KECE321 Communication Systems I

(Haykin Sec. 4.5 - Sec. 4.6)

Lecture #15, May 9, 2012

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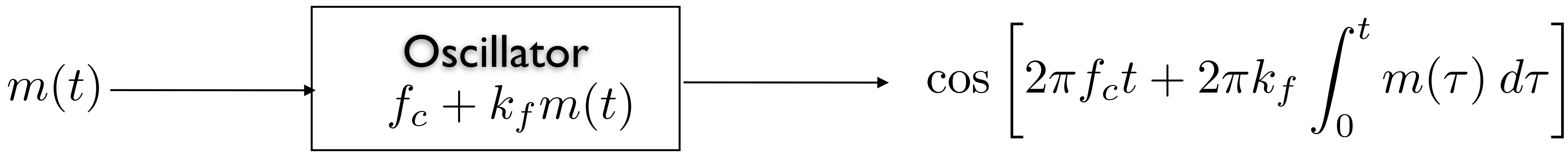
Summary

- Generation of FM waves
  - Direct method
  - Indirect method
- Demodulation of FM signals
  - Frequency discriminator
  - Phase locked loop (PLL)

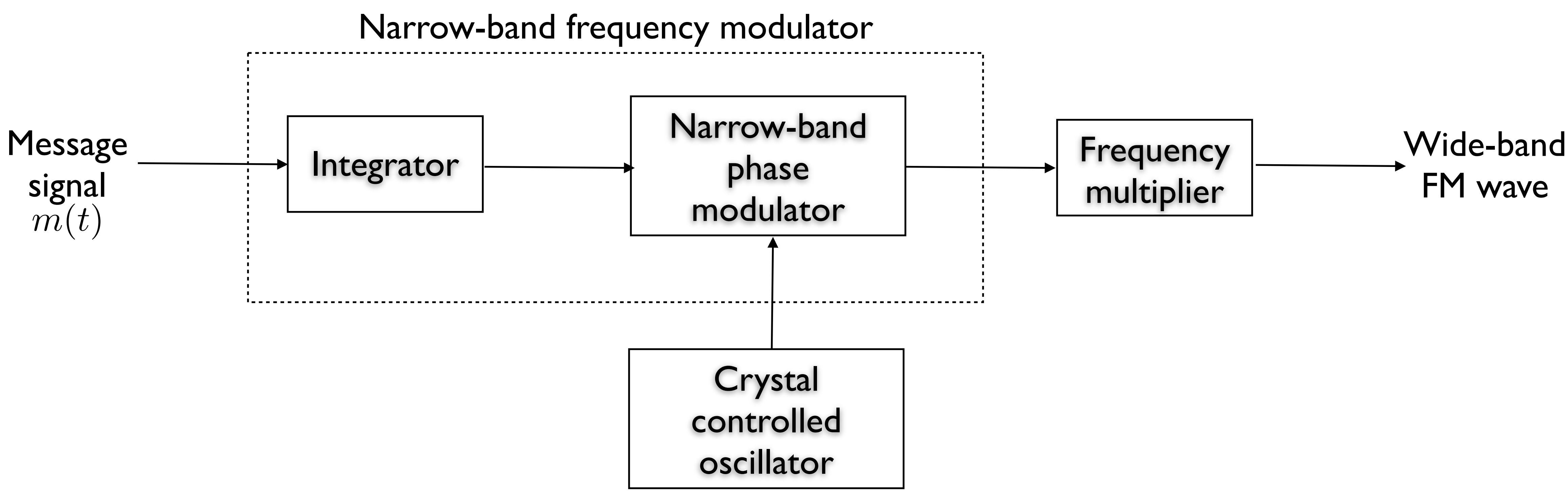
Generation of FM Waves: Direct Method

- Oscillator to be controlled by the message signal

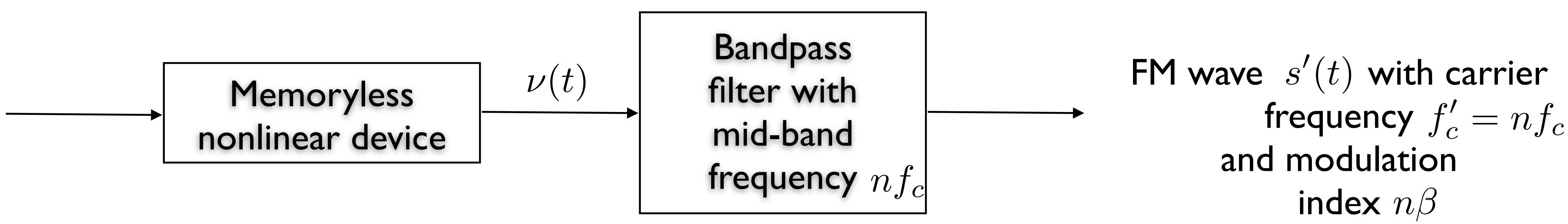


Generation of FM Waves: Indirect Method

- Block diagram of the indirect method of generating a wide-band FM wave



Frequency multiplier



$$\nu(t) = a_1 s(t) + a_2 s^2(t) + \cdots + a_n s^n(t)$$

