Digression on Symmetry Operations

$$|\alpha\rangle \longrightarrow S|\alpha\rangle = |\tilde{\alpha}\rangle \qquad S: \text{symmetry operation}$$
$$|\beta\rangle \longrightarrow S|\beta\rangle = |\tilde{\beta}\rangle.$$

$$\langle\alpha|\beta\rangle \longrightarrow \langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle\alpha|S^{\dagger}S|\beta\rangle = \langle\alpha|\beta\rangle.$$

we require $S^{\dagger}S = 1$ unitary

- Rotation $R^{T}R = 1$
- translation $\left(e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{a}}\right)^{\dagger}\left(\quad " \quad\right) = 1$
- rotation $\left(e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\phi}}\right)^{\dagger}\left(\quad\quad\right) = 1$

$\rightarrow$ all of them are unitary operators.

In the case of time reversal, we impose a weaker requirement:

$$|\langle\tilde{\beta}|\tilde{\alpha}\rangle| = |\langle\beta|\alpha\rangle|$$

example) $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^{*} = \langle\alpha|\beta\rangle$

Definition) $|\alpha\rangle \rightarrow |\tilde{\alpha}\rangle = \theta|\alpha\rangle$

$|\beta\rangle \rightarrow |\tilde{\beta}\rangle = \theta|\beta\rangle.$

θ is antiunitary if

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^{*}$$

and $\theta(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^{*}\theta|\alpha\rangle + c_2^{*}\theta|\beta\rangle.$

θ: antiunitary operator

$\rightarrow$: antilinear operator.

We claim that an antiunitary operator θ can be expressed as

$$\theta = UK$$

U: unitary operator
K: complex-conjugate operator

$$K\, c|\alpha\rangle = c^*(K|\alpha\rangle)$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$\rightarrow |\tilde{\alpha}\rangle = K|\alpha\rangle = K\sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$
$$= \sum_{a'} K|a'\rangle\langle a'|\alpha\rangle^*$$
$$= \sum_{a'} |a'\rangle\langle a'|\alpha\rangle^*$$

☆ $K|\alpha\rangle = |\alpha\rangle$. $\quad |\alpha\rangle$: matrix representation $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ real

$$\theta\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = UK\left(c_1|\alpha\rangle + c_2|\beta\rangle\right)$$
$$= U\left(c_1^* K|\alpha\rangle + c_2^* K|\beta\rangle\right)$$
$$= c_1^* UK|\alpha\rangle + c_2^* UK|\beta\rangle$$
$$= c_1^*\theta|\alpha\rangle + c_2^*\theta|\beta\rangle$$

satisfies the antilinear property

$$= \Theta|\alpha\rangle$$

$$|\alpha\rangle \to |\tilde{\alpha}\rangle = UK|\alpha\rangle$$
$$= UK\sum_{a'}|a'\rangle\langle a'|\alpha\rangle$$
$$= \sum_{a'}\left(UK|a'\rangle\right)\left(\langle a'|\alpha\rangle^*\right)$$
$$= \sum_{a'} \langle\alpha|a'\rangle\, U|a'\rangle$$

$$\langle\alpha|\beta\rangle \to \langle\tilde{\alpha}|\tilde{\beta}\rangle = \sum_{a'}\langle a'|\alpha\rangle\langle a'|U^\dagger \sum_{b'}\langle\beta|b'\rangle U|b'\rangle$$
$$= \sum_{a'b'} \cancel{\langle\ \rangle}\langle\beta|b'\rangle\langle a'|\alpha\rangle \langle a'|U^\dagger U|b'\rangle$$
$$= \sum_{a'b'}\langle a'|b'\rangle\langle\beta|b'\rangle\langle a'|\alpha\rangle$$
$$= \sum \delta_{a'b'}\langle\beta|b'\rangle\langle a'|\alpha\rangle$$
$$= \langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

Time-Reversal Operator

$|\alpha\rangle \to \Theta|\alpha\rangle$ $\quad\Theta$: time-reversal operator

→ time-reversed state.
(motion-reversed)

$$\Theta|p\rangle = e^{i\delta}|-p\rangle \qquad \longrightarrow \vec{p} \qquad \longleftarrow -\vec{p}$$

$$\Theta|\vec{J}\rangle = e^{i\delta}|-\vec{J}\rangle \qquad \vec{J} \qquad -\vec{J}.$$

$$\underset{\delta}{|\alpha(t)\rangle} = \overset{U(\delta t)}{\left(1-\frac{i}{\hbar}H\delta t\right)}|\alpha(0)\rangle$$

