Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #7

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School of Electrical Engineering

Korea University

Prof. Young-Chai Ko

Outline

- Autocorrelation and Cross-correlation
- Random process in LTI system
- Power spectral density
- Gaussian random processes
- White processes
- White Gaussian noise processes

Autocorrelation and Cross-Correlation Functions

- Autocorrelation function for a stationary process is an even function.

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = R_X(t_2, t_1)$$

Let $\tau = t_1 - t_2$, we have

$$R_X(\tau) = R_X(-\tau)$$

- Cross-correlation function for a stationary process is an odd function.

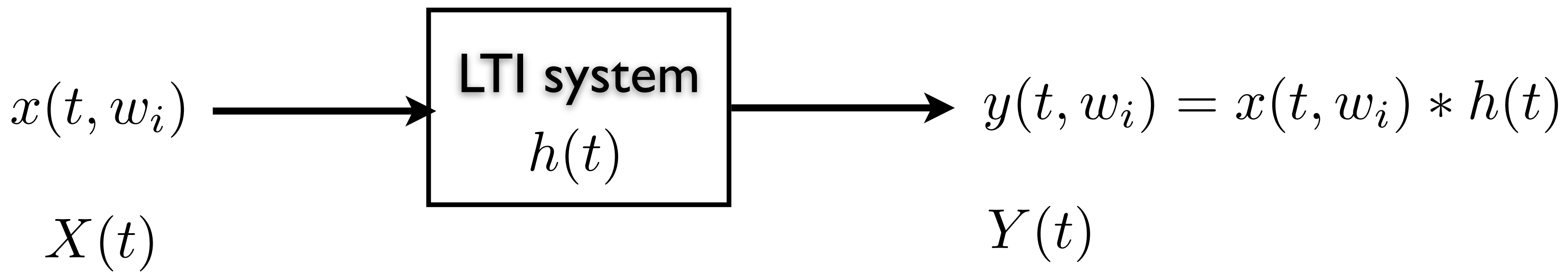
$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = R_{YX}(t_2, t_1)$$

Let $\tau = t_1 - t_2$, we have

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Multiple Random Process

- Example in the LTI system



- Independent processes

  Two random processes $X(t)$ and $Y(t)$ are independent if, for all $t_1,\ t_2$, the random variables $X(t_1)$ and $Y(t_2)$ are independent. Similarly, $X(t_1)$ and $Y(t_2)$ are uncorrelated if $X(t_1)$ and $Y(t_2)$ are uncorrelated for all $t_1,\ t_2$.

- Cross-Correlation

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

From the definition of cross-correlation, in general, we have

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = R_{YX}(t_2, t_1)$$

- Jointly wide-sense stationary if

1) $X(t)$ and $Y(t)$ are individually stationary

2) $R_{XY}(t_1, t_2)$ depends only on $\tau = t_1 - t_2$

- For jointly stationary process, it follows that

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Example

$X(t),\ Y(t)$: Jointly stationary RPs

Determine the autocorrelation of the process $Z(t) = X(t) + Y(t)$

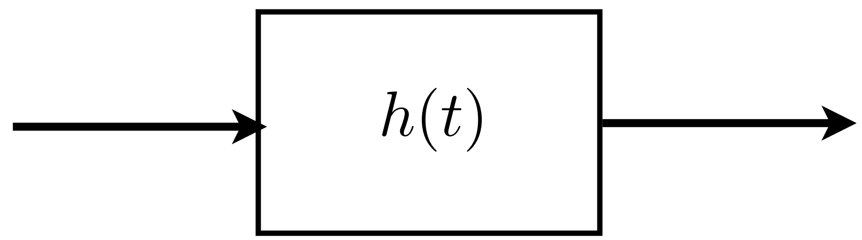
$$R_Z(t+\tau, t) = E[Z(t+\tau)Z(t)]$$

$$= E[(X(t+\tau) + Y(t+\tau))(X(t) + Y(t))]$$

$$= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{XY}(-\tau)$$

Random Processes and Linear Systems

Stationary process $X(t)$



$Y(t)$

$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(\lambda) h(t-\lambda)\, d\lambda$$

If $X(t)$ is stationary process, $Y(t)$ is also stationary!!!