Demodulation of FM Signals: Frequency Discriminator

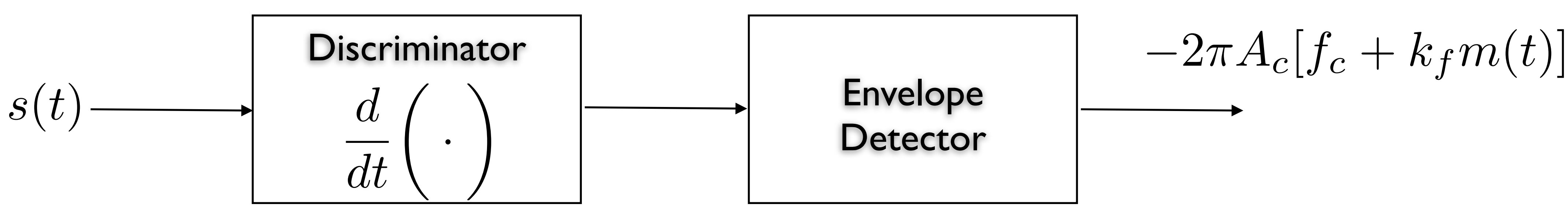
- Recall the FM signal

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right)$$

- Derivative of the FM signal with respect to time

envelope

$$\frac{ds(t)}{dt} = -2\pi A_c[f_c + k_f m(t)] \sin\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right)$$



Fourier transform of differentiation

$$g(t) \longleftrightarrow G(f)$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df \qquad G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt$$

Then

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df \\ &= \int_{-\infty}^{\infty} G(f)(j2\pi f) e^{j2\pi ft}\, df \\ &= \int_{-\infty}^{\infty} \Big[j2\pi f G(f) \Big] e^{j2\pi ft}\, df \end{aligned}$$