$\Theta|\alpha(0)\rangle$ and then apply $U(\delta t)$

$$U(\delta t)\Theta|\alpha(0)\rangle = \left(1-\frac{i}{\hbar}H\delta t\right)\Theta|\alpha(0)\rangle$$
$$= \Theta|\alpha(-\delta t)\rangle$$
$$= |\alpha(\delta t)\rangle$$

$$U(\delta t)\Theta|\alpha(0)\rangle = \Theta U(-\delta t)|\alpha(0)\rangle$$
$$= \Theta\left[1-\frac{i}{\hbar}H(-\delta t)\right]|\alpha(0)\rangle$$

$$H\Theta = -\Theta H. \Rightarrow \{H, \Theta\} = 0.$$

$$H(\Theta|n\rangle) = -\Theta H|n\rangle = -\Theta E_n|n\rangle$$
$$= (-E_n)(\Theta|n\rangle)$$

$\Rightarrow$ $\Theta|n\rangle$ is an eigenket with the eigenvalue $-E_n$.

$\Rightarrow$ nonsense

($\because$ free particle's energy = K.E. > 0)

$\vec{p}$ at $t=0$. $|\vec{p}\rangle$ $\Rightarrow$ at $t=0$ $-\vec{p} = \Theta\vec{p}$, $\Theta|\vec{p}\rangle$ $\Rightarrow$ $U(\delta t)\Theta|\vec{p}\rangle = U(\delta t)|-\vec{p}\rangle$

time evolution.

Try again.

$$\left(1-\frac{i}{\hbar}H\delta t\right)\Theta|\alpha\rangle = \Theta\left(1-\frac{i}{\hbar}H(-\delta t)\right)|\alpha\rangle.$$

require antilinear property

previously $-iH\Theta = \Theta iH = i\Theta H$

$\{\Theta, i\}=0$ $\Theta i = -i\Theta$ $\Leftarrow$ $\underbrace{\Theta i = i\Theta}_{\text{wrong!}}$.

Correct.

Then $-iH\Theta = -i\Theta H \Rightarrow [H,\Theta]=0$

$$H(\Theta|n\rangle) = \Theta H|n\rangle = \Theta E_n|n\rangle = E_n(\Theta|n\rangle)$$

$\therefore \Theta|n\rangle$ is still an eigenstate of H.

$\Theta|n\rangle = e^{i\delta}|n\rangle$.

$\langle\beta|\Theta|\alpha\rangle$ let Θ act on kets

avoid to let Θ act to the left.

$$\langle\beta|\Theta|\alpha\rangle = \langle\beta|(\Theta|\alpha\rangle)$$
$$\neq (\langle\beta|\Theta)|\alpha\rangle.$$

$|\tilde{\alpha}\rangle = \Theta|\alpha\rangle$

theorem

$$\langle\beta|\otimes|\alpha\rangle = \langle\tilde{\alpha}|\Theta\otimes^{+}\Theta^{-1}|\tilde{\beta}\rangle$$

We will prove this on the next page.

linear operator

Define $|\gamma\rangle \equiv X^{\dagger}|\beta\rangle$

dual correspondence

$$|\gamma\rangle \to \langle\gamma| = \langle\beta|X$$

$$\langle\beta|X|\alpha\rangle = (\langle\beta|X)|\alpha\rangle$$

$$= \langle\gamma|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\gamma}\rangle \leftarrow |\tilde{\gamma}\rangle = H|\gamma\rangle = H X^{\dagger}|\beta\rangle$$

$$= \langle\tilde{\alpha}|H X^{\dagger}|\beta\rangle$$

$$= \langle\tilde{\alpha}|H X^{\dagger} H^{-1}\, \underbrace{H|\beta\rangle}$$

$$= \langle\tilde{\alpha}|H X^{\dagger} H^{-1}|\tilde{\beta}\rangle \quad ?$$

◎ if X is a Hermitian

$$X = A, \quad A^{\dagger} = A$$

$$\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|H A H^{-1}|\tilde{\beta}\rangle$$

◎ Even or Odd.

$$H A_+ H^{-1} = A_+ : \text{T-even}$$

$$H A_- H^{-1} = -A_- : \text{T-odd}$$

$$H A_\pm H^{-1} = \pm A_\pm$$

$$\langle\beta|A_\pm|\alpha\rangle = \langle\tilde{\alpha}|H A_\pm H^{-1}|\tilde{\beta}\rangle$$

$$\langle\alpha|A_\pm|\beta\rangle^{*} = \pm\langle\tilde{\alpha}|A_\pm|\tilde{\beta}\rangle$$ ($A_\pm$ is a Hermitian)

expectation value:

$$\langle\alpha|A_\pm|\alpha\rangle^{*} = \pm\langle\tilde{\alpha}|A_\pm|\tilde{\alpha}\rangle$$

↓

$$\langle\alpha|A_\pm|\alpha\rangle = \pm\langle\tilde{\alpha}|A_\pm|\tilde{\alpha}\rangle$$