$$E[X(t)] = m_X$$

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$E[Y(t)] = m_Y$$

$$R_Y(\tau) = E[Y(t+\tau)Y(t)]$$

- Mean of $Y(t)$

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,d\tau\right] f_{X(\tau)}(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(\tau)f_{X(\tau)}(x)\,dx\right] h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} E[X(\tau)]h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} m_X h(t-\tau)\,d\tau$$

change of variable

$$\lambda = t - u$$

$$= m_X \int_{-\infty}^{\infty} h(\lambda)\,d\lambda \quad \equiv m_Y$$

- Cross-correlation

$$E[X(t_1)Y(t_2)] = E[X(t_1)Y(t_2)] = E\left[X(t_1)\int_{-\infty}^{\infty} X(s)h(t_2 - s)\,ds\right]$$

$$= \int_{-\infty}^{\infty} E[X(t_1)X(s)]h(t_2 - s)\,ds$$

$$= \int_{-\infty}^{\infty} R_X(t_1 - s)h(t_2 - s)\,ds$$

change of variable

$u = s - t_2$

$$= \int_{-\infty}^{\infty} R_X(t_1 - t_2 - u)h(-u)\,du$$

$$= \int_{-\infty}^{\infty} R_X(\tau - u)h(-u)\,du$$

$$= R_X(\tau) * h(-\tau) \equiv R_{XY}(\tau)$$

- Auto-correlation

$$E[Y(t_1)Y(t_2)] = E[Y(t_1)Y(t_2)]$$

$$= E\left[\left(\int_{-\infty}^{\infty} X(s)h(t_1 - s)\, ds\right) Y(t_2)\right]$$

$$= \int_{-\infty}^{\infty} R_{XY}(s - t_2)h(t_1 - s)\, ds$$

$$= \int_{-\infty}^{\infty} R_{XY}(u)h(t_1 - t_2 - u)\, du$$

$$= R_{XY}(\tau) * h(\tau)$$

$$= R_X(\tau) * h(-\tau) * h(\tau) \equiv R_Y(\tau)$$

Power Spectral Density of Stationary Process

- Power spectral density
  - A function that determines the distribution of the power of the random process at different frequencies

Theorem

For a stationary random process $X(t)$, the power spectral density is the Fourier transform of the autocorrelation function, i.e.,

$$S_X(f) = \mathcal{F}[R_X(\tau)]$$

Since $R_X(\tau) = R_X(-\tau)$, we can show $S_X(f) = S_X^*(f)$, that is, $S_X(f)$ is a real function.

Example

- Consider a random process

$$X(t) = A\cos(2\pi f_0 t + \Theta), \qquad \text{where} \quad \Theta \sim \mathcal{U}[0, 2\pi]$$

  - Autocorrelation function

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau),$$

  - Power spectral density

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

Power

- Power from Power Spectral Density

$$P_X = \int_{-\infty}^{\infty} S_X(f)\, df$$

- Power from autocorrelation

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau}\, d\tau \quad \Longrightarrow \quad R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, d\tau = P_X$$

- Power in the previous example

$$\begin{aligned} P_X &= \int_{-\infty}^{\infty} S_X(f)\, df = \int_{-\infty}^{\infty} \left[\frac{A^2}{4} [\delta(f-f_0)_+ \delta(f+f_0)] \right] df \\ &= 2 \times \frac{A^2}{4} = \frac{A^2}{2} \end{aligned}$$

Power Spectra in LTI System

- Input-Output relation in LIT system

$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\, d\tau$$

- Mean of $Y(t)$

$$m_Y = m_X \int_{-\infty}^{\infty} h(t)\, dt \quad \Longleftrightarrow \quad m_Y = m_X H(0)$$

since $H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}\, dt$

- Autocorrelation and Power spectral density

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau) \quad \Longleftrightarrow \quad S_Y(f) = S_X(f)|H(f)|^2$$

$S_Y(f)$ is a real function

- Crosscorrelation and Power spectral density

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau) \quad \Longleftrightarrow \quad S_{XY}(f) = S_X(f)H^*(f)$$

$S_{XY}(f)$ is in general, a complex function

- Also note that

$$R_{YX}(\tau) = R_{XY}(-\tau) \quad \Longleftrightarrow \quad S_{YX}(f) = S_{XY}^{*}(f) = S_X(f)H(f)$$

Since $S_X(f)$ is a real function

- Variance

$$\sigma^2 = E[X^2(t)] = E[X(t)X(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df$$

Gaussian Processes

- Gaussian processes
  - A random process $X(t)$ is a Gaussian process if for all n and all $(t_1, t_2, \ldots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^{n}$ have a jointly Gaussian density function.