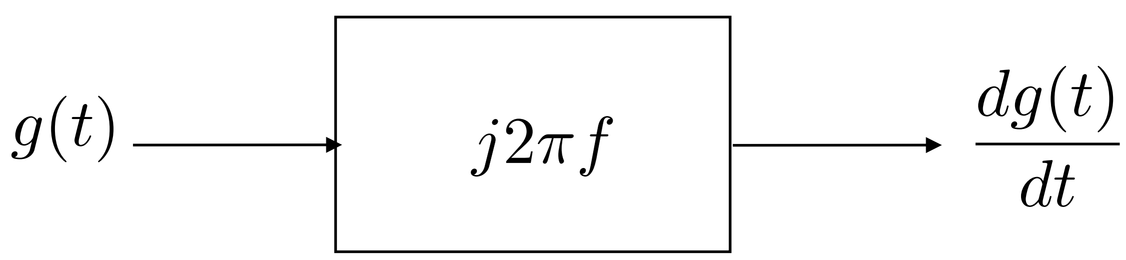
Hence,

$$\mathcal{F}\left[\frac{d}{dt} g(t)\right] = j2\pi f G(f)$$

- Now we want to design the filter for the differentiator

  - Note that

$$\frac{d}{dt} \longleftrightarrow j2\pi f$$



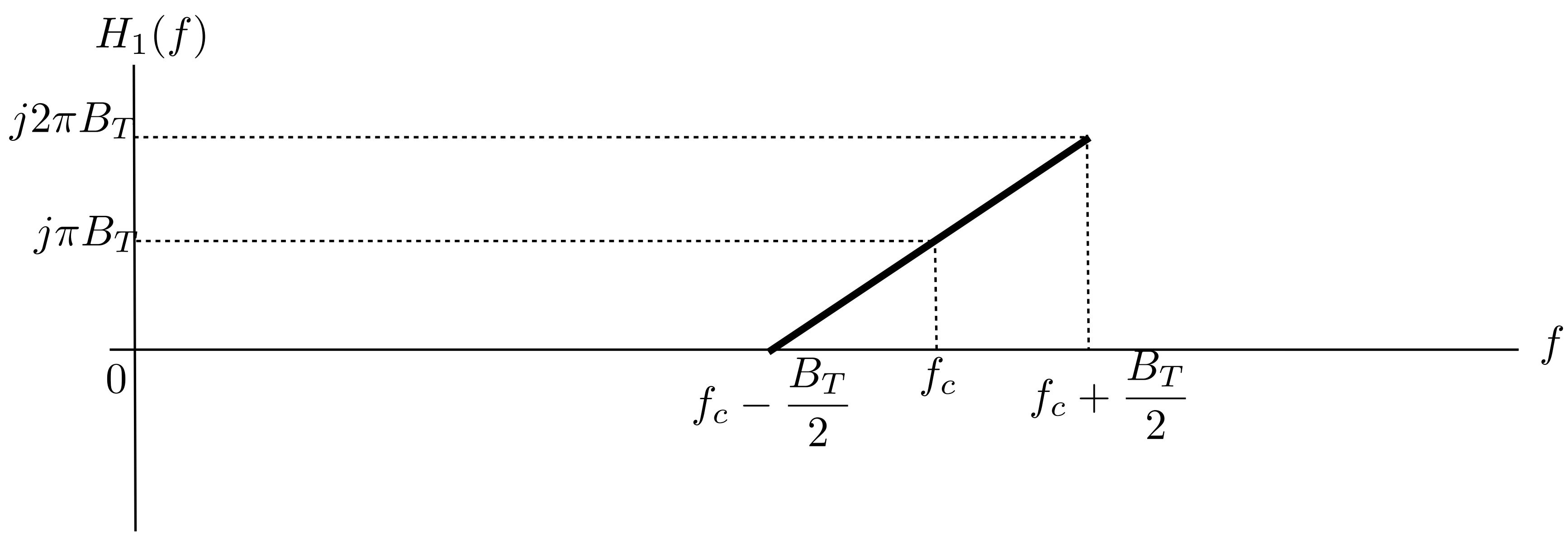
  - For the demodulation of FM signal, we need the differentiator operating over the frequency range

$$f_c - (B_T/2) \leq |f| \leq f_c + (B_T/2)$$

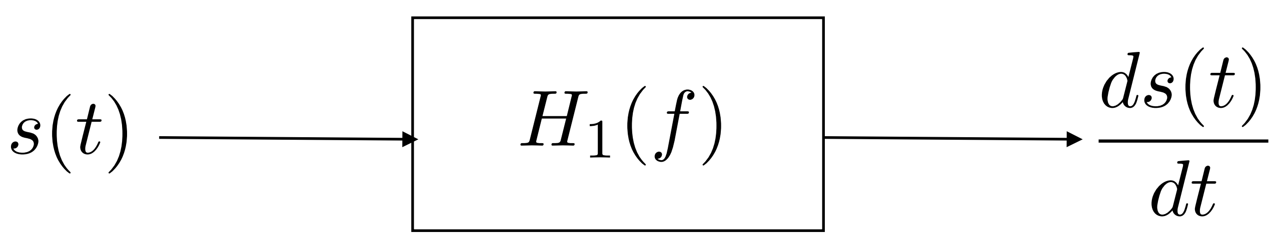
where B_T is the transmission bandwidth of the incoming FM signal $s(t)$.