Example) $\vec{p}$ $\vec{p}$: T-odd

$$\langle\alpha|\vec{p}|\alpha\rangle = -\langle\tilde{\alpha}|\vec{p}|\tilde{\alpha}\rangle$$

$$\Theta\vec{p}\Theta^{-1} = -\vec{p} \Rightarrow \Theta\vec{p} = -\vec{p}\Theta \quad : \{\Theta, \vec{p}\} = 0$$

$$\vec{p}\Theta|\vec{p}'\rangle = [\underbrace{-\Theta\vec{p}\Theta^{-1}}_{\vec{p}}]\Theta|\vec{p}'\rangle$$

$$= -\Theta\vec{p}|\vec{p}'\rangle = -\Theta\vec{p}'|\vec{p}'\rangle$$

$$= (-\vec{p}')\Theta|\vec{p}'\rangle \qquad \therefore \Theta|\vec{p}'\rangle = e^{i\delta}|\underline{-\vec{p}'}\rangle$$

In summary

$$\Theta\vec{p}\Theta^{-1} = -\vec{p}$$

$$\Theta|\vec{p}'\rangle = e^{i\delta}|-\vec{p}'\rangle$$

Example) $\vec{x}$: T-even

$$\Theta\vec{x}\Theta^{-1} = \vec{x}$$

$$\Theta|\vec{x}'\rangle = |\vec{x}'\rangle.$$

$$\langle\alpha|\vec{x}|\alpha\rangle = \langle\tilde{\alpha}|\vec{x}|\tilde{\alpha}\rangle.$$

Θ is antiunitary

$$\Theta[x_i, p_j]\Theta^{-1}\Theta = \Theta i\hbar\delta_{ij}\Theta^{-1}\Theta$$

x_i: even, p_j: odd

$$= -i\,\Theta\Theta^{-1}\Theta\,\hbar\delta_{ij}$$

$$\Theta x_i p_j \Theta^{-1} = \Theta x_i \Theta^{-1}\Theta p_j \Theta^{-1} = x_i(-p_j) = -x_i p_j$$

$$= -[x_i, p_j]\Theta\Theta^{-1}\Theta = -i\hbar\delta_{ij}\Theta.$$

Example) $\vec{J}$: T-odd.

$$\Theta J \Theta^{-1} = -J$$

$$\Theta [J^i, J^j] \Theta^{-1} = [J^i, J^j]$$

$$\Theta \, i\hbar \varepsilon^{ijk} J^k \Theta^{-1} = (-i\hbar)\, \varepsilon^{ijk} (-J^k)\, \Theta \Theta^{-1}$$

$$= i\hbar \varepsilon^{ijk} J^k$$

o.K.

Wavefunction

① $|\alpha\rangle$: a state of a spinless single-particle system at $t=0$.

$$|\alpha\rangle = \int d^3x' \, |\vec{x}'\rangle \langle \vec{x}' | \alpha \rangle$$

Time-reversal operation:

$$\Theta |\alpha\rangle = \int d^3x' \, \Theta |\vec{x}'\rangle \langle \vec{x}' | \alpha \rangle^*$$

Because $\Big\}$ $\Theta \vec{x} \Theta^{-1} = \vec{x}$, we choose $\Theta |x'\rangle = e^{i\delta}|x'\rangle = |x'\rangle$

$$= \int d^3x' \, |x'\rangle \langle x' | \alpha \rangle^*$$

$\therefore$ $\psi(\vec{x}') \xrightarrow{\text{T-reversal}} \psi^*(\vec{x}')$

the angular part

$$Y_{lm}(\theta, \phi) \longrightarrow [Y_{lm}(\theta, \phi)]^*$$

$$\langle \hat{n} | l\, m \rangle \qquad (e^{im\phi})^* = e^{-im\phi}$$

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_{lm}(\theta)\, e^{im\phi}.$$

$$P^*_{lm}(\theta) = P_{lm}(\theta)$$

$$[Y_{lm}(\theta, \phi)]^* = (-1)^m \sqrt{\frac{\boxed{(2l+1)(l-m)!}}{4\pi\, (l+m)!}}\, P_{lm}(\theta)\, e^{-im\phi}.$$

We can verify that $(Y_{lm}(\theta\phi))^* = (-1)^m Y_{l-m}$.

$$\Rightarrow \Theta | l\, m \rangle = (-1)^m | l, -m \rangle$$

$$[\circlearrowleft_{(m)}]^* \xrightarrow{T} [\circlearrowleft_{(m)}]^* = \circlearrowright_{(-m)} \; (-1)^m$$

(Theorem) $\Theta H \Theta^{-1} = H \rightarrow [\Theta, H] = 0$

$H|n\rangle = E_n|n\rangle$ $\quad \{|n\rangle\}$; nondegenerate.

then $\langle x|n\rangle$ is real.