- Example:

$$X(t): \text{ Zero-mean stationary Gaussian RP with } S_X(f) = 5\prod\left(\frac{f}{1000}\right)$$

Determine the PDF of the random variable $X(3)$.

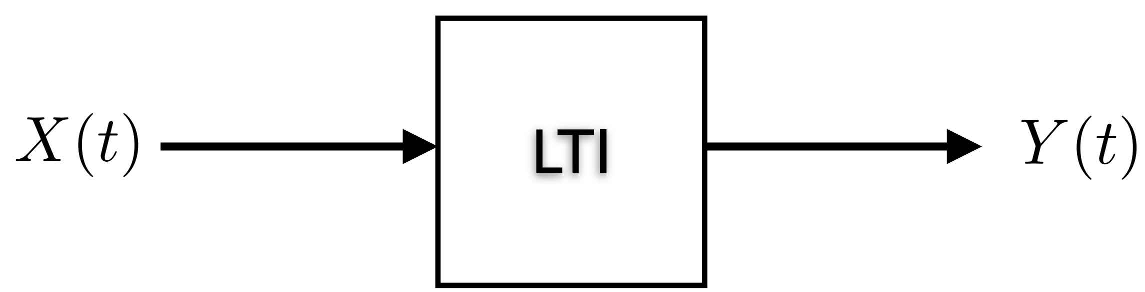
$$\sigma^2 = \int_{-\infty}^{\infty} S_X(f)\, df = 5000$$

$$f_{X(3)}(x) = \frac{1}{\sqrt{10000\pi}} e^{-\frac{x^2}{10000}}$$

Properties of Gaussian Processes

Property 1

In LTI system, the input signal, $X(t)$, which is a Gaussian random process, the output signal, $Y(t)$ is also a Gaussian random process.

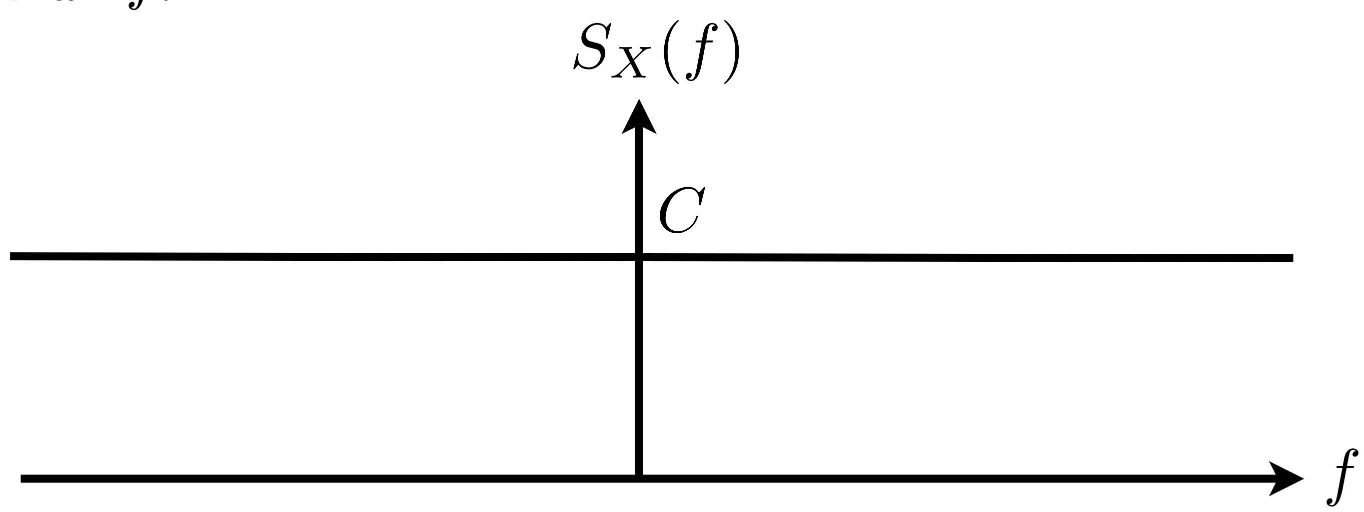


Property 2

For jointly Gaussian processes, uncorrelatedness and independence are equivalent.

White Processes

- Definition
  - A process $X(t)$ is called a white process if it has a flat spectral density, i.e., if $S_X(f)$ is a constant for all f.