- A transfer function of the differentiator operating over the frequency range aforementioned can be described by

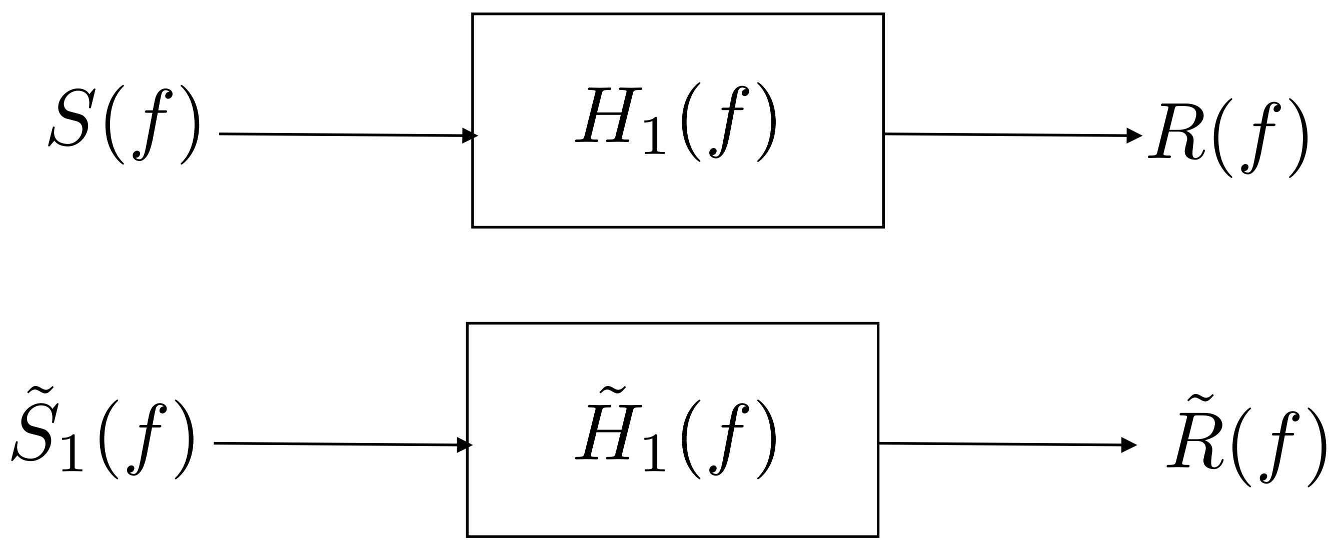
$$H_1(f) = \begin{cases} j2\pi[f - (f_c - B_T/2)], & f_c - (B_T/2) \le |f| \le f_c + B_T/2 \\ 0, & \text{otherwise} \end{cases}$$



- Differentiator for the demodulation of FM signal



- Now we want to learn that the pass-band signal can be represented in its equivalent low-pass form such as