$$H\Theta|n\rangle = \Theta H|n\rangle = \Theta E_n|n\rangle = E_n(\Theta|n\rangle)$$

$$[\Theta, H] = 0$$

$\Rightarrow |n\rangle$ an $\Theta|n\rangle$ have the same energy

Because $\{|n\rangle\}$ are nondegenerate,

$\Theta|n\rangle = c|n\rangle$, c is a complex number

$\Rightarrow$ ~~$|x\rangle$ projection~~

~~$\langle x|n\rangle$~~

~~$\langle x|\Theta|n\rangle = \int dx$~~

$$|n\rangle = \int dx\, |x\rangle\langle x|n\rangle$$

$$\Theta|n\rangle = \int dx\, \Theta|x\rangle\langle x|n\rangle^* = \int dx\, |x\rangle\langle x|n\rangle^*$$

$\Rightarrow \langle x|n\rangle = \langle x|n\rangle^* \Rightarrow$ wavefunction is real.

$\Rightarrow$ differ at most by a phase factor $e^{i\delta}$ independent of $\vec{x}$.

Hydrogen atom - $|n, l, m\rangle$; complex $(l \neq 0, m \neq 0)$

not a contradiction because

$|n, l, m\rangle$ are degenerate

For a state for $l=0$, $\psi(x) \to \psi^*(x)$

$$\Theta|\alpha\rangle = \Theta \int dx\, |x\rangle\langle x|\alpha\rangle = \int dx\, |x\rangle \langle x|\alpha\rangle^*.$$

if we expand in terms of the momentum eigenstate,

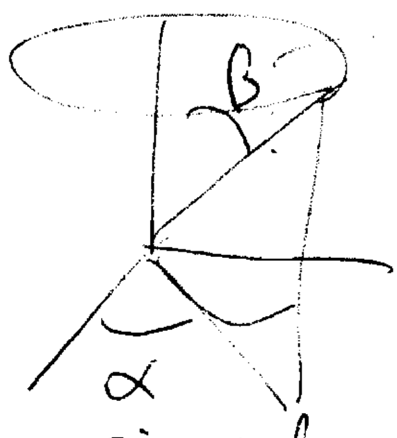
$$\Theta|\alpha\rangle = \Theta \int dp\, |p\rangle\langle p|\alpha\rangle = \int dp\, |-p\rangle\langle p|\alpha\rangle^*.$$

$$\underline{\underline{\phi(p) \to \phi^*(-p)}}$$

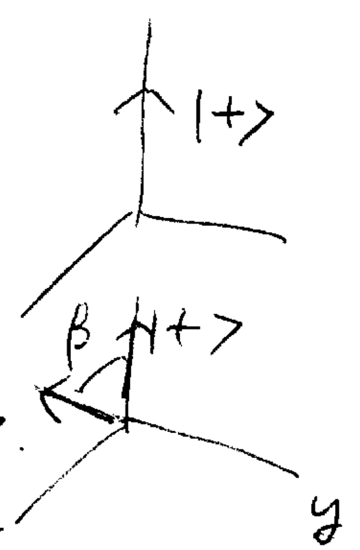
Time reversal for a spin $\frac{1}{2}$ system

Sec. 3.2 — polar angle.

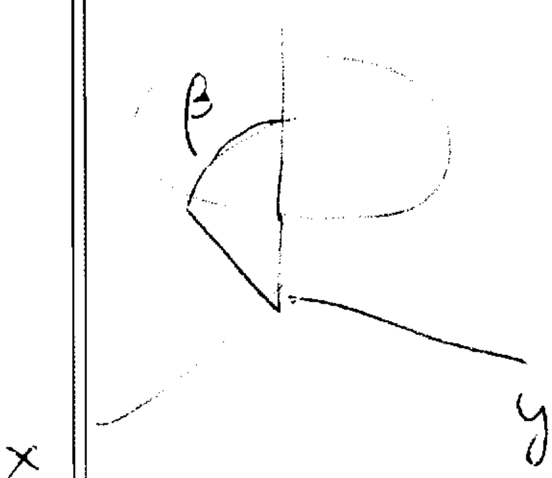
$|\hat{n}\rangle$

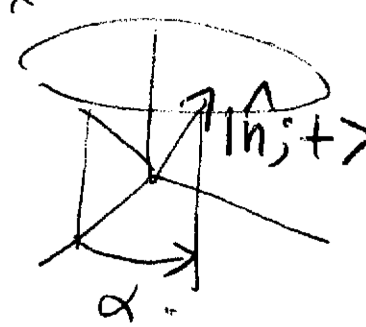


azimuthal angle



$e^{-\frac{i}{\hbar}S_y\beta}|+\rangle$





$$e^{-\frac{i}{\hbar}S_z\alpha}\, e^{-\frac{i}{\hbar}S_y\beta}|+\rangle$$