- Power of white processes

$$P_X = \int_{-\infty}^{\infty} S_X(f)\, df = \int_{-\infty}^{\infty} C\, df = \infty.$$

Thermal Noise

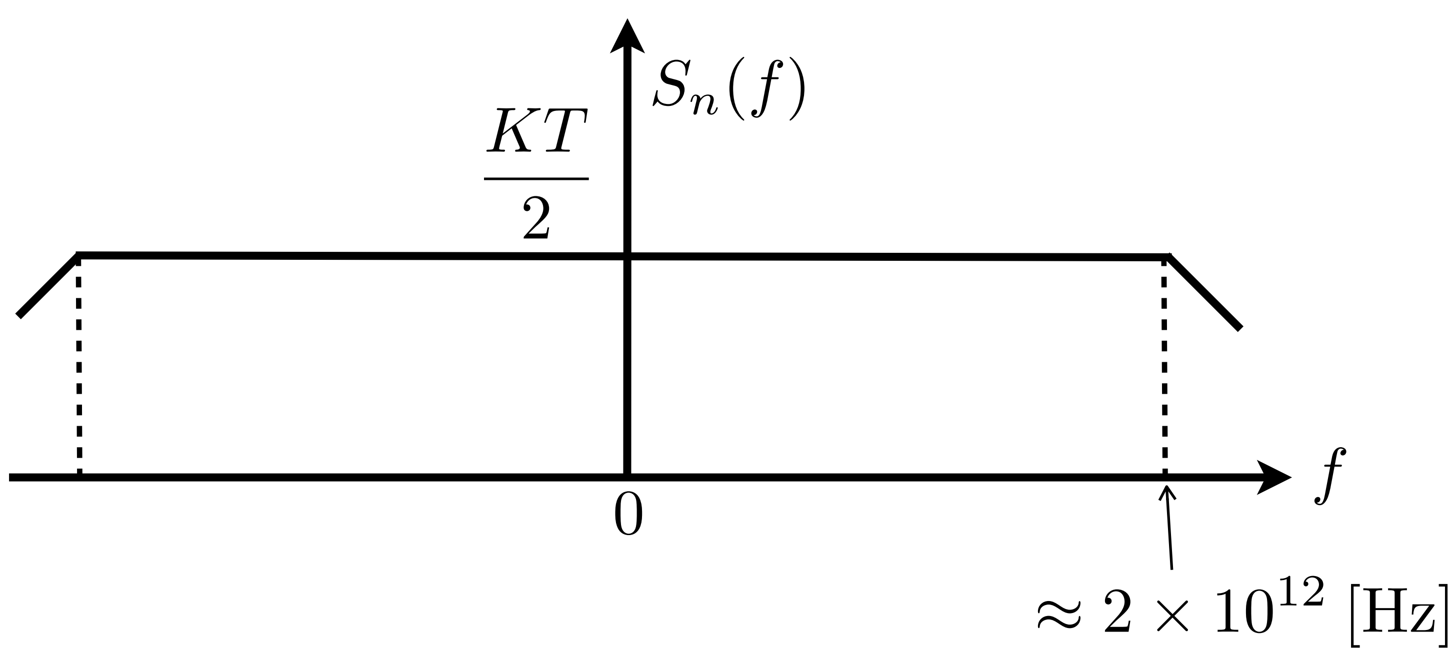
- Quantum mechanical analysis of thermal noise

