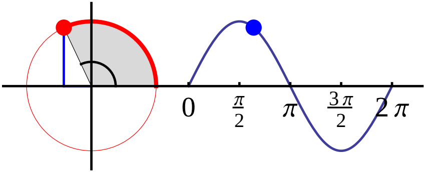
- It rotates $\boxed{\omega}$ radian per one second.
- It takes $\boxed{2\pi/\omega}$ seconds per one rotation.

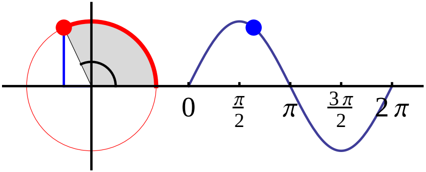
Oscillation in Time



$$x(t) = \cos(\omega t)$$
$$y(t) = \sin(\omega t)$$
$$z(t) = \exp(i\omega t)$$
ω= "angular frequency"

- It rotates ☐ radian per one second.
- It takes ☐ seconds per one rotation.

Oscillation in Time



$$x(t) = \cos(\omega t)$$
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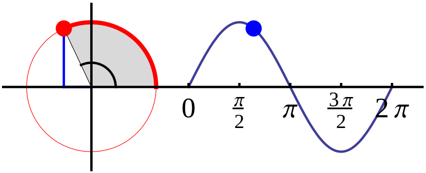
$$z(t) = \exp(i\omega t)$$
ω= "angular frequency"

- It rotates $\boxed{\omega}$ radian per one second.
- It takes $\boxed{2\pi/\omega}$ seconds per one rotation.

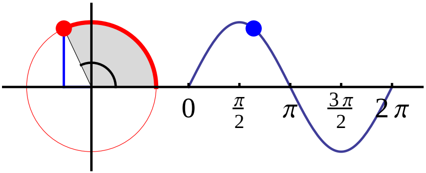
Oscillation in Space



$$x(t) = \cos(kx)$$
$$y(t) = \sin(kx)$$
$$z(t) = \exp(ikx)$$
k= "wave number"

- It rotates [ ] radian per one meter.
- It takes [ ] meters per one rotation.

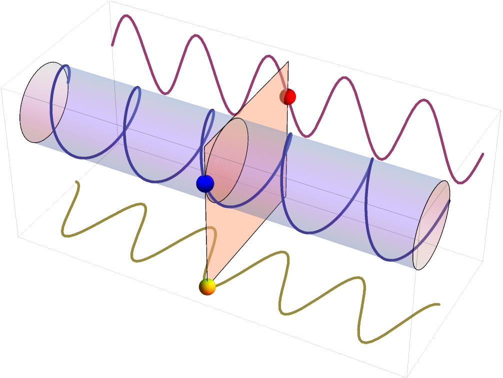
Oscillation in Space



$$x(t) = \cos(kx)$$
$$y(t) = \sin(kx)$$
$$z(t) = \exp(ikx)$$
k= "wave number"

- It rotates $\boxed{k}$ radian per one meter.
- It takes $\boxed{2\pi/k}$ meters per one rotation.

Wave



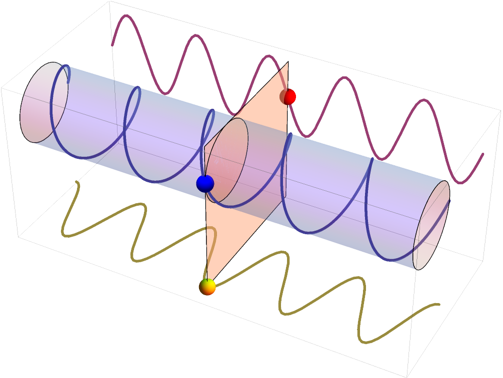
$$\psi(z,t) = \exp[i(\omega t - kz)]$$

ω = "angular frequency"

k = "wave number"

- It rotates [ ] radian per one second.
- It rotates [ ] radian per one meter.
- It takes [ ] seconds per one rotation.
- It takes [ ] meters per one rotation.

Wave



$$\psi(z,t) = \exp[i(\omega t - kz)]$$

ω = "angular frequency"

k = "wave number"

- It rotates $\boxed{\omega}$ radian per one second.
- It rotates $\boxed{k}$ radian per one meter.
- It takes $\boxed{2\pi/\omega}$ seconds per one rotation.
- It takes $\boxed{2\pi/k}$ meters per one rotation.

Summary

- π, i, e share the same spirit.
- Rotation, oscillation, and wave are essentially the same.
- $\sin(\theta)$, $\cos(\theta)$, and $\exp(i\theta)$ are essentially the same.

References

R. Feynman, R. B. Leighton & M. L. Sands, *The feynman lectures on physics*, Vol. I (Addison-Wesley, Redwood City, 1989).

D. Halliday, R. Resnick & J. Walker, *Fundamentals of physics*, 7th (John Wiley and Sons, Inc, New York, 2005).