$$R(f) = S(f)H_1(f)$$
$$\tilde{R}(f) = \tilde{S}(f)\tilde{H}_1(f)$$

$$r(t) = s(t) * h(t)$$

$$\tilde{r}(t) = \tilde{s}(t) * \tilde{h_1}(t)$$

$$s(t) = \Re[\tilde{s}(t)e^{j2\pi f_c t}]$$

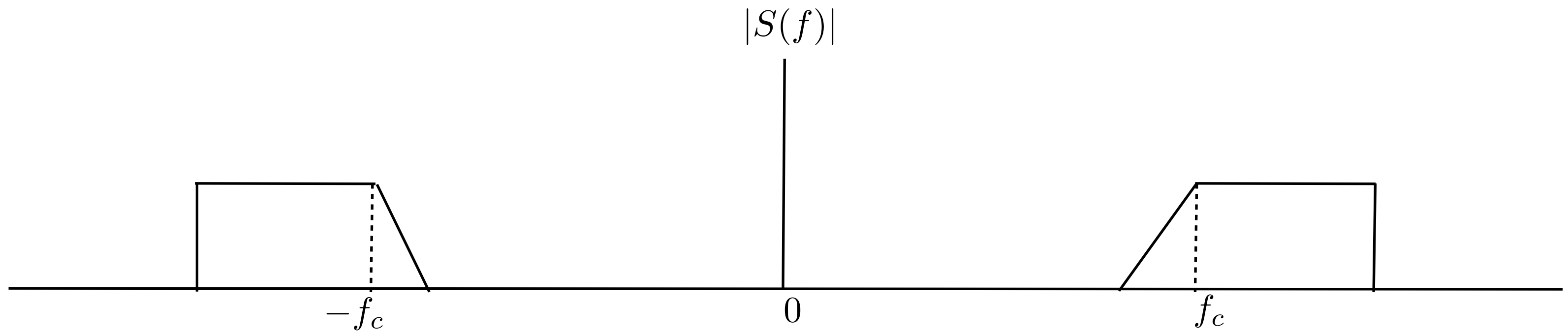
$$r(t) = \Re[\tilde{r}(t)e^{j2\pi f_c t}]$$

$$h_1(t) = 2\Re[\tilde{h}_1(t)e^{j2\pi f_c t}]$$

Representation of Band-Pass Signal

- Recall the positive signal component

$$S_+(f) = 2u(f)S(f)$$



- Analytic signal

$$\begin{aligned} s_+(t) &= \int_{-\infty}^{\infty} S_+(f)\, df \\ &= \mathcal{F}^{-1}[2u(f)] * \mathcal{F}^{-1}[S(f)] \end{aligned}$$

$$\mathcal{F}^{-1}[2u(f)] = \delta(t) + \frac{j}{\pi t}$$

$$= \left[\delta(t) + \frac{j}{\pi t}\right] * s(t) = s(t) + j\frac{1}{\pi t} * s(t)$$

- We can define

$$\hat{s}(t) = \frac{1}{\pi t} * s(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} \, d\tau$$

$$s_+(t) = s(t) + j\hat{s}(t)$$

- Define

$$\tilde{S}(f) = S_+(f + f_c)$$

- The equivalent time domain relation is

$$\begin{aligned} \tilde{s}(t) &= s_+(t)e^{-j2\pi f_c t} \\ &= [s(t) + j\hat{s}(t)]e^{-j2\pi f_c t} \end{aligned}$$

  - or equivalently

$$s(t) + j\hat{s}(t) = \tilde{s}(t)e^{j2\pi f_c t}$$

- Hence,

$$s(t) = \Re[\tilde{s}(t)e^{j2\pi f_c t}]$$

- Lowpass signal representation in frequency domain

  - Previously we proved the following three equivalent form

$$s(t) = \Re[\tilde{s}(t)e^{j2\pi f_c t}]$$

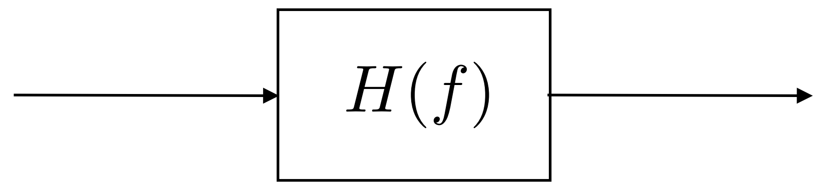
  - Its Fourier transform is

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} \left\{\Re[\tilde{s}(t)e^{j2\pi f_c t}]\right\} e^{-j2\pi ft}\,dt$$