$$S_n(f) = \frac{hf}{2(e^{\frac{hf}{kT}} - 1)}$$

h : Planck's constant, equal to 6.6×10^{-34} J× sec

k : Boltzmann's constant, equal to 1.38×10^{-23} J/ K

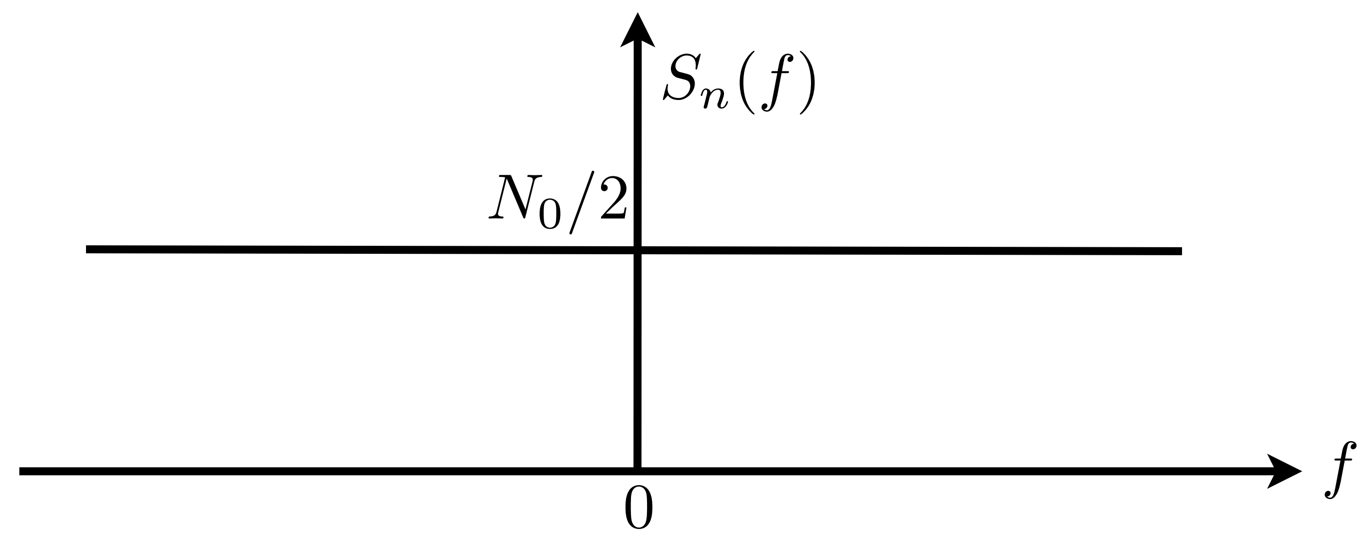
T : Temperature in Kelvin, $T = 300°$ at room temperature



Notation of White Gaussian Process

- Power spectral density

$$S_n(f) = \frac{N_0}{2}$$



- Autocorrelation function

$$R_n(\tau) = \mathcal{F}^{-1}\left[\frac{N_0}{2}\right] = \frac{N_0}{2}\delta(\tau)$$

Properties of the Thermal Noise

Property 1

Thermal noise is a stationary process.

Property 2

Thermal noise is a zero-mean process.

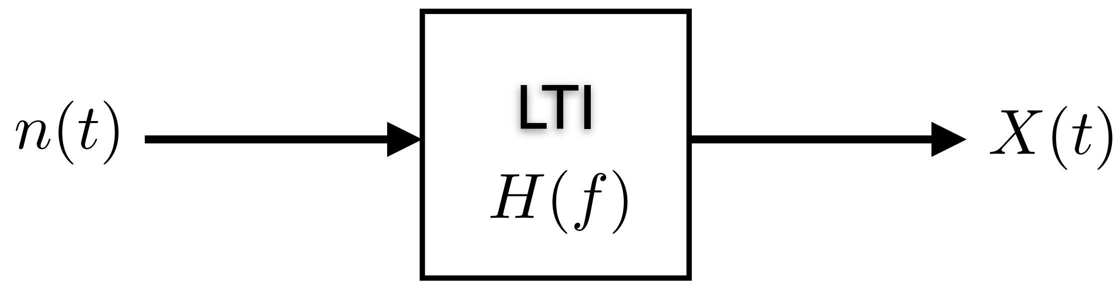
Property 3

Thermal noise is a Gaussian process.

Property 4

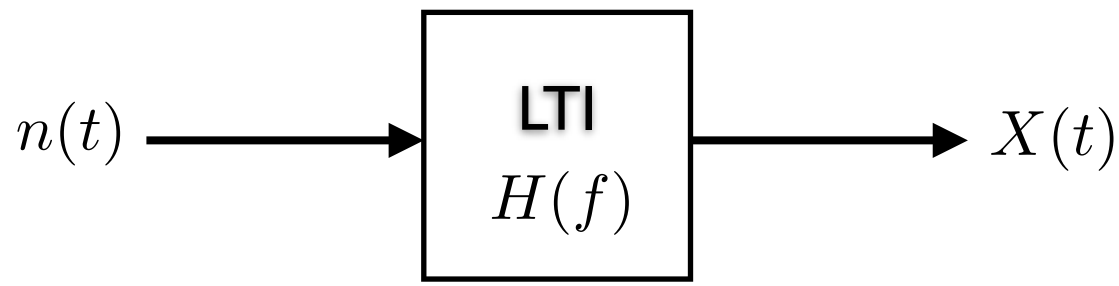
Thermal noise is a white process with a power spectral density $S_n(f) = \frac{kT}{2} = \frac{N_0}{2}$.