- Using $\Re\{z\} = \frac{1}{2}(z + z^*)$, we have

$$\begin{aligned} S(f) &= \frac{1}{2}\int_{-\infty}^{\infty} [\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}(t)e^{-j2\pi f_c t}]e^{-2j\pi ft}\,dt \\ &= \frac{1}{2}\left[\tilde{S}(f - f_c) + \tilde{S}^*(f + f_c)\right] \end{aligned}$$

- Representation of linear band-pass systems

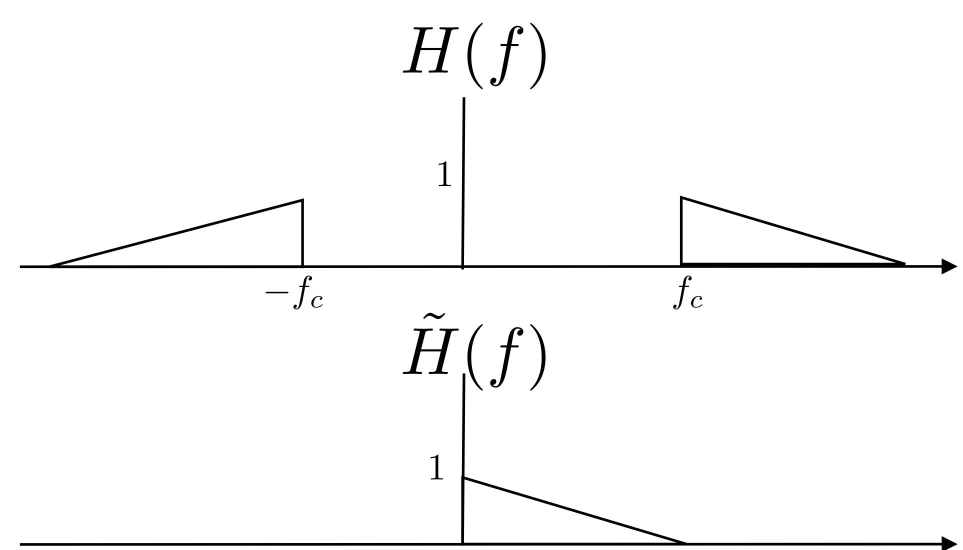


$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t}\, dt$$

$$H^*(-f) \quad = \quad \left[\int_{-\infty}^{\infty} h(t) e^{-j2\pi (-f) t}\, dt \right]^* = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t}\, dt = H(f)$$

- Let us define

$$\tilde{H}(f - f_c) = \begin{cases} H(f) & (f > 0) \\ 0 & (f < 0) \end{cases}$$



- then

$$\tilde{H}^*(-f - f_c) = \begin{cases} 0 & (f > 0) \\ H^*(-f) & (f < 0) \end{cases}$$

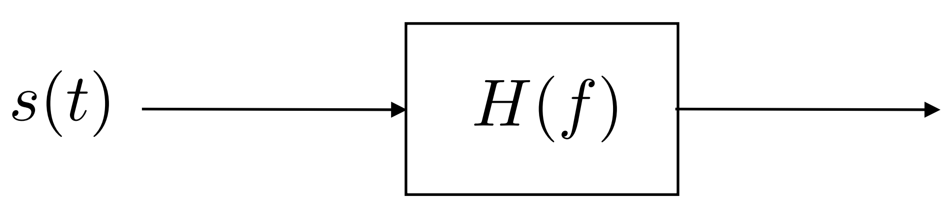
- Finally, the Fourier transform of $H(f)$ is

$$H(f) = \tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c)$$

- Taking the inverse Fourier transform, we have

$$h(t) = 2\Re[\tilde{h}(t)e^{j2\pi f_c t}]$$

- Response of a band-pass system to a band-pass signal



$$r(t) = \int_{-\infty}^{\infty} s(\tau)h(t - \tau)\, d\tau$$

$$R(f) = S(f)H(F)$$

- Then, we can write

$$\begin{aligned} R(f) &= S(f)H(F) \\ &= \frac{1}{2}[\tilde{S}(f-f_c)+\tilde{S}^*(-f-f_c)][\tilde{H}(f-f_c)+\tilde{H}^*(-f-f_c)] \\ &= \frac{1}{2}[\tilde{S}(f-f_c)\tilde{H}(f-f_c)+\tilde{S}^*(-f-f_c)\tilde{H}^*(-f-f_c)] \\ &= \frac{1}{2}[\tilde{R}(f-f_c)+\tilde{R}^*(-f-f_c)] \end{aligned}$$

where $\tilde{R}(f)=\tilde{S}(f)\tilde{H}(f)$

$$r_l(t)=\int_{-\infty}^{\infty}\tilde{s}(\tau)\tilde{h}(t-\tau)\,d\tau$$

- Now let us go back to the filter of the differentiator to be used for the demodulation of the FM signal.

  - The transfer function of the filter is

$$H_1(f) = \begin{cases} j2\pi[f - (f_c - B_T/2)], & f_c - (B_T/2) \le |f| \le f_c + B_T/2 \\ 0, & \text{otherwise} \end{cases}$$

  - Then its equivalent low pass form is

$$\tilde{H}_1(f) = \begin{cases} j2\pi[f + (B_T/2)], & -B_T/2 \le f \le B_T/2 \\ 0, & \text{otherwise} \end{cases}$$

  - Then,

$$\begin{aligned} \tilde{S}_1(f) &= \tilde{H}_1(f)\tilde{S}(f) \\ &= \begin{cases} j\pi\left(f + \frac{1}{2}B_T\right)\tilde{S}(f), & -\frac{1}{2}B_T \le f \le \frac{1}{2}B_T \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

- For $-\frac{1}{2}B_T \le f \le \frac{1}{2}B_T$, we have

$$\frac{d}{dt}\tilde{s}(t)$$

$$\tilde{S}_1(f) = j2\pi f \tilde{S}(f) + j\pi B_T \tilde{S}(f)$$

- Taking the inverse Fourier transform yields

$$\tilde{s}_1(t) = \frac{d}{dt}\tilde{s}(t) + j\pi B_T \tilde{s}(t)$$

- Noting that

$$\tilde{s}(t) = A_c \exp\left(j2\pi k_f \int_0^t m(\tau)\, d\tau\right)$$

- we can rewrite $\tilde{s}_1(t)$ as

$$\tilde{s}_1(t) = j\pi A_c B_T \left[1 + \left(\frac{2k_f}{B_T}\right) m(t)\right] \exp\left(j2\pi k_f \int_0^t m(\tau)\, d\tau\right)$$

- Finally,

$$
\begin{aligned}
s_1(t) &= \Re[\tilde{s}_1(t)\exp(j2\pi f_c t)] \\
&= \pi A_c B_T \left[1 + \left(\frac{2k_f}{B_T}\right) m(t)\right] \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau + \frac{\pi}{2}\right)
\end{aligned}
$$

- Then the envelope detector recovers the message signal $m(t)$, except for a bias. Specifically, under the ideal conditions, the output of the envelope detector is given by

$$
\nu_1(t) = \pi A_c B_T \left[1 + \left(\frac{2k_f}{B_T}\right) m(t)\right]
$$

- To remove the bias, we may use a second slope circuit followed by an envelope detector of its own which gives the output signal at the envelope detector as

$$
\nu_2(t) = \pi A_c B_T \left[1 - \left(\frac{2k_f}{B_T}\right) m(t)\right]
$$

- Now summing $\nu_1(t)$ and $\nu_2(t)$ removes the bias term such as

$$\nu(t) = \nu_1(t) + \nu_2(t) = cm(t)$$

where c is a certain constant.