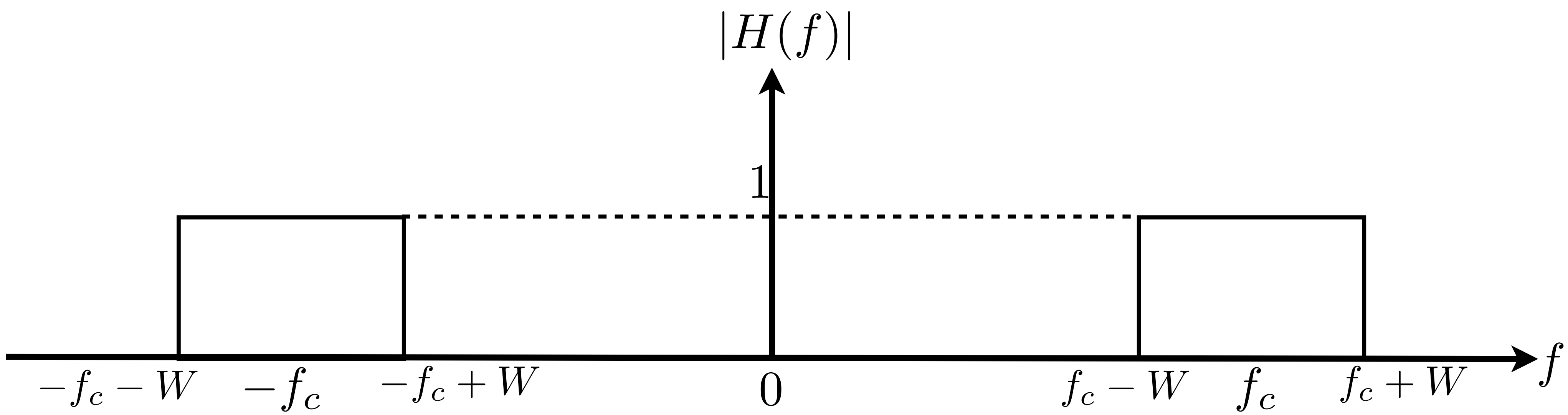
Filtered Noise Processes

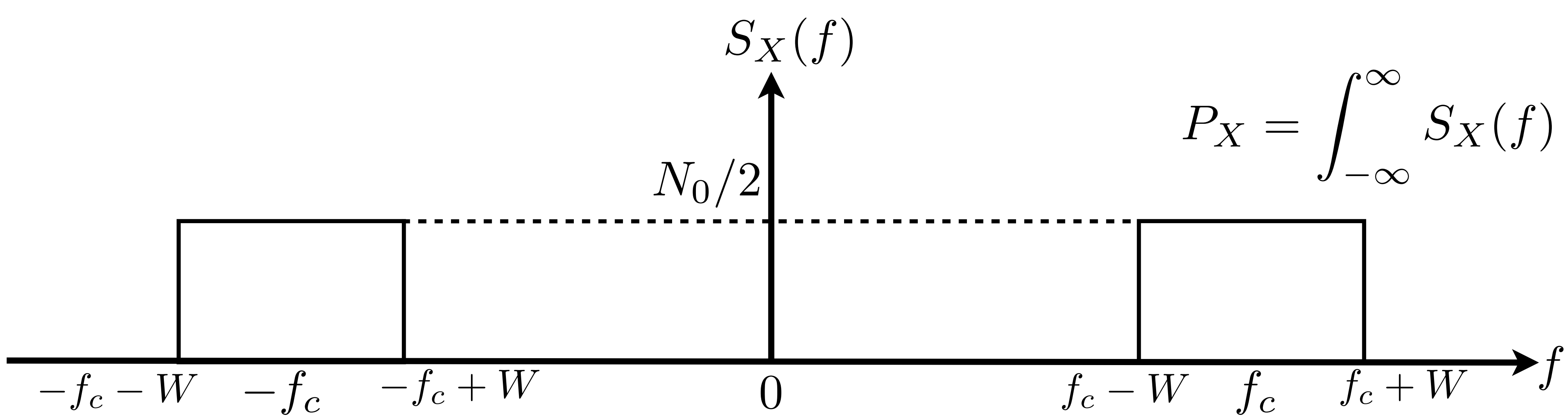


$$S_X(f) = S_n(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

Example of Filtered White Gaussian Process







$$P_X = \int_{-\infty}^{\infty} S_X(f)\, df = 2N_0W$$